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Small and Large Price Changes and the Propagation of Monetary Shocks

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Abstract

We present new evidence on the presence of both small and large price changes in individual price records from the CPI both in France and in the US. After correcting for measurement error and cross-section heterogeneity we find that the size distribution of price changes has a positive excess kurtosis, with a shape that lies between a Normal and a Laplace distribution. We construct a menu-cost model that is capable to reproduce the observed empirical patterns. The model, which features multiproduct firms and randomness in menu cost, has only 4 parameters, two of which are pinned down by the average frequency and by the standard deviation of price changes. Very different propagation mechanism, spanning the models of Taylor (1980), Calvo (1983) and Golosov and Lucas (2007), are nested under different combination of the remaining two parameters. We discuss the identification of these parameters using observations on the shape of the size-distribution of price changes (e.g. its kurtosis) and the actual cost of price adjustments borne by firms. We characterize analytically the response of the aggregate economy to a monetary shock, and how it depends on the variance and kurtosis, as well as on the frequency, of price changes.

JEL Classification Numbers: E3, E5

Key Words: price setting, micro evidence, distribution of price changes, menu cost, Calvo rule, monetary shocks

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1 Introduction and summary

This paper uses new micro evidence, and a new menu-cost model, to study the propagation of monetary shocks in an economy with sticky prices. We use the cross-section evidence to discipline the parametrization of our model which features multiproduct firms and random menu costs. The model has only 4 parameters, two of which are pinned down by the average frequency and by the standard deviation of price changes. Very different propagation mechanisms, spanning the models of Taylor (1980), Calvo (1983), Golosov and Lucas (2007) and Alvarez and Lippi (2013), are nested under different combinations of the remaining two parameters. The unified framework allows us to compare their effects and unveils the assumptions that are required to obtain each of them as an optimal mechanism. These assumptions can be tested empirically. We discuss the identification of these parameters using observations on the shape of the size-distribution of price changes, e.g. its kurtosis, and the actual cost of price adjustments borne by firms.

Our empirical contribution is to document the presence of small and large price changes using a large and comprehensive dataset of price records underlying the French CPI. The presence of small price changes is pervasive in the size-distribution of price changes and this finding persists even at a very disaggregate level of product-outlet-type, ruling out an explanation based on pure cross-section heterogeneity. These patterns are very similar to the ones that Klenow and Kryvtsov (2008) detect for the US. We moreover acknowledge that the CPI data may contain measurement error that tends to distort the measure of peakedness of the distribution of price changes. We propose a correction for this measurement error and conclude that the shape of the size-distribution of price changes is in between a Normal and a Laplace distribution.

We develop an analytical model that matches these patterns, at least qualitatively, with small price changes and positive excess kurtosis. The model yields some new results (like the fact that for large menu costs, or large fraction of free adjustments, the decision rule is not the quartic root and it has no option value). The model is parsimonious: for a given economy, as defined by a frequency and the scale of price changes, N_a and $Std(\Delta p_i)$ respectively, there are only two remaining parameters completely determining the cross-sectional behavior of the economy and the impulse response to shocks: the fraction of free adjustments λ/N_a and the number of goods sold in the bundle to which the menu costs applies: n.

We use the model to solve for the impulse responses of the aggregate economy to a once-and-for-all unexpected monetary shock analytically. We show how real effect of a monetary shock depend on the shock size and on the fundamental "parameters". Surprisingly, fixing the frequency N_a and scale $Std(\Delta p_i)$ of price adjustments, the real effects of monetary policy are an increasing function of *only one variable*, namely the Kurtosis of the price changes. We characterize how the kurtosis depends on the remaining two parameters, λ/N_a and n, showing that kurtosis ranges from 1, the value associated to the canonical menu cost model with $\lambda/N_a = 0$ and $n = 1$, to 6, the value in a Calvo-type model where $\lambda/N_a = 1$ (in this extreme case n becomes irrelevant). In general, for any given fraction of free adjustments λ/N_a the level of kurtosis is increasing in n, and so are the real effects of monetary policy.

The paper is organized as follows: the next section presents the cross section evidence on price setting behavior using data for France and the USA taken from various sources. Section 3 presents the theoretical model and the cross section predictions: it is shown that the model has fundamentally four parameters and we discuss the mapping between those and observable measures of price setting behavior. Section 4 derives the model predictions on the effect of an unexpected monetary shock.

2 The distribution of price changes: micro-evidence

A vast amount of research has investigated the patterns of price changes at the microeconomic level in the past decade. A recurring fact that emerges from those studies is that the size distribution of price changes exhibits a large amount of small price changes, as noted by Klenow and Malin (2010); Cavallo (2010); Klenow and Kryvtsov (2008) and Midrigan (2011) using selected samples of micro data from the US as well as many other industrial countries. This section revisits this evidence using a detailed dataset of price quotes underlying the French Consumer Price Index (about 65% of the CPI weights from 2003 to 2011), and comparing it with other micro datasets that are not affected by measurement error, as well as with comparable data from the US.

Two issues that are discussed in details concern heterogeneity and measurement error. Heterogeneity across type of goods and of outlets is pervasive in price data. A well known result related to mixtures of distribution is that under heterogeneity, the pooled data will have a spuriously large kurtosis.[1] For this reason, we standardize the data at levels at which we suspect that there is heterogeneity. We define the standardized price changes, z, by demeaning and dividing by the standard deviation of price changes at fine cell levels. A cell is a category of good and of outlet type. We then compute the statistics for the pooled standardized data. We discuss the theoretical set-ups (i.e. type of heterogeneity) where results can be obtained by aggregating across categories or goods differences. The nature of the correction for measurement error is to compare the CPI statistics with scanner data for similar goods and outlet types for which both sources are available. Our analysis shows that, after correcting for measurement error and removing the (time invariant) cross section heterogeneity, the size distribution of price changes features a large frequency of very large and very small price changes relative to what the standard menu cost model implies.[2]

We find it useful to compare the empirical distribution of price changes to three parametric distributions ordered in terms of increasing frequency of extreme price changes: the binomial, the Normal, and the Laplace distribution. Overall we conclude that, after taking into account heterogeneity and measurement error, the shape of the empirical distribution of price changes

[1] Formally, let Δp be a mixture of the $i = 1, ..., M$ distributions Δp_i with strictly positive weights, where we assume that $E[\Delta p_1] = \cdots = E[\Delta p_N] = 0$. Denote by k_i the kurtosis $k_i = E[\Delta p_i^4]/(Std(\Delta p_i)^4)$ and by r_i the ratio $r_i = E[|\Delta p_i|]/Std(\Delta p_i)$ for each distribution i. Assume that $k_1 = k_2 = \cdots = k_M \equiv k$ and that $r_1 = r_2 = \cdots = r_M \equiv r$. Then the statistics for the pooled data satisfy: $E[\Delta p_i^4]/(Std(\Delta p_i)^4) \geq k$ and $E[|\Delta p_i|]/Std(\Delta p_i) \geq r$, with equality iff $Std(\Delta p_1) = Std(\Delta p_2) \cdots = Std(\Delta p_M)$. Thus, standardizing the M distributions will preserve the values of k and r.

[2] We underline that, without correcting for measurement error and heterogeneity, the raw CPI data feature even more extreme price changes.

lays in "between" a Normal and a Laplace distribution. To quantify the presence of "extreme price changes" we focus on 3 statistics that are informative about the shape of the size distribution. These 3 statistics are appropriate for symmetric, zero-mean, distributions and are scale-free. The first 2 statistics measure the frequency of extreme (i.e. large and small) observations relative to the standard deviation of the distribution. The first one is kurtosis (see Balanda and MacGillivray (1988)), $\mathbb{E}[\Delta p_i^4]/Std(\Delta p_i)^4$, where for now we assume that price changes Δp_i are centered . As a benchmark, we note that for the Binomial, Normal and Laplace distribution the Kurtosis is 1, 3 and 6 respectively. The second statistic measuring extreme price changes is $\mathbb{E}[|\Delta p_i|]/Std(\Delta p_i)$. The main difference with respect to Kurtosis is that this metric is less sensitive to extreme outliers (since the squares of large (small) numbers are larger (smaller) than absolute values). For the Binomial, Normal and Laplace distributions the reference values are: 1, 0.80 and 0.70. The third statistic we consider is $P(|\Delta p_i| < (1/4)\,\mathbb{E}[|\Delta p_i|])$, a straightforward measure of the share of *small* price changes used in several previous studies. For the Binomial, Normal and Laplace distribution this statistic is 0, 0.16 and 0.22 respectively.

2.1 The French Data

In this section we describe the French data and construct summary statistics on the size distribution of price changes using a standardized measure that removes the cross industry heterogeneity. We also discuss measurement error by comparing the CPI data with another source presumably immune from measurement error: the scraped data from Cavallo (2010). Finally, we compare our evidence on the French data with existing comparable results that are available for the US.

The data are a longitudinal dataset of monthly price quotes collected by the INSEE (Institut National de la Statistique et des Etudes Economiques) in order to compute the French CPI, over the period 2003:4 to 2011:4.[3] Each record relates to a precisely defined

[3] The dataset is documented in details in Berardi, Gautier, and Le Bihan (2013).

product sold in a particular outlet in a given year and month. It contains the price level of the product, as well as limited additional information such as an outlet identifier, an index (when relevant) for package size (say 1 liter) and flags indicating the presence of sales. The raw dataset contains around 11 million price quotes and covers about 65% of the CPI weights.[4] The dataset also includes CPI weights, which we use to compute aggregate statistics. Price changes are computed as 100 times the log-difference in prices per unit. To minimize the presence of measurement errors we discarded observations with item substitutions (which might give rise to spurious price changes) and removed "outliers" which, in our baseline analysis, we defined as price changes larger than 0.1 percent, or lower than ln(10/3) (both in absolute value). See Appendix A for more information and several robustness checks.

An important issue with the data on price changes is the treatment of sales. The relevance of dealing with sales in analyzing price stickiness was emphasized by Nakamura and Steinsson (2008); Kehoe and Midrigan (2007) and Midrigan (2011) inter alia. The INSEE dataset contains an indicator variable that identifies whether a given observed price corresponds to a sales promotion discount (either seasonal sale or temporary discounts).[5] Price changes that result from sales (including price changes from a sales price to a regular one) account for approximately 17% of all the price changes . Overall, the incidence of sales on the frequency of price change is less important than in the US where according to Nakamura and Steinsson (2008) the share of price change due to sales is 21.5%. In the following, as a robustness check, we report results both with and without sales observations.

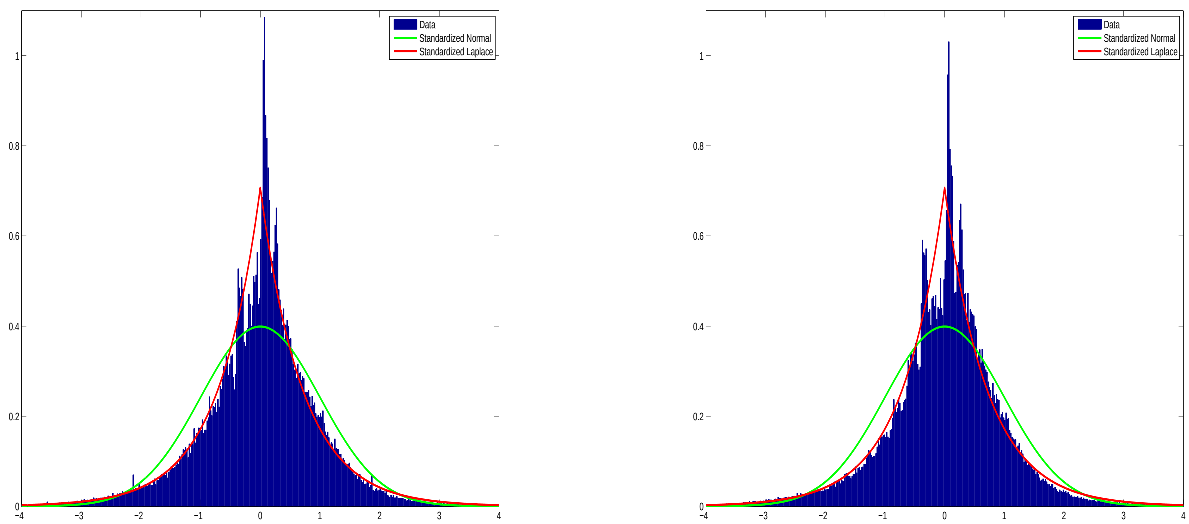
We now document the patterns on the peakedness and thick tails of the distribution of price changes. As those patterns vary considerably across sectors and outlet type, a concern already mentioned is that a large variance and kurtosis of price changes may essentially

[4]Some categories of goods and services are not available in our sample: fresh foods, rents, and prices centrally collected by the statistical institute - among which car prices and administered and public utility prices (e.g. electricity). Note that, while rents are out of our dataset, cost of owner-occupied housing is not incorporated in the French CPI, so the share of housing is the CPI is lower than in some other countries.

[5] The flag is documented directly by the field agent recording prices rather than constructed using a statistical filter. Baudry et al. (2007) investigate the extent of "undetected" sales and conclude this is a limited concern.

Figure 1: Histogram of Standardized Price Adjustments: French CPI 2003-2011

All data | Excluding sales



The figures uses the elementary CPI data from France (2003-2011). Price changes are the log difference in price per unit, standardized by good category (272) and outlet type (11) and pooled. Price changes equal to zero are discarded.

reflect that observations of price changes are drawn from a mixture of distributions, and thus be artefacts. In what follows we address this concern by considering the distribution of standardized price change.[6] We consider a breakdown of the data into J categories (for instance, one category will be bread in supermarkets). In each category j the standardized price change for an item i at date t is defined as $z_{ijt} = (\Delta p_{ijt} - m_j)/\sigma_j$ where m_j and σ_j are the mean and standard deviation of price changes in category j, and price changes equal to zero are disregarded. We will here use the finest partition possible in our data (each category is a COICOP category at the 6-digit level in an outlet type) and have around 1,500 categories.[7] Figure 1 is a weighted histogram of the standardized price changes. On the same graph we superimpose the density of the standard normal distribution as well as the standardized Laplace distribution (both have unit variance). The Laplace distribution has a kurtosis of 6 and is thus more peaked than the normal. It is apparent that the empirical distribution of standardized price changes is closer to the Laplace distribution than to the Normal.[8]

Table 1 reports the frequency of price changes as well as selected moments of the distribution of price changes. The frequency of price change is around 17% per month. The fraction of price decreases among price changes is around 40%. The average *absolute* price change is sizeable (9.19%), as is the standard deviation of price change (16.6%). These patterns match those documented by Alvarez et al. (2006) for the Euro area. With the qualification that frequency of price change is typically found to be smaller in the Euro area than in the US, they also broadly match US evidence provided by e.g. Nakamura and Steinsson (2008). The kurtosis and peakedness of the distribution of price changes have not been quantitatively documented so far on European data. The kurtosis of non-standardized price changes is huge: 12.81. This level of kurtosis is of same order of magnitude as that documented by Klenow

[6] This follows Klenow and Kryvtsov (2008), as well as Midrigan (2011).

[7] There are 11 outlet types and 272 CPI categories; but not every category of good is sold in a given outlet type, resulting in less than 2,992 cells.

[8] In an online appendix, we provide similar histograms by groups of good at a disaggregated level. Most of them have the same pattern as Figure 1, that is a distribution that is more peaked than the gaussian, and often more peaked than the Laplace.

Table 1: Selected moments from the size distribution of price changes (French CPI)

	Data		Benchmarks	
	all records	exc.sales	Normal	Laplace
Frequency of price changes	17.09	14.70		
Fraction of price changes that are decreases	39.23	35.73		
Moments for the size of price changes: Δp				
Average	0.33	1.06		
Standard deviation	16.60	8.01		
Kurtosis	12.81	20.86		
Moments of standardized price changes: z				
Kurtosis	8.89	10.40	3	6
Moments for the absolute value of standardized price changes: $\lvert z \rvert$				
Average: $\mathbb{E}(\lvert z \rvert)$	0.70	0.69	0.80	0.70
Fraction of observations $< 0.25 \cdot \mathbb{E}(\lvert z \rvert)$	22.2	20.7	15.8	22.1
Fraction of observations $< 0.5 \cdot \mathbb{E}(\lvert z \rvert)$	39.3	38.6	31.0	39.4
Fraction of observations $> 2 \cdot \mathbb{E}(\lvert z \rvert)$	12.9	12.5	11.1	13.5
Fraction of observations $> 4 \cdot \mathbb{E}(\lvert z \rvert)$	1.8	2.0	0.0	1.8
Number of obs. with $\Delta p \neq 0$	1,544,829	1,080,183		

Source is INSEE, monthly price records from French CPI, data from 2003:4 to 2011:4. Coverage is around 65% of CPI weight since rents, and prices of fresh food and centrally collected items (e.g. electricity, train and airplane tickets) are not included in the dataset. Frequency of price change is monthly, in percent. Size of price change are the first-difference in the logarithm of price per unit, expressed in percent. Observations with imputed prices or quality change are discarded. Observations such that $0.1/100 \leq |\Delta p| \leq \ln(10/3)$ are removed as outliers. "Exc. sales" exclude observations flagged as sales by the INSEE data collectors *and* observations such that $0.1/100 \leq |\Delta p| \leq \ln(10/3)$. Moments are computed aggregating all prices changes using CPI weights at the product level. The third and fourth panels report moments for the standardized price change $z_{jit} = \frac{\Delta p_{jit} - m_i}{\sigma_{\Delta p_i}}$ where m_i and $\sigma_{\Delta p_i}$ are the mean and standard deviation of price changes in category j (see the text). The Normal and Laplace distributions used in the last two columns have a zero mean and, without loss of generality, standard deviation equal to one.

and Malin (2010) for the US.[9] Considering standardized price changes delivers a similar picture: the kurtosis is 8.89. The fraction of small price changes is also large. The fraction of absolute standardized price changes lower than one fourth of the mean is 22.2 percent. Also 12.9 percent of absolute normalized price changes are larger than 2 times the mean of the absolute standardized price change. Overall, it appears that these figures are very close to the ones that would be produced by a (standardized) Laplace distribution. Consistently, the size of the average absolute standardized price change in the data is equal to 0.70, the same

[9] They report a kurtosis of 10 for posted prices and 17.4 for regular prices.

value that obtains for the statistic $\mathbb{E}[|\Delta p|]/Std(\Delta p_i)$ if Δp follows a Laplace distribution.

Removing sales has a large effect on the variance of absolute price change, as indicated by the results reported in the second column of Table 1.[10] However, removing sales does not affect our findings on the peakedness of the distribution. Kurtosis actually increases when sales observations are removed both in the raw data as well as in the standardized data. This is also visible in the right panel of Figure 1 which plots the distribution of standardized non sales-related price changes.

2.2 Quantifying measurement error

Eichenbaum et al. (2012) have warned that the small price changes recorded in the data may reflect measurement error. Appendix A.2 explores the concerns raised by Eichenbaum et al. (2012) and concludes that they only partially apply to the French data we analyze. However, we analyze below the consequences of one particular type of measurement error, arising from unrecorded product substitutions. We show that a small amount of this measurement error, inconsequential for measuring the aggregate the cost of living, may have sizeable consequences for the measurement of the descriptive statistics displayed in Table 1, such as Kurtosis, and suggest a procedure to correct for it.

A simple model of measurement error is useful in interpreting the data. We let Δp_m measure the observed price changes which are given by a mixture of two distributions:

$$\Delta p_m = \begin{cases} \Delta p_u & \text{with prob. } \zeta \\ \epsilon & \text{with prob. } 1-\zeta \end{cases}$$

where we interpret that ϵ is a measurement error and Δp_u is a "true" price change. This assumption aims to capture that, even at the finest level of disaggregation, some price changes in the CPI data are the consequence of small product substitution (e.g. different brands for a

[10]When removing "sales" price changes we remove any observation flagged as sales, as well as the subsequent observation of price increase on the way back to a "regular" price. Note also that for computed standardized non-sales related price change, we first discard sales related price change, then standardize the data

given good being recorded) which do not reflect an actual change in the good's price. Assume the distribution of Δp_u has standard deviation σ_u and kurtosis k_u, assumed independent of σ_u. Likewise the distribution of ϵ has kurtosis k_e and standard deviation σ_e. Both distributions are assumed to have zero expected value. One interpretation is that quality changes (not recorded by the statistical office) generate "artificial" price changes. We assume that these price changes are small, i.e. that σ_e is small, and that the process for the unreported changed in quality is independent of the "true" changes in prices. The kurtosis of the observed price changes is then equal to:

$$Kurt[\Delta p_m] = k_u \, \frac{\zeta\sigma_u^4 + (k_\epsilon/k_u)\sigma_e^4}{\zeta^2\sigma_u^4 + (1-\zeta)^2\sigma_e^4 + 2\zeta(1-\zeta)\sigma_e^2\sigma_u^2}$$

Letting σ_e go to zero we obtain that Kurtosis measured over the (observed) price changes is:

$$\lim_{\sigma_e \downarrow 0} Kurt[\Delta p_m] = \frac{k_u}{\zeta} \tag{1}$$

Thus, if the sample includes a fraction ζ of true price changes and the rest are spuriously imputed small price changes the kurtosis will increase by a factor $1/\zeta$, relative to the kurtosis of the true distribution.[11] Thus equation (1) may allow us to quantify ζ by comparing the observed kurtosis across a sample with measurement error and one without. We now turn to addressing this issue empirically.

We match a subset of our French CPI data with the prices for several French retailers taken from the Billion Price Project (BPP) dataset (see Cavallo (2010)). The BPP data are "scraped" on-line, thus they are arguably less contaminated by measurement errors.[12] We compare the results obtained using the scraped BPP data from two large retailers with our

[11]Under this interpretation the number of measured price changes, denoted by N_{am} will be higher than the number of true price changes per unit of time, say N_{au}. Let's denote $N_{a\epsilon}$ the expected number of incorrectly imputed price changes. We have: $N_{am} = N_{au} + N_{a\epsilon} = \zeta N_{am} + (1-\zeta)N_{am}$. Thus if we have two estimates of $kurt[\Delta p]$ and of N_a and we assume that one has no measurement error and the other has a fraction ζ of small imputed price changes as described above, can estimate ζ using either the ratio of the two estimates of kurtosis or the ratio of the two estimates of the number of price changes per unit of time.

[12] We are extremely grateful to Alberto Cavallo for sharing part of his data with us.

Table 2: Comparison of the CPI vs. the BPP data in France

Statistic	BPP retailer 1	BPP retailer 5	CPI Hypermarkets	BPP retailer 4	CPI Large ret. electr.
duration	8.58	8.06	4.82	6.44	7.24
		Statistics for standardized price changes: z			
mean $\|z\|$	0.71	0.70	0.65	0.78	0.70
% below 0.5 mean $\|z\|$	37.85	40.93	45.48	29.17	41.69
% below 0.25 mean $\|z\|$	17.46	25.26	26.19	15.33	23.10
kurtosis of z	5.50	4.30	10.15	2.82	6.33

Note: The BBP data are documented in Cavallo (2010). Results were communicated by the author. For CPI data source is INSEE, monthly price records from French CPI, data from 2003:4 to 2011:4. Sub-sample in column (3) is price records in outlet type "hypermarkets". Sub-sample in column (5) is goods in the category of appliances and electronic , as identified using the Coicop nomenclature, collected in the following outlets type: "hypermarkets","supermarkets", and "large area specialists". Data are standardized within each subsample using Coicop categories.

results based on the CPI data for a similar type of outlet: to this end we restrict our dataset to CPI price records in "hypermarkets", excluding gasoline. We also compare with the BBP data from a large French retailer specialized in electronic and appliances. In that case we restrict the CPI dataset to goods in the category of appliances and electronic using the Coicop nomenclature, collected in outlets type "hypermarkets","supermarkets", and "large area specialists".

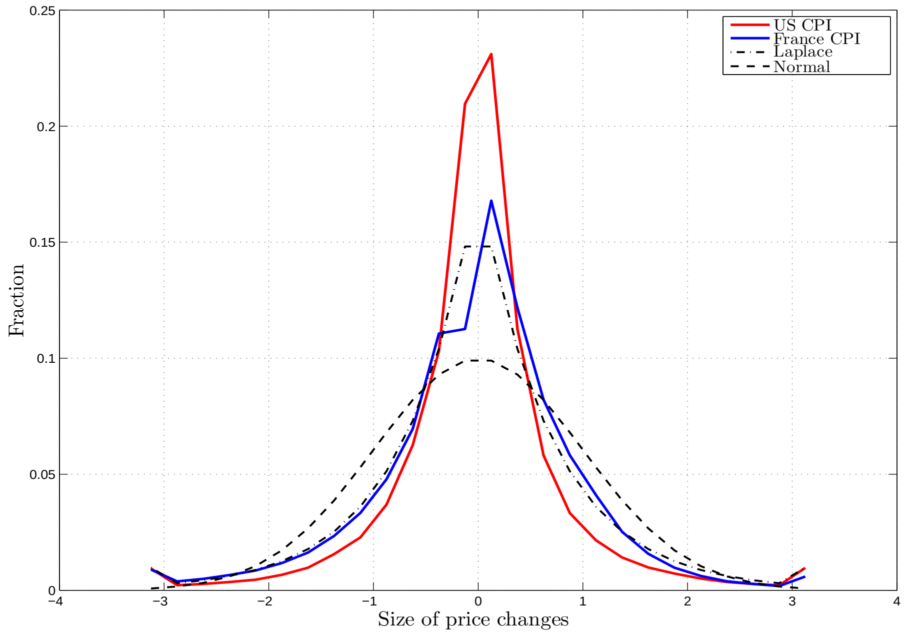
Comparing the values of kurtosis from both data sets suggests that $\zeta \cong 0.5$. We can apply this magnitude to the full sample of CPI data of Table 1, for which no "measurement error-free" counterpart like the BPP exists, to obtain a corrected kurtosis. The number thus obtained for the Kurtosis ranges between 4 and 5 (using the kurtosis of 8.89 of standardized price changes), so it lays in between the kurtosis of the Normal and the Laplace distribution.

2.3 A comparison with the US data

To assess whether the patterns documented above are specific to France we compare our data with the US figures presented, respectively, in Klenow and Kryvtsov (2008) and in Eichenbaum et al. (2012).

Figure 2 plots four histograms: two are price changes from the US and France, while the

Figure 2: Histogram of Standardized Price Adjustments: US and French CPI



Sales data are excluded. Data for France are from the CPI as in Figure 1. The CPI data for the US are taken from Figure 3 in Klenow and Kryvtsov (2008). Price changes equal to zero are discarded.

other two are theoretical benchmarks. The first one (in red) is the distribution of standardized (weighted) price changes (excluding sales) for the US based on Figure 3 of Klenow and Kryvtsov (2008).[13] Since the distribution is truncated at -3 and +3, its standard deviation is 0.83 instead of 1, its kurtosis is 6.95. The second histogram (in blue) is the distribution of the standardized price changes (excl. sales) for the French CPI, constructed using the trimming criteria used for the US. This distribution has a standard deviation 0.95 and a kurtosis of 4.42.[14] The figure also reports, for comparison, the standardized Normal and Laplace distributions (discretized and truncated). The main outcome of Figure 2 is that the histogram of standardized, non-sales, price change are very similar in France and the US. Furthermore, in both cases the shape is closer to that of a Laplace distribution than to a Gaussian one (and consistently with previous sub-section, in both cases we conjecture measurement error explains why these distribution are actually more peaked than the Laplace).

Table 3 uses the same thresholds of Eichenbaum et al. (2012) to measure the fraction of small price changes. The presence of small price changes (in absolute value) is at first sight a more prominent fact in France than in the US. One factor that may contribute to explaining this pattern is the fact that sales are less prevalent in France. Measurement errors, as discussed above, may play a role, but we see no obvious reason to presume that measurement errors are larger in the French CPI data. We observe that, if we define small price change as relative to the mean average price change, rather than with an absolute threshold, the fraction of small price change appears to be lower in France than in the US, as shown in Table 3.

Table 4 provides a further comparison based on datasets presumably less subject to measurement errors. For France we use data from the BPP, and those from hypermarkets in the CPI dataset. For the US we use the results on scanner data reported by Midrigan (2011), as

[13]The histogram has twenty four bins, spaced every 0.25 units, of the distribution of standardized regular price changes (excl. sales). The standardization was done by ELI, the narrowest categories of goods. After standardization the distributions are weighed according to the CPI weight.

[14]The smaller standard deviation and much smaller kurtosis than in Table 1 are due to the discretization and truncation.

Table 3: Fraction of small price changes: US and French CPI

Moments for the absolute value of price changes: $\lvert\Delta p\rvert$	France	US	Normal	Laplace
Average $\lvert\Delta p\rvert$	9.2	14.0		
Fraction of $\lvert\Delta p\rvert$ below 1%	11.8	12.5		
Fraction of $\lvert\Delta p\rvert$ below 2.5%	32.5	24.0		
Fraction of $\lvert\Delta p\rvert$ below 5%	57.1	40.6		
Fraction of $\lvert\Delta p\rvert$ below $(1/14) \cdot \mathbb{E}(\lvert\Delta p\rvert)$	2.4	12.5	4.5	6.9
Fraction of $\lvert\Delta p\rvert$ below $(2.5/14) \cdot \mathbb{E}(\lvert\Delta p\rvert)$	13.5	24.0	11.3	16.4
Fraction of $\lvert\Delta p\rvert$ below $(5/14) \cdot \mathbb{E}(\lvert\Delta p\rvert)$	28.7	40.6	22.4	30.0
Number of obs	1,542,586	1,047,547		

For France, source is INSEE, monthly price records from French CPI, data from 2003:4 to 2011:4. Coverage is aroud 65% of CPI weight since rents, and prices of fresh food and centrally collected items (e.g. electricity, train and airplane tickets) are not included in the dataset. Frequency of price change is monthly, in percent. Size of price change are the first-difference in the logarithm of price per unit, expressed in percent. Data are trimmed as in the baseline of Table 1. Observations with imputed prices or quality change are discarded. Moments are computed aggregating all prices changes using CPI weights at the product level. The US data are taken from Eichenbaum et al. (2012) Table 1, and refer to "Posted price changes" from 1998:1 to 2011:6. The mean absolute size of price changes is taken from Klenow and Kryvtsov (2008) table III where data are from 1998:1 to 2005:1. Figures for the US are weighted and cover around 70% of the CPI (US CPI includes owners equivalent rents, while French CPI does not). In the third panel we compute the threshold for defining small price changes as fraction of the mean so as to match the US figures in column 2 of the second panel. The Normal and Laplace distributions used in the last two columns have a zero mean and, without loss of generality, standard deviation equal to one.

Table 4: Comparison across datasets for large Hypermarkets in France and the US

	France		**US**	
	CPI	BPP data	scanner data	Midrigan (2009)
	Statistics for standardized price changes: z			
mean of $\lvert z\rvert$	0.65	0.70	0.80	-
% below 0.50 mean $\lvert z\rvert$	45	39	31	29
% below 0.25 mean $\lvert z\rvert$	24	21	20	13
kurtosis of z	10	5	2.8	3.5

Percentages. All price changes, including sales. The BPP statistics for France are an average of the ones reported in Table 2. The US scanner data in the third column are from a large US supermarket chain. The data from Midrigan (2009) are taken from his Table 1 and 2b, using simple averages of the AC Nielsen and Dominick's scanner data.

well as results on scanner data from a large US supermarket chain. As reflected by our three summary statistics, the distribution is somewhat more peaked in France; for instance the kurtosis is 5 in the BPP against 3.5 in Midrigan (2011). However, these results still support the notion that the share of small price changes is sizeable in both countries.

Overall we conclude that, after accounting for heterogeneity and measurement errors, the prevalence of small price change appears relevant both in France and in the US. We also obtain that the shape of the empirical distribution of price changes lays in "between" a Normal and a Laplace distribution even though the distribution appears close to a Normal in the US and closer to Laplace in France.

3 A tractable menu cost model

This section presents a simple menu cost model aimed at qualitatively matching some of the patterns documented above. In the canonical menu cost model the price changes when a threshold is hit, so that the implied distribution of price changes fails to generate the small changes that appear in the data (see the discussion in Midrigan (2011); Cavallo (2010); Alvarez and Lippi (2013)). We thus propose a model that is able to produce a large mass of small price changes and a positive excess Kurtosis of the distribution of price changes. Two ingredients are key to this end: that the menu costs are random and that the menu cost faced by the firm, ψ, applies to a bundle of n goods, so that after paying the fixed cost the firm can reprice one or all goods at no extra cost. Each of these assumptions individually is capable to generate small price changes and higher kurtosis than in a canonical model where $n = 1$ and there menu costs are constant. The combination of the two assumptions will give us more flexibility to parameterize the model vis a vis the data, as we discuss below. For ease of exposition we first illustrate the model where the firm sells a single good (i.e. $n = 1$) and then extend this model to include any number of goods $n > 1$.

The model for a firm selling $n = 1$ good. Consider a firm whose profit-maximizing price at time t, $p^*(t)$, follows the process $\mathrm{d}p^*(t) = \sigma \, \mathrm{d}W(t)$ where $W(t)$ is a standard brownian motion with no drift and i.i.d. innovations with standard deviation σ. The technology to change prices is as follows: to change the price at will the firm needs to incur a fixed menu cost of size ψ. However, with some probability the firm receives an opportunity to adjust the price "for free". Assume this probability is Poisson, i.e. that the free-adjustments have a constant hazard rate per unit of time, equal to λ. Let $p(t)$ denote the "price gap" at time t, i.e. the difference between the actual sale price $P(t)$ and the profit maximizing price $p^*(t)$, i.e. $p(t) \equiv P(t) - p^*(t)$. The instantaneous firm losses (i.e. reduction in profits) created by the price gap are given by the quadratic: $B\, p^2(t)$. Let $v(p)$ be the present-value cost function for a firm with price gap p.

Upon the arrival of a free adjustment opportunity the firm optimally resets the price gap to zero, hence the Bellman equation for the range of inaction reads:

$$r\ v(p) = Bp^2 + \lambda\,[v(0) - v(p)] + \frac{\sigma^2}{2}\ v''(p), \qquad \text{for } p \in (0, \bar{p})\ ,$$

where $\bar{p}$ is the boundary of the region in which inaction is optimal. The value matching and smooth pasting conditions are given by $v(\bar{p}) = v(0) + \psi$ and $v'(\bar{p}) = 0$.

Next we describe the optimal decision rules and some key statistics implied by the model with $n = 1$ (see Appendix D for the derivation). A Taylor expansion of the value function yields the following approximate optimal threshold $\bar{p} = \left(\frac{6\psi\sigma^2}{B}\right)^{\frac{1}{4}}$ which is accurate when ψ/B is small.[15] We comment on two properties of the decision rule of this problem which are proved later for the more general case: the value function, and the optimal decision rules, are a function of $\lambda + r$, as opposed to each of them separately. Intuitively this is because when a free adjustment opportunity occurs the price gap is adjusted, so that λ acts as an addition to the discount factor. Second, for a small value of ψ/B or a small value of $r + \lambda$, the value of $\bar{p}$ is insensitive to $r + \lambda$, as the previous approximation shows. More precisely, the derivative of $\bar{p}$ with respect to $\lambda + r$ is zero as ψ/B or $r + \lambda$ tend to zero. This property, which was known for the case of $\lambda = 0$, extends to the case where $\lambda + r > 0$ using the first property of the decision rule.

[15] Exactly the same expression was established by Barro (1972); Dixit (1991) for the case in which $\lambda = 0$. Below we discuss an approximate threshold for the case in which ψ is large.

Computing the expected time between adjustments yields an expression for the average number of adjustments per period, N_a

$$N_a = \lambda \ \frac{e^{\sqrt{2\phi}} + e^{-\sqrt{2\phi}}}{e^{\sqrt{2\phi}} + e^{-\sqrt{2\phi}} - 2} \qquad \text{where we define} \quad \phi \equiv \frac{\lambda \bar{p}^2}{\sigma^2}$$

which shows that the fraction of free adjustments λ/N_a depends only on the parameter ϕ. The parameter ϕ can be interpreted as the ratio between λ, the number of free adjustments, and $\sigma^2/\bar{p}^2$, the number of adjustments in a model where $\lambda = 0$ and the threshold policy $\bar{p}$ is followed.

The distribution of price changes $w(\Delta p_i)$ is symmetric around $\Delta p_i = 0$. This distribution has a mass point at $\Delta p_i = \pm\bar{p}$ with probability $1 - \lambda/N_a$, i.e. this is the fraction of price changes that occurs because the price gap reaches the boundaries of the inaction region. The remaining fraction of price changes, λ/N_a occurs when a free adjustment opportunity arrives, at which time the price gap is set to zero. Price changes in the range $p \in (-\bar{p}, \bar{p})$ have a density $\lambda/N_a \ h(p)$ where $h(p)$ denotes the density of the invariant distribution of price gaps

$$h(p) = \frac{\sqrt{2\phi}}{2\bar{p}\left(e^{\sqrt{2\phi}} - 1\right)^2} \left(e^{\sqrt{2\phi}\left(2 - \frac{|p|}{\bar{p}}\right)} - e^{\sqrt{2\phi}\frac{|p|}{\bar{p}}}\right) \quad \text{for} \quad p \in [-\bar{p}, \bar{p}] \ .$$

Thus the distribution of price changes is given by

$$\begin{cases} \Pr\left(\Delta p_i = \{-\bar{p}, \bar{p}\}\right) & = \frac{1}{2}\left(1 - \frac{\lambda}{N_a}\right) \\ \Pr\left(\Delta p_i \in \ dp\right) & = \frac{\lambda}{N_a} \ h(p)dp \equiv w(\Delta p_i)dp \qquad \text{for } p \in (-\bar{p}, \bar{p}) \end{cases}$$

which is a symmetric "tent shaped" distribution in the $(-\bar{p}, \bar{p})$ interval with the two mass points at the boundaries $\pm\bar{p}$. As detailed below the kurtosis of this distribtuion is increasing in λ, and in particular the distribution of price changes is more peaked than that of a standard menu cost model $\lambda = 0$.

We make two remarks about this simple model which will hold, and be generalized, in the more general model developed next. The first one is that the shape of the distribution of price changes depends *only* on the ϕ (or, equivalently, on the fraction of free adjustments λ/N_a). This means that two economies, or sectors, that differ in the standard deviation of price changes $Std(\Delta p_j)$ and/or in the frequency of price adjustment N_a will display a distribution of price changes with exactly the same shape (once its scale is adjusted) provided that the have the same value of ϕ. This property is useful to aggregate the sectors of an economy that are heterogenous in their steady state features ($Std(\Delta p_j)$,N_a). Because of this property the ratio of moments from the size distribution of price changes, such as kurtosis, are scale free and can be used to retrieve information on ϕ.

The second property, which we state here and prove below for the more general economy, is that the "shape" of the impulse response function of this economy to a (once and for all) monetary shock depends *only* on ϕ. We will show how one can simply scale (or relabel) one or both axes of an impulse function to analyze economies with the same ϕ that differ in either N_a or $Std(\Delta p_j)$.

3.1 Extending the model to multi-product firms

This section incorporates into the model with free adjustment opportunities discussed above the model of Alvarez and Lippi (2013) where the firm is selling n goods, as opposed to a single good, but pays a single fixed adjustment cost to change the n prices. We incorporate this feature for several reasons. First, as explained above, in the model with $n = 1$ good, there is a mass point on price changes of size $|\Delta p_i| = \bar{p}$. There is no evidence of this in any data set we can find. Second, and related to the previous point, in the model with $n = 1$ a simple estimate of $\bar{p}$ will be the highest price change. We propose to use a different one, since this order statistic is both difficult to measure in practice and its role to measure $\bar{p}$ is very sensitive to the specification of the model. Third, the model with $\lambda = 0$ has a kurtosis that increases with n, hence providing and alternative to randomness on fixed cost, as discussed below. Fourth, for large n and $\lambda = 0$ the distribution of price changes tends to normal, which is both a nice benchmark and an accurate description of the price changes for some sectors.

We now briefly describe the setup of the firm problem with n products. As before the free adjustment opportunities are independent of the driving processes $\{W_i(t)\}$ for price gaps, and arrive according to a Poisson process with constant intensity λ. In between price adjustments each of the price gaps evolves according to a Brownian motion $\mathrm{d}p_i(t) = \sigma \, \mathrm{d}W_i(t)$. It is assumed that all price gaps are subject to the same variance σ and that the innovations are independent across price gaps. We assume that, when the opportunity arrives, the firm can adjust all prices without paying the cost ψ. The analysis of the multi product problem can be greatly simplified by using $y \equiv ||p||^2$ as a state, as shown in Alvarez and Lippi (2013). The scalar y summarizes the state because the period objective function can be written as a function of it and because, from an application of Ito's lemma one can derive one dimensional diffusion which describes its behavior, namely

$$\mathrm{d}y = n\sigma^2 \, \mathrm{d}t \, + \, 2\,\sigma\sqrt{y}\,\mathrm{d}W$$

where W is a standard BM.

Using N_a and $Var(\Delta p_i)$ to denote the frequency and variance of the price changes of product i the next proposition establishes a useful relationship that holds in a large class of models for *any policy* for price changes. We describe a policy for price changes by a stopping time.

Proposition 1 *Let τ describe the time at which a price change takes place, so that all price gaps are closed. Assume the stopping time treats each of the n price gaps symmetrically. For any finite stopping time τ we have:*

$$N_a \cdot Var(\Delta p_i) = \sigma^2 \ . \tag{2}$$

The proposition highlights the trade-off for the firm's policy: more frequent adjustments are required to have smaller price gaps. We underline that equation (2) holds for any stopping rule, not just for the optimal one. See Appendix B for the proof, where the reader can verify that the key assumptions are the random walks and symmetry, indeed equation (2) holds for a larger class of models, for instance those with correlated price gaps and a richer class of random adjustment cost.

Upon the arrival of a free adjustment opportunity the firm will set the price gap to zero, hence the Bellman equation for the range of inaction reads:

$$r\ v(y) = By + \lambda\left[v(0) - v(y)\right] + n\sigma^2\ v'(y) + 2\sigma^2 y\ v''(y), \qquad \text{for } y \in (0, \bar{y})\ , \tag{3}$$

where By is the sum of the deviation from the optimal profits from the n goods. The use of the one dimensional y instead of the vector $(p_1, ..., p_n)$ simplifies the problem substantially.

We note that given the symmetry of the problem after an adjustment of the n prices the firm will set each of the price gap to zero, i.e. will set $||p||^2 = y = 0$. The value matching condition is then $v(0) + \psi = v(\bar{y})$, which uses that when y reaches a critical value, denoted by $\bar{y}$, by paying the fixed cost ψ the firm can change the n prices. The smooth pasting condition is $v'(\bar{y}) = 0$, as derived in Alvarez and Lippi (2013). The next proposition gives an explicit closed form solution to the value function $v(y)$ in the inaction region, i.e. for $y \in (0, \bar{y})$ subject to $v(0) < \infty$. The solution is parameterized by $\beta_0 = v(0)$, see Appendix E for a discussion of this solution.

Proposition 2 *Let $\sigma > 0$. The ODE in equation (3) is solved by the analytical function: $v(y) = \sum_{i=0}^{\infty}\ \beta_i\ y^i$, for $y \in [0, \bar{y}]$ where, for any β_0, the coefficients $\{\beta_i\}$ solve: $\beta_0 = \frac{n\sigma^2}{r}\beta_1$, $\beta_2 = \frac{(r+\lambda)\beta_1 - B}{2\sigma^2(n+2)}$, $\beta_{i+1} = \frac{r+\lambda}{(i+1)\sigma^2\ (n+2i)}\ \beta_i$ for $i \geq 2$.*

The function described in this proposition allows to fully characterize the solution of the firm's problem. One can use it to evaluate the two boundary conditions described above, value matching and smooth pasting, and define a system of two equations in two unknowns, namely β_0 and $\bar{y}$. The next lemma establishes how to solve for $\bar{y}$ using the solution of a simpler problem where $\lambda = 0$ discussed in Alvarez and Lippi (2013). It turns out that a simple change of variables allows us to use the solution for the case of $\lambda = 0$ with the solution for the case of interest in this paper. The change of variables consists on using $r + \lambda$ as the interest rate in the solution of the problem with $\lambda = 0$. We have:

Lemma 1 *Let $\bar{y}(r, \lambda)$ and $v(y; r, \lambda)$ be the optimal threshold and value function for a problem with discount rate r and arrival rate λ. Then $\bar{y}(r, \lambda) = \bar{y}(r + \lambda, 0)$ and $v(y; r, \lambda) = v(y; r + \lambda, 0) - \lambda\psi/r$.*

The proof of this lemma follows immediately from a guess and verify strategy. The lemma allows us to use the characterization of $\bar{y}$ with respect to r given in Proposition 4 of Alvarez and Lippi (2013) to study the effect of $r + \lambda$ on $\bar{y}$. The next proposition summarizes that result and extends the characterization of the optimal threshold to the case where ψ is large, a case that is useful to understand the behavior of an economy with a lot of free adjustments opportunity as in a Calvo mechanism.

Proposition 3 *Assume $\sigma^2 > 0$, $n \geq 1$, $\lambda + r > 0$ and $B > 0$, and let $\bar{y}$ be the threshold for the optimal decision rule. We then have that:*

1. *As $\psi \to 0$ then $\frac{\bar{y}}{\sqrt{2(n+2)\sigma^2 \frac{\psi}{B}}} \to 1$ or $\bar{y} \approx \sqrt{2(n+2)\sigma^2 \frac{\psi}{B}}$.*
2. *As $\psi \to \infty$ we have $\frac{\bar{y}}{\psi} \to B(r + \lambda)$ or $\bar{y} \approx \frac{\psi}{B}(r + \lambda)$. Moreover this also holds for large n and large $\frac{\psi}{n}$, namely $\lim_{\psi/n \to \infty} \lim_{n \to \infty} \frac{\bar{y}/n}{\psi/n} = B(r + \lambda)$ or $\frac{\bar{y}}{n} \approx \frac{\psi/n}{B}(r + \lambda)$.*

The proposition shows that $\bar{y}$ is approximately constant with respect to λ for small values of ψ, so that for small menu costs the result is the well known quartic root formula (recall that y has the units of a squared price gap) and the inaction region is increasing in the variance of the shock, due to the higher option value. Interestingly, and novel in the literature, the second part of the proposition shows that for large values of the adjustment cost the rule becomes a square root and that the option value component of the decision becomes neglible.

We now turn to the discussion of the model implications for the frequency of price changes. We let $N_a(\bar{y}; \lambda)$ be the expected number of adjustments per unit of time of a model with a given λ and $\bar{y}$. We establish the following (see Appendix B for the proof):

Proposition 4 *The fraction of free adjustments is $\lambda/N_a = \mathcal{L}_n(\phi)$, where*

$$\mathcal{L}_n(\phi) \equiv \frac{\phi\left[1+\sum_{i=1}^{\infty}\left(\prod_{k=1}^{i}\frac{n}{(k+1)\ (n+2k)}\right)\phi^i\right]}{1+\phi\left[1+\sum_{i=1}^{\infty}\left(\prod_{k=1}^{i}\frac{n}{(k+1)\ (n+2k)}\right)\phi^i\right]} \quad \textit{where} \quad \phi \equiv \frac{\lambda\bar{y}}{n\sigma^2} \tag{4}$$

The proposition shows that λ/N_a, a key parameter for the behavior of the model, is a function only of two variables: n and ϕ. As for the $n = 1$ model, the parameter ϕ can be interpreted as the ratio between λ, the number of free adjustments, and $n\sigma^2/\bar{p}^2$, the number of adjustments in a model where $\lambda = 0$ and the threshold policy $\bar{y}$ is followed. A second order approximation of $\mathcal{L}_n(\phi)$ shows that λ has a negligible effect on the frequency of adjustment N_a when $\bar{y}$ is small, i.e. the first order term is the same as the one for the model with $\lambda = 0$.[16]

We now turn to characterize the invariant distribution of y for the case where $\lambda > 0$, a key ingredient to compute the size-distribution of price changes. The density of the invariant distribution solves the Kolmogorov forward equation: $\frac{\lambda}{2\sigma^2}f(y) = f''(y)y - \left(\frac{n}{2}-2\right)f'(y)$ for $y \in (0,\bar{y})$, with the two boundary conditions $f(\bar{y}) = 0$ and $\int_0^{\bar{y}} f(y)dy = 1$. It is clear from these conditions that $f(\cdot)$ is uniquely defined for a given triplet: $\bar{y} > 0$, $n \geq 1$ and $\lambda/\sigma^2 \geq 0$. The general solution of this ODE is

$$f(y) = \left(\frac{\lambda y}{2\sigma^2}\right)^{\left(\frac{n}{4}-\frac{1}{2}\right)}\left[C_1\ I_\nu\left(2\sqrt{\frac{\lambda y}{2\sigma^2}}\right) + C_2\ K_\nu\left(2\sqrt{\frac{\lambda y}{2\sigma^2}}\right)\right] \tag{5}$$

where I_ν and K_ν are the modified Bessel functions of the first and second kind, C_1, C_2 are two arbitrary constants and $\nu = |\frac{n}{2} - 1|$, see Zaitsev and Polyanin (2003) for a proof. The constants C_1, C_2 are chosen to satisfy the two boundary conditions.[17] While the density in equation (5) depends on 3 constants n, ϕ and $\bar{y}$, its shape depends only on 2 constants, namely n and ϕ, as formally stated in Lemma 2 in Appendix B. The lemma shows that one can normalize $\bar{y}$ to 1 and compute the density for the corresponding ϕ.

[16] The expansion gives $\frac{1}{N_a} = \frac{\bar{y}}{n\sigma^2}\left[1-\lambda\frac{\bar{y}}{n\sigma^2}\frac{(n+4)}{(2n+2)}\right] + o\left(\lambda\left(\frac{\bar{y}}{\sigma^2}\right)^2\right)$ which shows that $1/N_a = \bar{y}/(n\sigma^2) + o(\bar{y})$.

[17] We note that both modified Bessel functions are positive, that $I_\nu(y)$ is exponentially increasing with $I_\nu(0) \geq 0$, and that $K_\nu(y)$ is exponentially decreasing with $K_\nu(0) = +\infty$.

We denote the marginal distribution of price changes by $w(\Delta p_i)$. Recall that firms change prices either when y first reaches $\bar{y}$ or when they get a free adjustment opportunity even though $y < \bar{y}$. Thus to construct the distribution of price changes we need three objects: the fraction of free adjustments $\frac{\lambda}{N_a}$, the invariant distribution $f(y)$ and the marginal distribution of price changes *conditional on a value of* y, $\omega(\Delta p_i; y)$ which, following Proposition 6 of Alvarez and Lippi (2013) when $n \geq 2$, is

$$\omega(\Delta p_i; y) = \begin{cases} \frac{1}{Beta\left(\frac{n-1}{2}, \frac{1}{2}\right)\sqrt{y}} \left(1 - \left(\frac{\Delta p_i}{\sqrt{y}}\right)^2\right)^{(n-3)/2} & \text{if } \left(\Delta p_i\right)^2 \leq y \\ 0 & \text{if } \left(\Delta p_i\right)^2 > y \end{cases} \tag{6}$$

where $Beta(\cdot, \cdot)$ denotes the Beta function. In this case the standard deviation of the price changes is $Std\left(\, \Delta p_i \,; y\right) = \sqrt{y/n}$. The marginal distribution of price changes $w(\Delta p_i)$ is given by

$$w(\Delta p_i) = \omega(\Delta p_i; \bar{y}) \left(1 - \frac{\lambda}{N_a}\right) + \left[\int_0^{\bar{y}} \omega(\Delta p_i; y) f(y) dy\right] \frac{\lambda}{N_a} \qquad \text{for } n \geq 2 \ . \tag{7}$$

For the case when $n = 2$ the density of the price changes diverges at the boundaries of the domain where $\Delta p_i = \pm\sqrt{\bar{y}/n}$, as can be seen in Figure 3. This feature echoes the two mass points that occur in the $n = 1$ case where a non-zero mass of price changes occurs exactly at the boundaries. For $n \geq 6$ the shape of the density takes a tent-shape, similar to the one that is seen in the data. As the fraction of free adjustments approaches 1 the shape of the density function converges to the shape of the Laplace distribution. The next proposition shows that n and λ/N_a completely determine the shape of the distribution of price changes. (see Appendix B for the proof):

Proposition 5 *Let* $w(\Delta p_i; n, \frac{\lambda}{N_a}, 1)$ *be the density function for the price changes* Δp_i *in an economy with* n *goods, a share* λ/N_a *of free adjustments, and a unit standard deviation of price changes* $Std(\Delta p_i) = 1$*. This density function is homogenous of degree -1 in* Δp_i *and*

$Std(\Delta p_i)$, which implies

$$w\left(S\,\Delta p_i; n, \frac{\lambda}{N_a}, S\right) = \frac{1}{S}\;\; w\left(\Delta p_i; n, \frac{\lambda}{N_a}, 1\right) \qquad \text{for all} \quad S > 0. \tag{8}$$

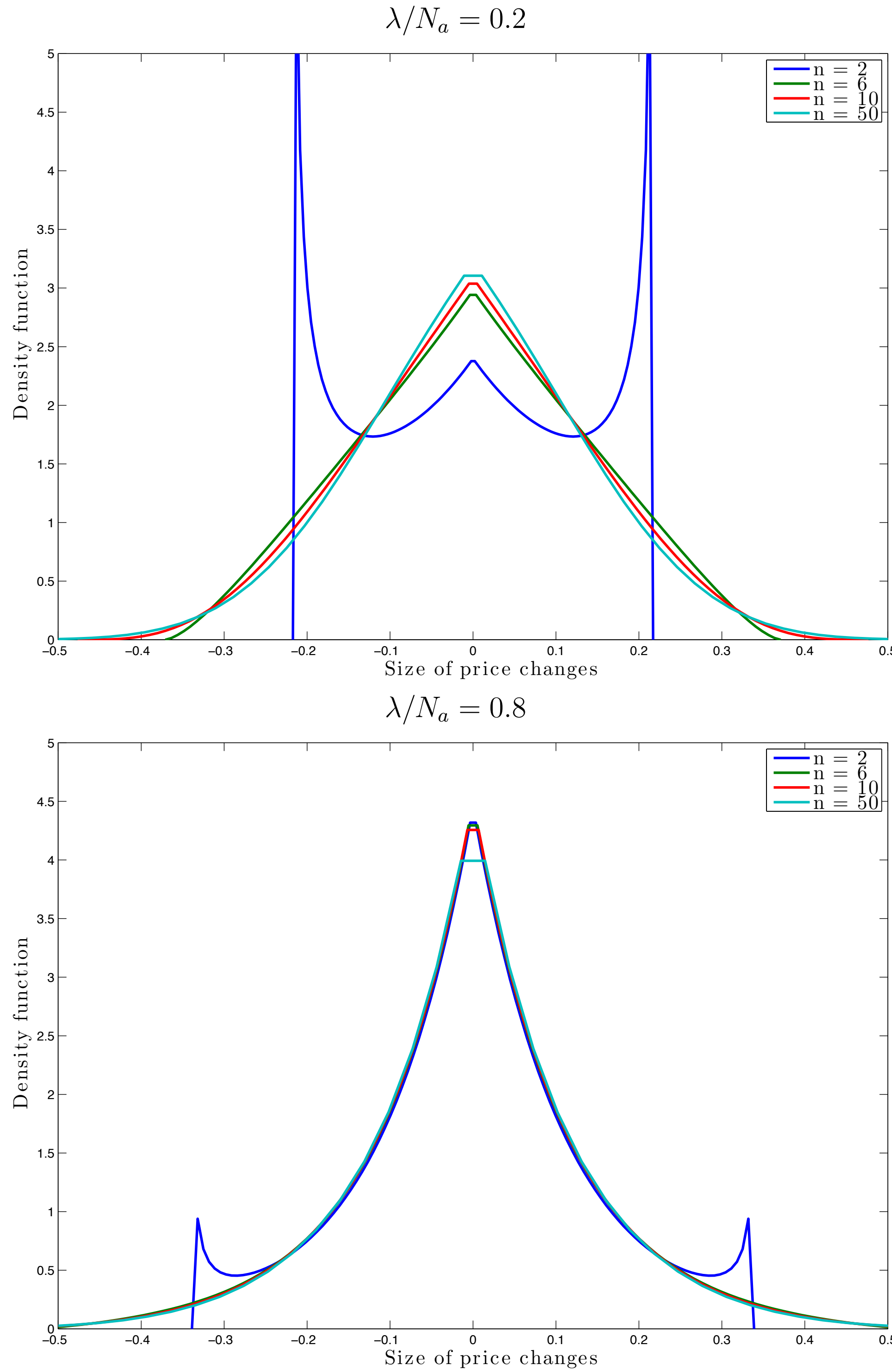
The proposition establishes that the "shape" of the size distribution of price changes has 2 parameters: n and λ/N_a. Every two economies sharing these parameters will have the same size distribution of price changes once the scale is adjusted. The proposition implies that we can aggregate firms or industries that are heterogenous in terms of frequency N_a and standard deviation of price changes $Std(\Delta p_i)$ provided that n and λ/N_a are the same. Notice in particular that the frequency of price changes N_a does not have an independent effect on the distribution of price changes as long as λ/N_a remains constant.

Notice that the distribution $w(\Delta p_i)$ is a mixture of the $\omega(\Delta p_i, y)$ densities. These densities are scaled versions of each other with different standard deviations. This increases the kurtosis of the distribution of price changes compared to the case where $\lambda = 0$. In particular Proposition 6 in Alvarez and Lippi (2013) shows that the variance and kurtosis of $\omega(\Delta p_i, y)$ are given by y/n and $3n/(n+2)$ respectively. Using that Δp_i is distributed as a mixture of the $\omega(\Delta p_i, y)$, we can compute several moments

$$\begin{aligned}
\mathbb{E}(|\Delta p_i|) &= \frac{\left(1-\frac{\lambda}{N_a}\right)\sqrt{\bar{y}} + \frac{\lambda}{N_a}\;\int_0^{\bar{y}} \sqrt{y}\; f(y)dy}{\frac{n-1}{2}\; Beta\left(\frac{n-1}{2}, \frac{1}{2}\right)} \\
Var(\Delta p_i) &= \left(1-\frac{\lambda}{N_a}\right)\frac{\bar{y}}{n} + \frac{\lambda}{N_a}\int_0^{\bar{y}} \frac{y}{n}\; f(y)dy \\
Kurt(\Delta p_i) &= \frac{3n}{2+n}\;\frac{\left(1-\frac{\lambda}{N_a}\right)\bar{y}^2 + \frac{\lambda}{N_a}\;\int_0^{\bar{y}} y^2\; f(y)dy}{\left[\left(1-\frac{\lambda}{N_a}\right)\bar{y} + \frac{\lambda}{N_a}\;\int_0^{\bar{y}} y\; f(y)dy\right]^2} > \frac{3n}{2+n}
\end{aligned}$$

It is immediate from Proposition 5 that the value of the kurtosis and the value of the ratio $\mathbb{E}(|\Delta p_i|)/Std(\Delta p_i)$ depend only on two parameters: n and $\frac{\lambda}{N_a}$. For instance, if one were to change the parameters $\psi/B, \lambda$ and σ^2 keeping the same values for $\frac{\lambda}{N_a}$ and n, the kurtosis of the price changes will be the same. The inequality that appears in the third line is a well

Figure 3: Size distribution of price changes



Note: All distributions are zero mean with a standard deviation equal to 0.14. As stated in Proposition 5 the shape of this distribution only depends on λ/N_a and n... TO BE FINISHED; – REDO THE BOTTOM FIGURE WITH VERTICAL ASYMPTOTES FOR THE n=2 CASE known result: the mixture of distributions with the same kurtosis but with different variances has higher kurtosis, which itself follows from Jensen's inequality. Moreover as $\bar{y} \to \infty$ (as it will happen if $\psi/B \to \infty$) then $N_a \to \lambda$ and one can show that $Kurt(\Delta p_i) \to 6$. This is because as $\bar{y} \to \infty$ the price changes in each coordinate are independent, and hence it has the same distribution than in the case of $n = 1$, i.e. a Laplace distribution. To reiterate, the maximum kurtosis that the model with free adjustments can produce is 6 which happens in the limiting case in which *all* adjustments are free (e.g. when $\lambda/N_a \uparrow 1$ and $\bar{y} \to \infty$) and is independent of the number of products that are priced by the firm, n. Table 5 computes the kurtosis of the model for the intermediate cases in which only a fraction of adjustments are free. The columns correspond to different values of n, the number of goods. Each line corresponds to a different proportion of free adjustments: λ/N_a. When the fraction of free adjustment is small (first and second line of the table) the model behaves essentially like the one described in Alvarez and Lippi (2013): kurtosis is increasing in n up to a level of about 3. As the fraction of free adjustments increases the kurtosis increases towards 6, and becomes less responsive to n.

Table 5: Model statistic for the Kurtosis of Price changes

% of free adjustments:	number of products n					
λ/N_a	1	2	4	6	10	50
0 %	1.0	1.5	2.0	2.3	2.5	2.9
10%	1.1	1.6	2.1	2.4	2.6	3.0
20%	1.2	1.7	2.2	2.5	2.7	3.1
50%	1.6	2.2	2.7	3.0	3.2	3.6
70%	2.1	2.8	3.3	3.5	3.7	4.1
80%	2.6	3.2	3.7	3.9	4.1	4.4
90%	3.4	3.9	4.3	4.5	4.7	4.9
95%	4.1	4.5	4.8	5.0	5.1	5.3
100%	6.0	6.0	6.0	6.0	6.0	6.0

More summary statistics on the model predictions, concerning e.g. the fraction of small price changes are provided in Appendix G.

3.2 Quantifying the magnitude of menu cost

The model presented above has essentially four independent parameters: the menu cost ψ/B, the volatility of shocks σ, the number of goods n and the rate of free adjustment opportunities λ. We find it convenient to pin down two of these parameters by matching the model to some observable statistics that are available in micro datasets: the frequency and the variance of price changes: N_a and $Var(\Delta p_i)$. Given these statistics, the model has essentially two residual parameters: λ/N_a and n, which determine the shape of the size distribution of price changes as discussed above. This section uses the model to quantify the size of one price adjustment using two complementary measures: the first one is the cost of a single price adjustment in units of profits per product: ψ/n. The second one is the average flow cost of price adjustment that a firm pays, given by :

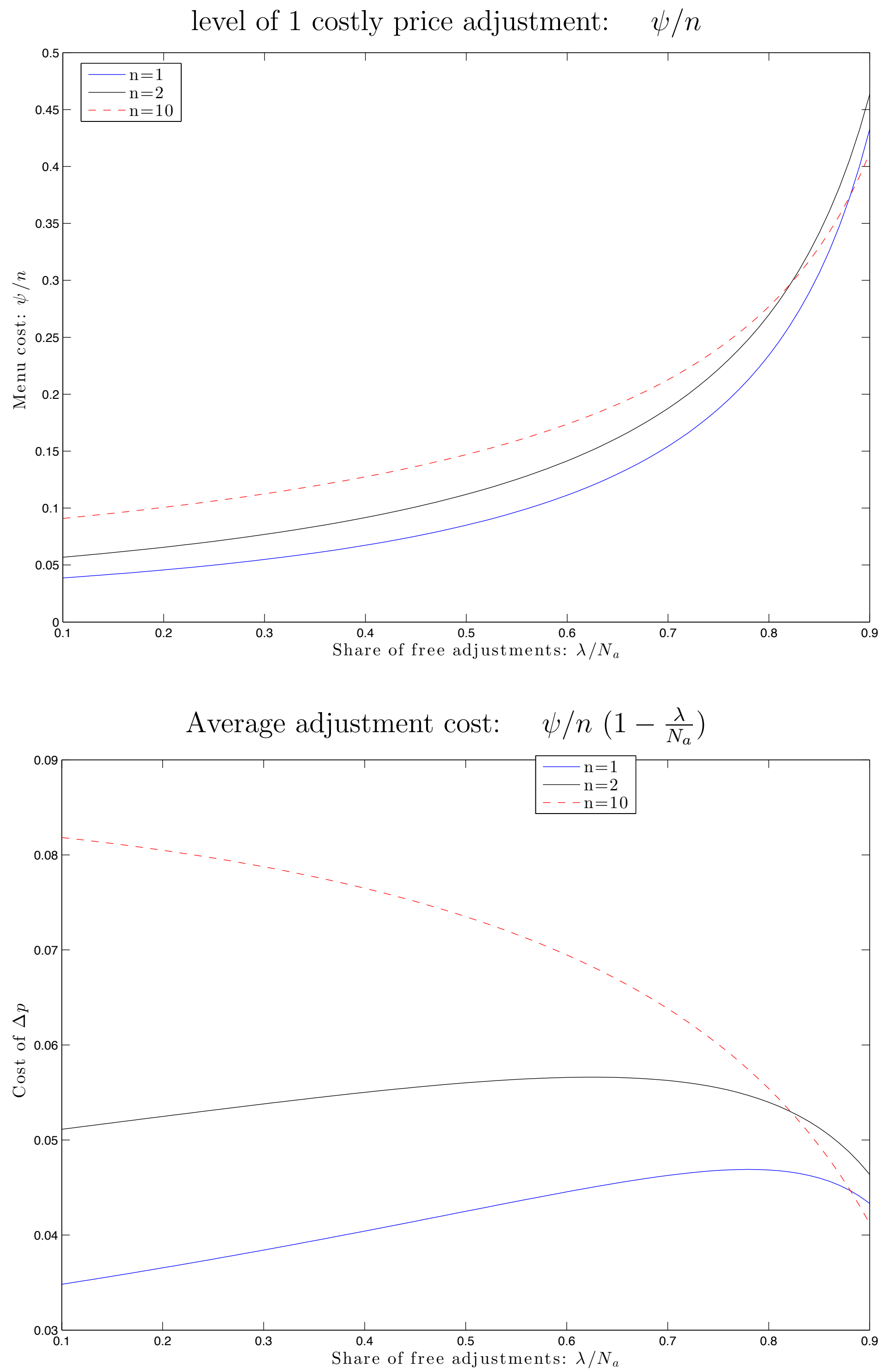
$$\frac{\psi}{n}\left(1-\frac{\lambda}{N_a}\right)=\frac{\psi}{n}\left(1-\mathcal{L}_n\left(\frac{\bar{y}\lambda}{\sigma^2 n}\right)\right)=\frac{\psi}{n}\left(1-\mathcal{L}_n\left(\left[\frac{\bar{y}}{\psi n}\frac{\lambda}{N_a}\frac{1}{Var(\Delta p_i)}\right]\psi\right)\right)$$

where we used equation (4) and that $\sigma^2 = N_a\ Var(\Delta p_i)$. We show that the limiting case of the Calvo model where $\frac{\lambda}{N_a} = 1$ (i.e. all adjustments obtain due to the free adjustment opportunity) obtains when $\frac{\psi}{n} \to \infty$ in which case we have that $\frac{\psi}{n}\left(1-\frac{\lambda}{N_a}\right) = 0$.

BOTTOMLINE: TO GET A LARGE FRACTION OF FREE ADJUSTMENTS, I.E. TO GET CLOSE TO THE DISTRIBUTION OF ADJUSTMENT TIMES IN A CALVO MODEL, IT TAKES VERY LARGE VALUES OF THE MENU COST FOR 1 ADJUSTMENT; HOWEVER NOTICE THAT IF ONE WANTS TO HAVE AN AVERAGE COST OF PRICE ADJUSTMENTS BORNE PER YEAR THAT IS IN LINE WITH THE DATA (AROUND 5% OF PROFITS (OR 0.7% OD REVENUES), ACCORDING TO LEVY ET AL.) THEN ONE MAY BE PUSHED TO A LAMBDA/NA ABOUT 0.8 PER CENT .

The next proposition analyzes the implications for scaled menu cost ψ consistent with a value of $\phi \in [0, \infty)$ or equivalently consistent with a value of $\lambda/N_a \in [0, 1)$.

Figure 4: Implied cost of price adjustment (in % of the profits per good)



The units are costs as a fraction of the profits per-good. All economies feature the same $N_a = 1$ and $std(\Delta p_i) = 0.10$.

Proposition 6 *Fix the number of products $n \geq 1$ and let $r \downarrow 0$. There is a unique triplet $(\sigma^2, \lambda, \psi)$ consistent with any triplet $\phi \in [0, \infty), Var(\Delta p_i) > 0$ and $N_a > 0$. Moreover, fixing any value ϕ, the menu cost $\psi \geq 0$ can be written as:*

$$\frac{\psi}{n} = B \, \frac{Var(\Delta p_i)}{N_a} \, \Psi \left(n \, , \, \phi\right) \tag{9}$$

where Ψ is only a function of (n, ϕ). The function $\Psi(n, \cdot)$ satisfies:

$$\Psi \left(n, \, 0\right) \quad = \quad \frac{n}{2\left(n+2\right)} \, , \; \frac{\partial \log \Psi \left(n, 0\right)}{\partial \log \phi} = 0 \, , \;\; \textit{and} \quad \lim_{\phi \to \infty} \Psi \left(n, \, \phi\right) = \infty \; .$$

Note for $\phi = 0$, the value of ψ/n is increasing in n, where this expression being *three* times larger as n goes from 1 to ∞.

4 The impulse response of prices to a monetary shock

In this section we give a description of the impulse response of prices and output an unexpected once and for all increase on the money supply of size δ, starting from a steady state with zero inflation. We first describe the general equilibrium set up which is, essentially, the one in Golosov and Lucas (2007), adapted to a multi-product firm as in Alvarez and Lippi (2013). Then we describe the impulse response of prices and output to a monetary shock.

General Equilibrium Setup. Briefly, this is an economy where each firm produces n goods, each with a linear labor only technology subject to independent idiosyncratic productivity shocks, whose logs follows a BM with instantaneous variance σ^2. As in the previous sections, a firm is subject to a random menu cost to simultaneously change the price of its n products. In a period of length dt this cost equals ψ_ℓ units of labor with probability $1 - \lambda dt$, or zero. Also each firm faces a demand with constant elasticity $\eta > 1$ for each of its n products, coming from households's CES utility function for the consumption aggregate. The $p_i(t)$ in our previous sections are the logs of the markups in each product of the firm relative to the static optimal markup, and our quadratic objective function can be taken to be a second order expansion on the firm's profits with $B = (1/2)\eta(\eta - 1)$. Households' period utility function is additively separable: log in real balances, linear in leisure, and has constant intertemporal elasticity of substitution $1/\eta$ for the consumption aggregate.

Impulse response to a monetary shock. The initial conditions are the steady state of the economy with constant money supply, and hence constant economy wide price index. The mechanics of the impulse response is that in this economy nominal wages jump on impact by the same percentage as money supply. In Proposition 7 of Alvarez and Lippi (2013) we show that, up to first order, the firm's optimal policy is to keep $\bar{y}$ unaltered during the transition, a result that can be extended to the the present case with $\lambda > 0$. Given this result the characterization of the impulse response is an exercise in aggregation: the steady state distribution of price gaps is perturbed by the common increase in cost across all firms, which will return to steady state slowly, in a process which we describe below. Letting $\mathcal{P}(t)$ the impulse response of the percentage change in aggregate price level at horizon t, the one for the percentage change in output is proportional to $\delta - \mathcal{P}(t)$, where the constant of proportionality is $1/\epsilon$.

To compute the IRF of the aggregate price level we find the contribution to the aggregate price level of each firm at the time of the shock. They start with price gaps distributed according to g, the invariant distribution. Then the monetary shock displaces them, by subtracting the monetary shock δ to each of them. After that we divide the firms in two groups. Those that adjust immediately and those that adjust at some future time. Note that it suffices to keep track only of the contribution to the aggregate price level of the first adjustment after the shock, because after that one the future contributions are all equal to zero in expected value. Now we develop the notation to define the impulse response of the aggregate price level.

Let $g\left(p; n, \lambda/\sigma^2, \bar{y}\right)$ be the density of firms with price gap vector $p = (p_1, ..., p_n)$ at time $t = 0$, just before the monetary shock, which corresponds to the invariant distribution with constant money supply. The density g equals the density f of the steady state square norms of the price gaps given by Lemma 2 evaluated at $y = p_1^2 + \cdots + p_n^2$ times a correction for area of sphere and the different variables.[18] In particular we have

$$g\left(p_1, ..., p_n\,; n, \frac{\lambda}{\sigma^2}, \bar{y}\right) = f\left(p_1^2 + \cdots + p_n^2\,; n, \frac{\lambda}{\sigma^2}, \bar{y}\right) \frac{\Gamma\left(n/2\right)}{\pi^{n/2}\ \left(p_1^2 + \cdots + p_n^2\right)^{(n-2)/2}}$$

To define the impulse response we introduce two extra pieces of notation. First we let $\{(\bar{p}_1(t,p), \ldots, \bar{p}_n(t,p))\}$ the process for n independent BM, each one with variance per unit of time equal to σ^2, which at time $t = 0$ start at p, so $\bar{p}_i(0,p) = p_i$. We also define the stopping time $\tau(p)$, also indexed by the initial value of the price gaps p as the minimum of two stopping times, τ_1 and $\tau_2(p)$. The stopping time τ_1 denotes the first time since $t = 0$ that jump occurs for a Poisson process with arrival rate λ per unit of time. The stopping time $\tau_2(p)$ denotes the fist time that $||\bar{p}(t,p)||^2 > \bar{y}$. Thus $\tau(p)$ is the first time a price change occur for a firm that start with price gap p at time zero. The stopped process $\bar{p}(\tau(0), p)$ is the vector of price gaps at the time of price change for such a firm.

We can write the impulse response function as:

$$\mathcal{P}(t, \delta; \sigma, \lambda, \bar{y}) \quad = \quad \Theta(\delta; \sigma, \lambda, \bar{y}) \, + \int_0^t \theta(\delta, s; \sigma, \lambda, \bar{y})\, ds\,,$$

where $\Theta(\delta)$ gives the impact effect, the contribution of the monetary shock δ to the aggregate price level on impact, i.e. at the time of the monetary shock. The integral of the θ's gives the remaining effect of the monetary shock in the aggregate price level up to time t, i.e. $\theta(\delta, s)ds$ is the contribution to the increase in the average price level in the interval of times $(s, s+ds)$ from a monetary shock of size δ. Instead the functions θ and Θ are easily defined in terms of the density g, the process $\{\bar{p}\}$ and the stopping times τ:

$$\Theta(\delta; \sigma, \lambda, \bar{y}) \equiv \int_{||p(0) - \iota\delta|| \geq \bar{y}} \left(\delta - \frac{\sum_{j=0}^{n} p_j(0)}{n}\right) g\left(p(0); n, \frac{\lambda}{\sigma^2}, \bar{y}\right) \, dp_1(0) \, \cdots \, dp_n(0)$$

[18] See Section 5 of Alvarez and Lippi (2013) for this result and the Online appendix for a derivation.

and $\theta(\delta, t; \sigma, \lambda, \bar{y})$ is the density, i.e. the derivative with respect to t of the following expression:

$$\int_{||p(0)-\iota\delta||<\bar{y}} \mathbb{E}\left[-\frac{\sum_{j=0}^{n} \bar{p}_j\left(\tau(p), p\right)}{n} \mathbf{1}_{\{\tau(p)\leq t\}} \,\middle|\, p = p(0) - \iota\delta\right] g\left(p(0); n, \frac{\lambda}{\sigma^2}, \bar{y}\right) dp_1(0) \cdots dp_n(0)$$

where ι is a vector of n ones. This expression takes each firm that has not adjusted price on impact, i.e. those with $p(0)$ satisfying $||p(0) - \iota\delta|| < \bar{y}$, weight them by the relevant density g, displace their initial price gap by the monetary shock, i.e. sets $p = p(0) - \iota\delta$, and then looks a the (negative) of the average price gap at the time of the first price adjustment, $\tau(p)$, provided that the price adjustment has happened before or at time t. We make a few remarks about this expression. First, price changes equal the negative of the price gaps because price gaps are defined as prices minus the ideal price. Second, we define θ as a density because, strictly speaking, there is no effect on the price level due to price changes at *exactly* time t, since in continuous time there is a zero mass of firms adjusting at any given time. Third, we can disregard the effect of any subsequent adjustment because each of them has an expected zero contribution to the average price level, they start with firms with zero price gaps in all dimensions. Fourth, the impulse response is based on the steady-state decision rules, i.e. adjusting only when $y \geq \bar{y}$ even after an aggregate shock occurs. This approximation is justified by Proposition 7 in Alvarez and Lippi (2013) which shows that disregarding the general equilibrium feedbacks has a second order impact on the results.

Given the results in Proposition 4 -Proposition 5 we can parametrize our model either in terms of $(n, \lambda, \sigma^2, \psi/B)$ or instead parametrize it, for each n, in terms of the implied observable statistics $(N_a, Std[\Delta p], \lambda/N_a)$. These propositions show that this mapping is indeed one-to-one and onto. We refer to λ/N_a as an "observable" statistic, because the have shown that the "shape" of the distribution of price changes depends only on it.

Proposition 7 *Fix an economy whose firm produce n products and with steady state statistics $(N_a, Std[\Delta p], \lambda/N_a)$. The cumulative proportional response of the aggregate price level $t \geq 0$ periods after a once and for all proportional monetary shock of size δ can be obtained* *from the one of an economy with one price change per year and with unitary standard deviation of price changes as follows:*

$$\mathcal{P}_n\left(t\,,\,\delta\,;\,N_a\,,\,Std[\Delta p]\,,\,\frac{\lambda}{N_a}\right) = Std[\Delta p]\,\mathcal{P}_n\left(t\,N_a\,,\,\frac{\delta}{Std[\Delta p]}\,;\,1\,,\,1\,,\,\frac{\lambda}{N_a}\right)\,. \tag{10}$$

This proposition extends a similar result in Proposition 8 in Alvarez and Lippi (2013) to the case of $\lambda > 0$. The proof in Alvarez and Lippi (2013) is constructive in nature, exploiting results from applied math on the characterization of hitting times for BM in hyper-spheres, which is not longer valid for $\lambda > 0$. Here we use a different strategy which relies on limits of discrete-time, discrete state approximations.

Corollary 1 *For small monetary shocks $\delta > 0$, the impulse response is independent of $Std[\Delta p]$. Differentiating equation (10) gives:*

$$\mathcal{P}_n\left(t\,,\,\delta\,;\,N_a\,,\,Std[\Delta p]\,,\,\frac{\lambda}{N_a}\right) = \delta\,\frac{\partial}{\partial\delta}\,\mathcal{P}_n\left(t\,N_a\,,\,0\,;\,1\,,\,1\,,\,\frac{\lambda}{N_a}\right) + o(\delta)$$

for all $t > 0$ and, since $f(\bar{y}) = 0$, then the initial jump in prices can be neglected, i.e.:

$$\mathcal{P}_n\left(0\,,\,\delta\,;\,N_a\,,\,Std[\Delta p]\,,\,\frac{\lambda}{N_a}\right) \equiv \Theta_n\left(0\,,\,\delta\,;\,Std[\Delta p]\,,\,\frac{\lambda}{N_a}\right) = o(\delta)\,.$$

We define the half life of monetary shock of size δ as the (smallest) time $\hat{T}_{1/2}$ which solves:

$$\frac{\delta}{2} = \mathcal{P}_n\left(\hat{T}_{1/2}\,,\,\delta\,;\,N_a\,,\,Std[\Delta p]\,,\,\frac{\lambda}{N_a}\right) \tag{11}$$

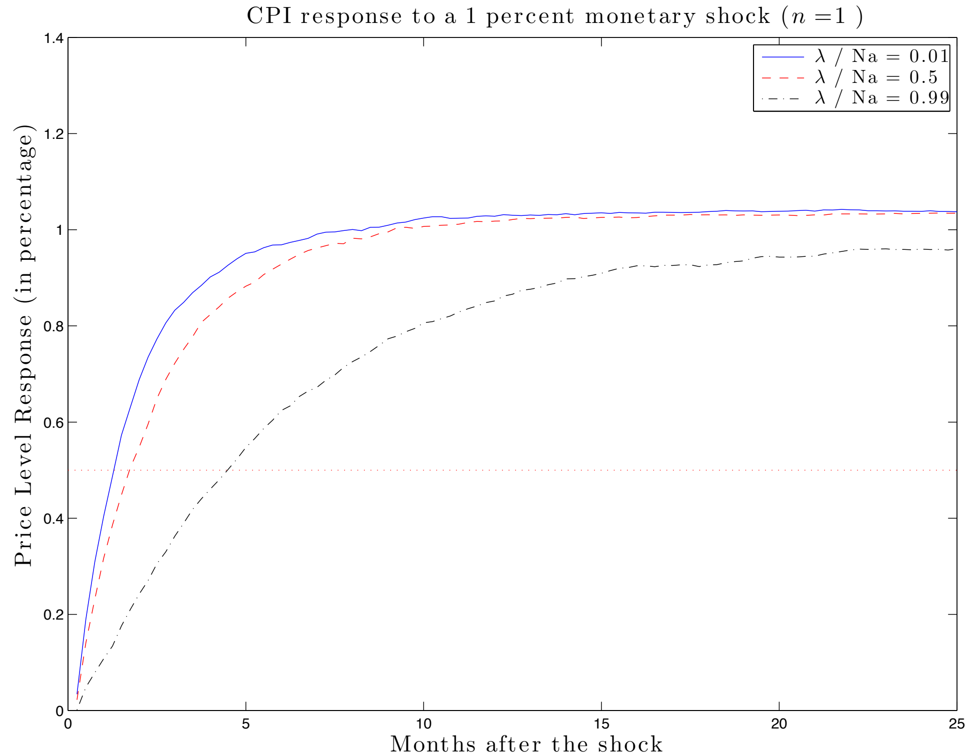
Using the time scaling property we obtain that:

$$\frac{\delta}{2} = \mathcal{P}_n\left(\hat{T}_{1/2}\,N_a\,,\,\delta\,;\,1\,,\,Std[\Delta p]\,,\,\frac{\lambda}{N_a}\right)$$

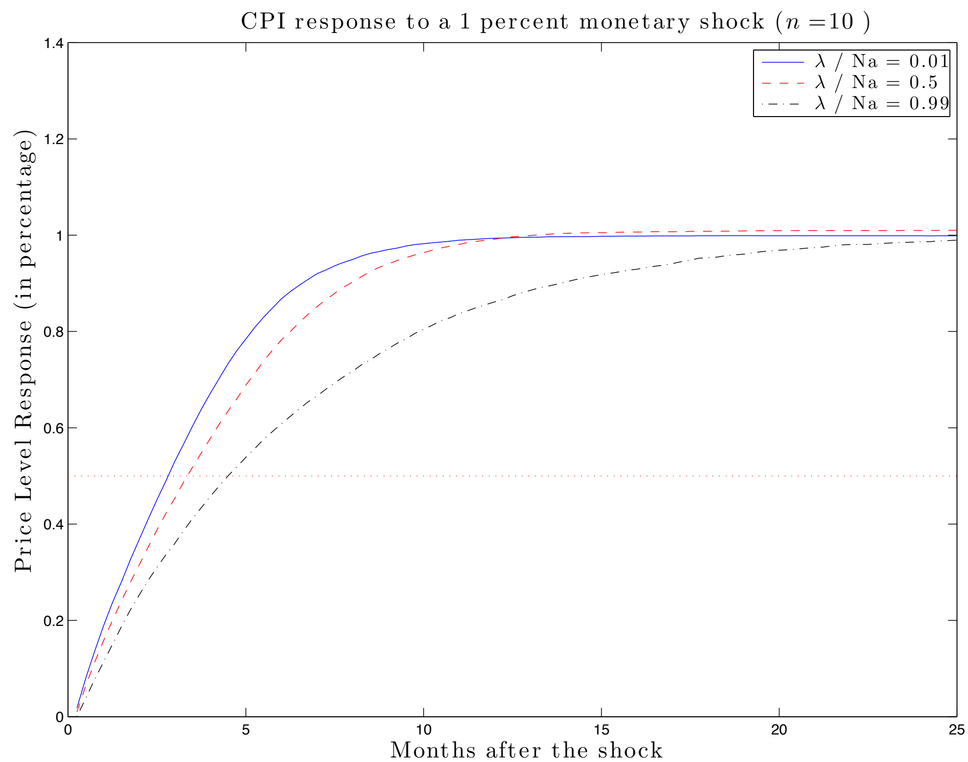
We define $T_{1/2}$ as the normalized half-time, i.e we measure it relative to the time it takes in

Figure 5: Impulse response of the CPI to a monetary shock of size $\delta = 1\%$ when $n = 1$

Economy with $n = 1$



Economy with $n = 10$



Impulse response for economies with $N_a = 2.0$ and $std(\Delta p_i) = 0.15$. (see the text).

steady state to change each price:

$$T_{1/2} \equiv \hat{T}_{1/2}\, N_a$$

we get that normalized half life do not depend on N_a. Finally for small monetary shocks we have the simpler expression:

$$\frac{1}{2} = \frac{\partial}{\partial \delta}\, \mathcal{P}_n \left(T_{1/2}\,,\, 0\,;\, 1\,,\, 1\,,\, \frac{\lambda}{N_a} \right) \tag{12}$$

which depends exclusively on n and λ/N_a, so in particular it does *not* depend on δ nor on $Std[\Delta p]$.

4.1 The cumulated output effect of a monetary shock

We give an analytical characterization of a summary measure for the effect of monetary shocks. The summary measure is the area under the impulse response for output, i.e. the sum of the output above steady state after a monetary shock of size $\delta > 0$, which we denote as:

$$\mathcal{M}_n(\delta) = (1/\epsilon) \int_0^\infty \left[\delta - \mathcal{P}_n(\delta, t)\right]\, dt \tag{13}$$

where ϵ is a the reciprocal of intertemporal elasticity of substitution, and where $\mathcal{P}_n(\delta, t)$ is the cumulative effect of monetary shock δ in the (log) of the price level after t periods. For large enough shocks, given the fixed cost of changing prices, the model display more price flexibility, so to isolate the effects from that, and because it is more realistic (see corollary Corollary 1) we consider the case of small shocks δ by taking the first oder approximation to equation (13), so we consider $\mathcal{M}_n(\delta) \approx \mathcal{M}_n'(0)\delta$. As a consequence of our characterization in Proposition 7, the derivative $\mathcal{M}_n'(0)$ is the product of $1/(\epsilon\, N_a)$ times a function that depends on n and λ/N_a only. The scaling for N_a is quite intuitive: the effect of monetary policy is inversely proportional to price flexibility. The scaling by ϵ represent the elasticity of the labor supply. To isolate these effects we write $\mathcal{M}_n'(0)\, \epsilon\, N_a$, and refer to it as the scaled effect of monetary shocks. The next results links the scaled effect of monetary policy with the kurtosis of the price changes.

Proposition 8 *Fix $n \geq 1$. Let $Kurt_n\left(\Delta p_i; \frac{\lambda}{N_a}\right)$ be the kurtosis of the steady state price changes for an economy where each firm sells n products and the fraction of free adjustments is $\frac{\lambda}{N_a}$. Then:*

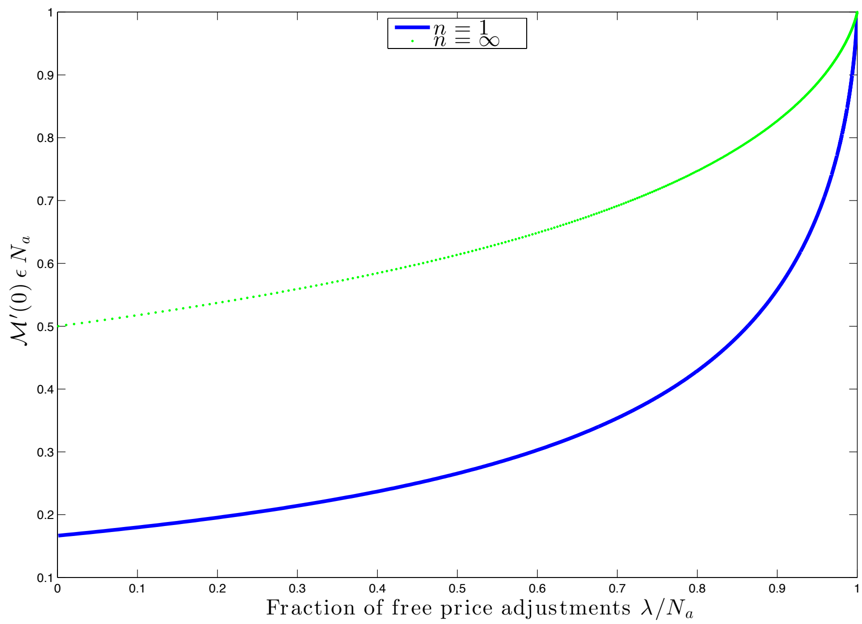
$$\mathcal{M}_n\left(\delta; N_a, Std[\Delta p_i], \frac{\lambda}{N_a}\right) \approx \mathcal{M}_n'\left(0; N_a, 1, \frac{\lambda}{N_a}\right)\delta = \frac{\delta}{\epsilon\, N_a} \frac{Kurt_n\left(\Delta p_i; \frac{\lambda}{N_a}\right)}{6} ,$$

where $\approx$ means the remainder is of order smaller than δ.

A few comments are in order. First recall that the shape of the distribution, and hence kurtosis, depends only on n and λ/N_a, or equivalently on the pair (n, ϕ). For a fixed n, kurtosis is increasing in λ/N_a. Indeed, as λ/N_a goes to 1 then kurtosis goes to 6, and hence we obtain that: $\mathcal{M}_n(\delta) \approx \delta/(\epsilon\, N_a)$, which is the same result as using Calvo's pricing model. On the other extreme, as $\lambda/N_a = 0$ we have that kurtosis equals $3n/(n+2)$. This implies that, for instance, in the *Golosov and Lucas* case of $n = 1$, the impact of monetary policy is 1/6 of *Calvo's*. Also, keeping $\lambda/N_a = 0$ and varying n the effect goes from 1/6 to 1/2 of *Calvo's*, as n diverges towards infinity. Indeed in the case of $\lambda/N_a = 0$ and $n = \infty$ the model becomes *Taylor's* staggered price model.

The cumulative output effect in two special cases: $n = 1$ and $n = \infty$. This section displays the expression for two special cases which are easier to derive analytically, namely the $n = 1$ case and the $n \to \infty$ case. These special cases bracket the possible range of output effects that are achievable by our model, i.e. they bracket the general cases of $2 \leq n < \infty$. For each case we derive the implications for prices and output while considering the full range of values for $\lambda/N_a \in (0, 1)$ while always keeping the frequency and variance of price changes constant.

Figure 6: Total cumulated output effect of a monetary shock



The expression for these effects are given by

$$\mathcal{M}'(0) = \begin{cases} \frac{1}{\epsilon\, N_a} \left[\frac{1-(1+\phi)e^{-\phi}}{\left(1-e^{-\phi}\right)^2} \right] & \text{for } n = \infty \\ \\ \frac{1}{\epsilon\, N_a} \frac{e^{2\sqrt{2\phi}}+1}{\left(e^{\sqrt{2\phi}}-1\right)^2} \left(\frac{e^{\sqrt{2\phi}}+e^{-\sqrt{2\phi}}-2(1+\phi)}{e^{\sqrt{2\phi}}+e^{-\sqrt{2\phi}}-2} \right) & \text{for } n = 1 \end{cases}$$

where different values of ϕ map monotonically into the fraction of free adjustments λ/N_a. Figure 6 shows that at any level of λ/N_a the real output effect of a monetary shock is smaller for $n = 1$ that for a very large n. As shown by Alvarez and Lippi (2013) a larger number of goods dampens the selection effect of monetary policy increasing the real output consequences of a monetary shock. Moreover, the figure shows that fixing n the output effect are increasing in λ/N_a. In the limit, as $\lambda/N_a \to 1$ the economy converges to a Calvo model where the real effects are largest and independent of n.

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A Data Appendix

A.1 Details on data treatment and further sectoral statistics

Some additional features of our data treatment are as follows.
Dealing with product replacement. The dataset contains flags for product replacement as well as imputed prices which we use as follows to design our dataset. First, we discarded observations with item substitution, as item substitution may result into spurious values for price changes, if quality adjustment is not accounted for or imperfectly measured (Berardi, Gautier, and Le Bihan (2013) investigate the inclusion of information on item substitutions). Second, we replaced any "imputed price" in the dataset, by the previous price of the same item in the same outlet present in the data, i.e. a carry–forward procedure. In the source dataset imputed prices are introduced by the INSEE when prices are missing.[19] Imputed prices are constructed either using the carry–forward procedure, or imputing the average price change of similar goods observed in the close area. The latter procedure makes sense from the aggregate CPI point of view but is obviously ill-suited for characterizing price change at the individual level. We used the flag for imputed prices to locate and replace them by carry-forward prices. This procedure amounts to discarding imputed prices when computing the distribution of (non-zero) price changes.
Computing price changes and dealing with outliers. Price changes were computed as 100 times the log-difference in prices per unit. We compute a consistent price per unit by, when relevant, dividing prices by the indicator of quantity sold (package size). We removed outliers, which in our baseline analysis we define as price changes larger in absolute value than 0.1 percent, or lower in absolute value than ln(10/3). These thresholds are set as a first crude ways to deal with measurement errors. Some robustness checks are presented in Table 7.

The upper threshold for outliers is set with sales in mind, as we informally observe that price rebates as large as 70% are sometime advertised in sales periods. Our threshold allows for a price to decrease by up to 70% and subsequently return to its former level without discarding the observation. Price changes larger than this threshold are discarded as being outliers.[20]
Identifying sales. The flag for sale allows to identify sales. Two kinds of sales-promotion discounts, that have a different status, exist in France: seasonal sales or temporary discounts. Seasonal sales ('soldes') are subject to administrative restrictions: the time period (twice a year) is decided by local authorities and price posting is subject to precise regulations. Temporary discounts are not subject to such restrictions but sales below cost are prohibited by commercial law. By contrast, selling below cost is allowed in the case of seasonal sales. On the sample period, seasonal sales are observed only in some specific categories of goods (mainly clothes). The proportion of price quotes that are flagged as seasonal sales is 0.76% and the proportion of temporary discounts amounts to 1.92%.

Some sectoral facts of price changes are as follows.

[19] Prices may be missing because of stock-outs, closed outlet due e.g. to holidays or seasonality in product availability, for instance.

[20] An example of outlier is the fee for parking in the street, which is free in some cities in summer.

Main facts at sectoral level. The different sectors in the CPI have very different pricing patterns, as well documented in recent research. The purpose of this appendix section is to illustrate that the peakedness of the price change distribution is a fact observed in all sectors. Table 6 documents pricing patterns fact using a breakdown 6 into broad economic sectors.[21] As previous research, we observe many sectoral specificities: prices change less often and rarely decrease in services; the size of price changes is smaller in services; energy prices change frequently and by small amounts; reflecting sales, the variance of price change is huge in clothes. However, noticeably a large kurtosis is observed in all sectors, one exception being clothes for which kurtosis (2.09) is lower than that of the Gaussian distribution. The fraction of small price changes, using one fourth of mean absolute price change as a threshold, ranges between 8% and 27% for all categories other than energy. An Online appendix, using a sector and type of good partition, further documents that this fact is consistently observed at higher levels of disaggregation.

Table 6: Results by type of goods

Good type	**Freq**	**Avg |dp|**	**Std |dp|**	**Kurt (dp)**	**Frac25**
Food	19.38	9.18	12.31	10.78	29.26
Durable goods	15.16	14.73	13.57	5.99	18.07
Clothing	11.00	42.48	24.71	2.16	10.21
Other manufactured goods	11.43	10.39	14.34	9.36	34.02
Energy	77.00	3.79	3.10	6.90	12.13
Services	6.53	7.80	10.29	17.58	21.29

Source is INSEE, monthly price records from French CPI, data from 2003:4 to 2011:4. Coverage is aroud 65% of CPI weight since rents, and prices of fresh food and centrally collected items (e.g. electricity, train and airplane tickets) are not included in the dataset. Freq. denotes monthly frequency of price change in percent. Size of price change Δp are the first-difference in the logarithm of price per unit, expressed in percent. Avg is average, Std standard devation, Frac25 the share of absolute price change that are inferior to 0.25 Avg$[|\Delta p|]$, Kurt denotes Kurtosis. Observations with imputed prices or quality change are discarded. Moments are computed aggregating all prices changes using CPI weigths at the product level.

A.2 Small price changes and measurement error

This appendix examines to what extent the arguments of Eichenbaum et al. (2012) apply to our data and investigates the robustness of our findings to various criteria for trimming the data. Measurement errors may arise for several reasons. Eichenbaum, Jaimovich, and Rebelo (2008) and Eichenbaum et al. (2012) articulate two concerns about the small price change. First they notice that in scanner data studies the price level of an item is typically computed as the ratio of recorded weekly revenues to quantity sold. To the extent that there are temporary or individual specific discounts (say coupons), this will generate spurious small

[21] The breakdown we use (food; durable goods; clothing & textile; other manufactured goods ; energy;services) is one we deem the most meaningful to capture price-setting idiosyncracies.

Table 7: Robustness to trimming

Type of trimming	Flag	Freq.	Avg($\lvert\Delta p\rvert$)	Std[$\lvert\Delta p\rvert$]	Frac25	Kurt[Δp_i]	Kurt[$\frac{\Delta p - m_i}{\sigma_{\Delta p_i}}$]
$\lvert\Delta p_i\rvert \leq \ln(2)$	1	17.10	8.51	12.00	28.93	10.23	7.35
Exc. flagged sales	2	14.82	5.05	5.90	18.77	13.59	8.60
$\lvert\Delta p_i\rvert \leq \ln(10/3)$	3	17.21	9.12	13.79	30.33	12.92	9.04
$0.1/100 \leq \lvert\Delta p_i\rvert \leq \ln(2)$	4	16.98	8.59	12.03	28.48	10.14	7.21
$0.5/100 \leq \lvert\Delta p_i\rvert \leq \ln(2)$	5	16.56	8.84	12.12	27.06	9.84	6.86
$0.1/100 \leq \lvert\Delta p_i\rvert \leq \ln(10/3)$ & ex.sales	6	14.70	5.15	6.23	18.21	20.86	10.40
$0.1/100 \leq \lvert\Delta p\rvert \leq \ln(10/3)$	8	17.09	9.19	13.82	29.91	12.81	8.89
$1/100 \leq \lvert\Delta p_i\rvert \leq \ln(2)$	11	15.27	9.66	12.44	22.46	8.94	6.33

(Table, continued) Moments of standardized price change

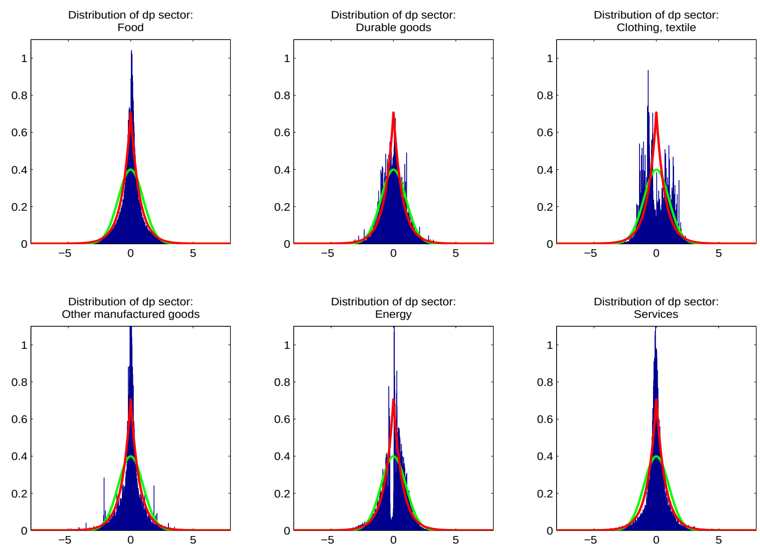
Type of trimming	Flag	Frac($< 0.25m$)	Frac($< 0.5m$)	Frac($> 2m$)	Frac($> 4m$)
$\lvert\Delta p_i\rvert \leq \ln(2)$	1	39.29	22.01	13.10	1.75
Exc. flagged sales	2	38.59	20.62	12.58	1.97
$\lvert\Delta p_i\rvert \leq \ln(10/3)$	3	39.55	22.25	12.95	1.82
$0.1/100 \leq \lvert\Delta p_i\rvert \leq \ln(2)$	4	39.10	21.90	13.07	1.72
$0.5/100 \leq \lvert\Delta p_i\rvert \leq \ln(2)$	5	38.36	20.91	12.85	1.61
$0.1/100 \leq \lvert\Delta p_i\rvert \leq \ln(10/3)$ & ex.sales	6	38.55	20.67	12.51	1.96
$0.1/100 \leq \lvert\Delta p\rvert \leq \ln(10/3)$	8	39.31	22.18	12.91	1.79
$1/100 \leq \lvert\Delta p_i\rvert \leq \ln(2)$	11	35.61	17.74	12.09	1.29

Source is INSEE, monthly price records from French CPI, data from 2003:4 to 2011:4. Coverage is aroud 65% of CPI weight since rents, and prices of fresh food and centrally collected items (e.g. electricity, train and airplane tickets) are not included in the dataset. Freq. denotes monthly frequency of price change in percent. Size of price change Δp are the first-difference in the logarithm of price per unit, expressed in percent. Avg is average, Std standard devation, Frac25 the share of absolute price change that are inferior to 0.25 Avg[$\lvert\Delta p\rvert$], Kurt denotes Kurtosis. Kurt[$\frac{\Delta p - m_i}{\sigma_{\Delta p_i}}$] denotes Kurtosis of the distribution of standardized price changes. Standardized price changes are computed at the category of good * type of outlet level. Observations with imputed prices or quality change are discarded. Moments are computed aggregating all prices changes using CPI weigths at the product level. Each row describes a sub-sample constructed applying the filter described by the column "type of trimming"."Ex. sales" exclude observations flagged as sales by the INSEE data collectors. "ex. euro" indicates the 12 month period (2001:7-2002:6)surrounding the euro cash change-over is discarded. Subsample described by the last row, with flag code 8, is taken as a baseline in the tables of the paper.

price changes.[22] Moreover Eichenbaum et al. (2012) highlight a related problem for some CPI items: they spot 27 items (named ELIS in the BLS terminology) that are problematic because these prices are typically computed as a Unit Value Index (a ratio of expenditure to quantity purchased), or they are not consistently recorded in the same outlet, or they are the price of a bundle of goods (for instance the sum of airplane fare and airport tax).

[22] Notice that in principle CPI data are immune from this type of measurement error, as these data are direct transaction prices observed by a field agent. Indeed, in the instance of a temporary discount, the CPI dataset will record either no price change, or the large price change of observed during the discount, if the field agent happens to be collecting data during the temporary discount. Further, the protocol of data collection requires that the field agent records the price faced by a regular customer, not benefiting from individual-specific discounts.

Figure 7: Distribution of standardized Price Adjustments by group of goods: France 2003-2011



The figures uses the elementary CPI data from France (see the text).

We were able to match these items with their counterparts in our French dataset. Out of the 27 problematic items 15 are not present in our data because in the French CPI those items are not recorded by a field agent but are centrally collected (thus not made available in the subset of CPI we have access to).[23] Concerning the 12 remaining items virtually no price record in the French CPI is computed as a Unit Value Index, which is hypothesized by Eichenbaum et al. (2012) as a major source of small price changes. Inspecting the patterns of price changes over these 12 potentially "problematic" items in our dataset shows that the amount of small price changes is not significantly different from the one detected over the rest of our sample. One exception is the price of "Residential water" where it can be suspected that many small variations in local taxes occur.[24]

A second investigation on measurement error was developed by varying the upper and lower thresholds of small and large price changes used to define outliers. Results are displayed in Table 7 of the Appendix. In each of the variants considered in Table 7, both kurtosis and the fraction of small price changes remain large. The lowest level of kurtosis obtains when we use the most stringent thresholds for outliers, discarding price change smaller than 0.5% and larger than $ln(2)$ (i.e. treating price decreases larger in absolute value than 50% and

[23]These items are Hospital room in-patient; Hospital in-patient services other than room ; Electricity; Utility natural gas service; Telephone services, local charges ; Interstate telephone services ; Community antenna or cable TV ; Cigarettes; Garbage and trash collection; Airline fares; New cars; New trucks; Ship fares; Prescription drugs and medical supplies; Automobile insurance.

[24]Otherwise, on the bulk of consumption items, there are no local taxes in France, and the main, nation-wide, rate of the Value Added Tax rate did not move over the sample period.

price increases larger than 100% as outliers). However, even in this case, the kurtosis of the standardized prices change is still as large as 6.33.

B Proofs

Proof. (of Proposition 1)

Let $p(0) = 0$. Define $x(t) \equiv ||p(t)||^2 - n\,\sigma^2\,t$ for $t \geq 0$. Using Ito's lemma we can verify that the drift of $||p||^2$ is $n\sigma^2$, and hence $x(t)$ is a Martingale. By the optional sampling theorem $x(\tau)$, the process stopped at τ, is also a martingale. Then

$$\mathbb{E}\left[x(\tau)\;\middle|\;p(0)\right] = \mathbb{E}\left[||p(\tau)||^2\;\middle|\;p(0)\right] - n\sigma^2\mathbb{E}\left[\tau\;\middle|\;p(0)\right] = x(0) = 0$$

and since

$$N_a = 1/\mathbb{E}\left[\tau\;\middle|\;p(0)\right] \text{ and } Var(\Delta p_i) = \mathbb{E}\left[||p(\tau)||^2\;\middle|\;p(0)\right]/n$$

we obtain the desired result. □

Proof. (of Proposition 3) The first part is straightforward given Lemma 1 and Proposition 3 in Alvarez and Lippi (2013). The second part is derived from the following implicit expression determining $\bar{y}$ (see the proof of Proposition 3 in Alvarez Lippi (2013) for a derivation):

$$\psi \;=\; \frac{B}{r+\lambda}\bar{y}\left[1 - \frac{\frac{2\sigma^2(n+2)}{r+\lambda}\bar{y} + \bar{y}^2 + \bar{y}^2\sum_{i=1}^{\infty}\kappa_i\,(r+\lambda)^i\,\bar{y}^i}{\frac{2\sigma^2(n+2)}{r+\lambda}\bar{y} + 2\bar{y}^2 + \bar{y}^2\sum_{i=1}^{\infty}\kappa_i\,(i+2)\,(r+\lambda)^i\,\bar{y}^i}\right] \tag{14}$$

where $\kappa_i = (r+\lambda)^{-i}\prod_{s=1}^{i}\frac{1}{\sigma^2(s+2)(n+2s+2)}$. So we can write this expression as:

$$\psi = \frac{B}{r+\lambda}\bar{y}\left[1 - \xi(\sigma^2, r+\lambda, n, \bar{y})\right]$$

where $\xi(\sigma^2, r+\lambda, n, \bar{y})$ is given by:

$$\xi(\sigma^2, r+\lambda, n, \bar{y}) \equiv \frac{\frac{2\sigma^2(n+2)}{r+\lambda}\bar{y} + \bar{y}^2 + \bar{y}^2\sum_{i=1}^{\infty}\kappa_i\,(r+\lambda)^i\,\bar{y}^i}{\frac{2\sigma^2(n+2)}{r+\lambda}\bar{y} + 2\bar{y}^2 + \bar{y}^2\sum_{i=1}^{\infty}\kappa_i\,(i+2)\,(r+\lambda)^i\,\bar{y}^i}$$

Since $\bar{y} \to \infty$ as $\psi \to \infty$ then we can define the limit:

$$\lim_{\psi\to\infty}\frac{\psi}{\bar{y}} = \frac{B}{r+\lambda}\left[1 - \lim_{\bar{y}\to\infty}\xi(\sigma^2, r+\lambda, n, \bar{y})\right]$$

Simple analysis can be used to show that $\lim_{\bar{y}\to\infty}\xi(\sigma^2, r+\lambda, n, \bar{y}) = 0$ which gives the expression in the proposition (see the Online Appendix F for a detailed derivation). □

Proof. (of Proposition 4).

To characterize N_a we write the Kolmogorov backward equation for the expected time between adjustments $\mathcal{T}(y)$ which solves (see Appendix E for a discussion of the solution to this equation): $\lambda \mathcal{T}(y) = 1 + \; n\, \sigma^2 \, \mathcal{T}'(y) + \; 2 \, y \, \sigma^2 \, \mathcal{T}''(y)$ for $y \in (0, \bar{y})$ and $\mathcal{T}(\bar{y}) = 0$. Then the expected number of adjustments is given by $N_a = 1/\mathcal{T}(0)$, subject to $\mathcal{T}(0) < \infty$.

We guess that the solution of the ODE equation (3) has a power series representation:

$$\mathcal{T}(y) = \sum_{i=0}^{\infty} \alpha_i \; y^i \;, \quad \text{for } y \in [0, \bar{y}] \;. \tag{15}$$

and then obtain the following conditions on its coefficients $\{\alpha_i\}$:

$$\alpha_1 = \frac{\lambda \alpha_0 - 1}{n \sigma^2} \quad , \quad \alpha_{i+1} = \frac{\lambda}{(i+1) \; \sigma^2 \; (n + 2i)} \; \alpha_i \;, \quad \text{for } i \geq 1 \;. \tag{16}$$

and where $0 < \alpha_0 < 1/\lambda$ is chosen to that $0 \geq \alpha_i$ for $i \geq 1$, $\lim_{i \to \infty} \frac{\alpha_{i+1}}{\alpha_i} = 0$ and $0 = \sum_{i=0}^{\infty} \alpha_i \; \bar{y}^i$. Moreover, $\mathcal{T}(0) = \alpha_0$ is an increasing function of $\bar{y}$ since α_0 solves:

$$\begin{aligned} & 0 = \alpha_0 + \frac{(\alpha_0 - 1/\lambda)}{n} \left(\frac{\bar{y}\lambda}{\sigma^2} \right) \left[1 + \sum_{i=1}^{\infty} \left(\prod_{k=1}^{i} \frac{1}{(k+1) \; (n+2k)} \right) \left(\frac{\bar{y}\lambda}{\sigma^2} \right)^i \right] \quad \text{or} \\ & \alpha_0 \left\{ 1 + \frac{1}{n} \left(\frac{\bar{y}\lambda}{\sigma^2} \right) \left[1 + \sum_{i=1}^{\infty} \left(\prod_{k=1}^{i} \frac{1}{(k+1) \; (n+2k)} \right) \left(\frac{\bar{y}\lambda}{\sigma^2} \right)^i \right] \right\} \\ = \; & \frac{1}{\lambda n} \left(\frac{\bar{y}\lambda}{\sigma^2} \right) \left[1 + \sum_{i=1}^{\infty} \left(\prod_{k=1}^{i} \frac{1}{(k+1) \; (n+2k)} \right) \left(\frac{\bar{y}\lambda}{\sigma^2} \right)^i \right] . \end{aligned}$$

Solving for α_0 gives the desired expression. The second order approximation follows from differentiating this expression twice. $\square$

We first state a lemma about the density $f(y)$.

Lemma 2 *Let $f(y; n, \frac{\lambda}{\sigma^2}, \bar{y})$ be the density of $y \in [0, \bar{y}]$ in equation (5) satisfying the boundary conditions. For any $k > 0$*

$$f\left(y; n, \frac{\lambda}{\sigma^2}, \bar{y}\right) = \frac{1}{k} \; f\left(\frac{y}{k}; n, \frac{\lambda k}{\sigma^2}, \frac{\bar{y}}{k}\right)$$

Proof. (of Lemma 2). Consider the function $f(y; n, \frac{\lambda}{\sigma^2}, \bar{y})$ solving equation (5) (and boundary conditions) for given $n, \frac{\lambda}{\sigma^2}, \bar{y}$. Without loss of generality set $\sigma' = \sigma$ and consider $\bar{y}' = \bar{y}/k$ and $\lambda' = \lambda k$. Notice that by setting $C_1' = C_1 k$ and $C_2' = C_2 k$ we verify that the boundary conditions hold (because $C_1'/C_2' = C_1/C_2$) and that (5) holds (which is readily verified by a change of variable). $\square$

Proof. (of Proposition 5) Let $w\left(\Delta p_i; n, \frac{\lambda}{N_a}, Std(\Delta p_i)\right)$ be the density function in equation (7). Next we verify equation (8). From the first term in equation (7) notice that

$$\left(1 - \frac{\lambda}{N_a}\right) \omega \left(\Delta p_i; \bar{y}\right) = s \left(1 - \frac{\lambda}{N_a}\right) \omega \left(s \Delta p_i; s^2 \bar{y}\right)$$

where the first equality uses the homogeneity of degree -1 of $\omega(\Delta p_i; y)$ (see equation (6)). From the second term in equation (7) for $n \geq 2$

$$\frac{\lambda}{N_a}\int_0^{\bar{y}} \omega(\Delta p_i; y) f(y; n, \frac{\lambda}{\sigma^2}, \bar{y}) dy = \frac{\lambda}{N_a}\int_0^{\bar{y}} s\ \omega\left(s\Delta p_i; s^2 y\right)\ s^2\ f\left(ys^2; n, \frac{\lambda}{s^2\sigma^2}, \bar{y}s^2\right) dy$$

where the first equality follows from Lemma 2 for $k = 1/s^2$, and the homogeneity of degree -1 of $\omega(\cdot,\cdot)$. Further we note

$$\frac{\lambda}{N_a}\int_0^{\bar{y}} s\ \omega\left(s\Delta p_i; s^2 y\right) s^2\ f\left(ys^2; n, \frac{\lambda}{s^2\sigma^2}, \bar{y}s^2\right) dy = s^3\frac{\lambda}{N_a}\int_0^{\bar{y}} \omega\left(s\Delta p_i; s^2 y\right) f\left(ys^2; n, \frac{\lambda'}{\sigma'^2}, \bar{y}'\right) dy$$

where we note that $\frac{\lambda' \bar{y}'}{\sigma'^2} = \frac{\lambda \bar{y}}{\sigma^2}$, so that λ/N_a is the same across the two economies. Using the change of variable $z = y\ s^2$

$$s^3\frac{\lambda}{N_a}\int_0^{\bar{y}} \omega\left(s\Delta p_i; s^2 y\right) f\left(ys^2; n, \frac{\lambda'}{\sigma'^2}, \bar{y}'\right) dy = s\ \frac{\lambda}{N_a}\int_0^{\bar{y}'} \omega\left(s\Delta p_i; z\right) f\left(z; n, \frac{\lambda'}{\sigma'^2}, \bar{y}'\right) dz\ .$$

where $\bar{y}' = s^2\bar{y}$, which completes the verification of equation (8). □

Proof. (of Proposition 6)

To obtain expression in equation (9) we use the characterization of $\lambda/N_a = \mathcal{L}_n\left(\frac{\lambda\bar{y}}{n\sigma^2}\right)$ of Proposition 4, it is equivalent to fix a value of $\phi \equiv \frac{\lambda\bar{y}}{n\sigma^2}$. We let the optimal decision rule be $\bar{y}\left(\psi/B, \sigma^2, r+\lambda, n\right)$ so that we have:

$$\bar{y}\left(\frac{\psi}{B}, \sigma^2, r+\lambda, n\right)\ \frac{\lambda}{n\,\sigma^2} = \phi$$

Moreover we have that to be consistent with $Var(\Delta p_i)$ and N_a we have, using Proposition 1 and $\lambda/N_a = \mathcal{L}_n(\phi)$:

$$N_a = \lambda/\mathcal{L}_n(\phi) \ \text{ and } \ \frac{\lambda}{\sigma^2} = \mathcal{L}_n(\phi)/Var(\Delta p_i)\ .$$

Thus, after taking $r \downarrow 0$ and using the expression above we can write:

$$\bar{y}\left(\frac{\psi}{B}\ ,\ N_a Var(\Delta p_i)\ ,\ \mathcal{L}_n(\phi)N_a\ ,\ n\right)\ \frac{\mathcal{L}_n(\phi)}{n\,Var(\Delta p_i)} = \phi$$

Fixing n and totally differentiating this expression with respect to $(\psi/B, N_a, Var(\Delta p_i))$, and denoting by η_ψ, η_{σ^2}, η_λ the elasticities of $\bar{y}$ with respect to ψ/B, σ^2, λ we have:

$$\eta_\psi\,\hat{\psi} + \eta_{\sigma^2}\,(\hat{N}_a + \hat{V}ar(\Delta p_i)) + \eta_\lambda\,\hat{N}_a = \hat{V}ar(\Delta p_i)$$

where a *hat* denotes a proportional change. Using Proposition 3-(iv) in Alvarez and Lippi (2013) and Lemma 1 we have that these elasticities are related by:

$$\eta_\lambda = 2\eta_\psi - 1 \ \text{ and } \ \eta_{\sigma^2} = 1 - \eta_\psi\ .$$

Thus

$$\eta_\psi \, \hat{\psi} + (1-\eta_\psi) \, (\hat{N}_a + \hat{V}ar(\Delta p_i)) + (2\eta_\psi - 1) \, \hat{N}_a = \hat{V}ar(\Delta p_i) \ .$$

Rearranging:

$$\eta_\psi \, \hat{\psi} + (1 - \eta_\psi + 2\eta_\psi - 1) \, \hat{N}_a + (1 - \eta_\psi - 1)\hat{V}ar(\Delta p_i) = 0 \, ,$$

and canceling terms:

$$\eta_\psi \, \hat{\psi} + \eta_\psi \, \hat{N}_a - \eta_\psi \hat{V}ar(\Delta p_i) = 0 \ .$$

Dividing by η_ψ we obtain that $\hat{\psi} = \hat{V}ar(\Delta p_i) - \hat{N}_a$. Additionally, since $\bar{y}$ is a function of B, then we can write $\psi/n = B \, (Var(\Delta p_i)/N_a) \ \Psi(n, \phi)$.

That $\psi \to \infty$ as $\lambda/N_a \to 1$ follows because $\mathcal{L}_n(\phi) \to 1$ as $\phi \to \infty$ and because, by Proposition 3-(i) in Alvarez and Lippi (2013), $\bar{y}$ is increasing in ψ and has range and domain $[0, \infty)$.

Taking logs and differentiating the definition of ψ we obtain:

$$\eta_\psi \, \hat{\psi} + \eta_\lambda \, \hat{\mathcal{L}}_n + \hat{\mathcal{L}}_n = 1$$

Replacing $\eta_{\sigma^2} = 1 - \eta_\psi$, we obtain:

$$\hat{\psi} = 1/\eta_\psi - 2\hat{\mathcal{L}}_n \ .$$

As discussed above, $\eta_\psi \to 1/2$. Using $\mathcal{L}_n(\phi) = h_n(\phi)/(1 + h)n(\phi))$. We get

$$\hat{\mathcal{L}}_n \equiv \frac{\phi}{\mathcal{L}_n(\phi)} \frac{\partial \mathcal{L}_n(\phi)}{\partial \phi} = \frac{1}{1 + h_n(\phi)} \, \frac{\phi h_n'(\phi)}{h_n(\phi)}$$

using the form of h_n and taking $\phi \to 0$ it is easy to see that this elasticity is one. Thus

$$\lim_{\phi \to 0} \frac{\phi}{\Psi(n, \phi)} \frac{\partial \Psi(n, \phi)}{\partial \phi} = 0 \ .$$

Finally, for $\lambda = 0$ and $N_a > 0$ we obtain:

$$\frac{\psi}{n} = B \, \frac{Var\,(\Delta p)}{N_a} \ \frac{n}{2\,(n+2)} \ .$$

This follows from using the square root approximation of $\bar{y}$ for small $\psi \, (\lambda + r)^2$, the expression for $N_a = n\sigma^2/\bar{y}$ and Proposition 1, i.e. $N_a \, Var(\Delta p_i) = \sigma^2$. □

Proof. (of Proposition 7) The proof has three parts. First we introduce a discrete time, discrete state version of the model, second we show the scaling of the time with N_a , and finally the homogeneity of degree one in terms of $Std[\Delta p]$ and δ.

Discrete Time Formulation. We start with discrete time version of the process for price gaps, with length of the time period Δ, which makes some of the arguments more accessible.

Let N be a

$$N(t+\Delta) = \begin{cases} N(t) & \text{with probability } (1-\lambda\Delta) \\ N(t)+1 & \text{with probability } \lambda\Delta \end{cases} \tag{17}$$

Thus, as $\Delta \downarrow 0$ this process converges to a continuous time Poisson counter with instantaneous intensity rate λ per unit of time. Let $\bar{p}_i$ follow n drift-less random walks

$$\bar{p}_i(t+\Delta, p) = \begin{cases} \bar{p}_i(t,p) \, + \, \sigma\sqrt{\Delta} & \text{with probability } 1/2 \\ \bar{p}_i(t,p) \, - \, \sigma\sqrt{\Delta} & \text{with probability } 1/2 \end{cases} \tag{18}$$

where the initial condition satisfies:

$$\bar{p}_i(0) = p_i \text{ for } i = 1,..,n \ ,$$

and where the n random walks are independent of each other and of the Poisson counter. As $\Delta \downarrow 0$ the process for $\bar{p}$ converges to a Brownian motion whose changes have variance σ^2 per unit of time. We define the stopping time of the first price adjustment $\tau(p)$, conditional on the starting at price gap vector p at time zero, as:

$$\begin{aligned} \tau_1 &\equiv \min\left\{t = 0, \Delta, 2\Delta, ... \ : N(j\Delta+\Delta) - N(j\Delta) = 1\right\}, \\ \tau_2(p) &\equiv \min\left\{t = 0, \Delta, 2\Delta, ... \ : \sum_{i=1}^{n} \left(\bar{p}_i(j\Delta+\Delta, p)\right)^2 \geq \bar{y}\right\} \text{ and} \\ \tau(p) &\equiv \min\left\{\tau_1 \ , \ \tau_2(p)\right\}. \end{aligned}$$

The function g is the density for the continuous time limit, i.e. the case where $\Delta \downarrow 0$. For small Δ, we can approximate the distribution of the *fraction* of firms with price gap vector p as the product of the density g and a correction to convert it into a probability, i.e a fraction. This gives:

$$g\left(p_1, ..., p_; n, \lambda/\sigma^2, \bar{y}\right) \, \left(\sigma\sqrt{\Delta}\right)^n$$

where the last term uses that in each dimension price gaps vary discretely in steps of size $\sigma\sqrt{\Delta}$. We can write the discrete time impulse response function as:

$$\mathcal{P}(t, \delta; \sigma, \lambda, \bar{y}, \Delta) \ = \ \Theta(\delta; \sigma, \lambda, \bar{y}, \Delta) \, + \sum_{s=\Delta}^{t} \theta(\delta, s; \sigma, \lambda, \bar{y}, , \Delta) \, \Delta \, ,$$

In this expression we can, without loss of generality, restrict t to be an integer multiple of Δ. We have divided the expression for θ by Δ, and hence multiplied its contribution back by Δ in $\mathcal{P}$, so that it has the interpretation of the contribution per unit of time to the IRF of price changes at time t, i.e. it has the units of a density. Moreover, in this manner the term has a non-zero limit, and the expression in $\mathcal{P}$ converges to an integral. Thus we get the $\mathcal{P} = \lim \mathcal{P}(\Delta)$ as $\Delta \downarrow \infty$. The functions θ and Θ are given by:

$$\Theta(\delta; \sigma, \lambda, \bar{y}, \Delta) \equiv \sum_{||p(0)-\iota\delta|| \geq \bar{y}} \left(\delta - \frac{\sum_{j=0}^{n} p_j(0)}{n}\right) g\left(p(0); n, \frac{\lambda}{\sigma^2}, \bar{y}\right) \, \left(\sigma\sqrt{\Delta}\right)^n \ , \text{ and}$$

$$\theta(\delta, t; \sigma, \lambda, \bar{y}, \Delta) \equiv$$
$$-\frac{1}{\Delta} \sum_{||p(0)-\iota\delta||<\bar{y}} \mathbb{E}\left[\frac{\sum_{j=0}^{n} \bar{p}_j(t,p)}{n}\, \mathbf{1}_{\{\tau(p)=t\}} \;\Big|\; p = p(0) - \iota\delta\right] g\left(p(0); n, \frac{\lambda}{\sigma^2}, \bar{y}\right) \left(\sigma\sqrt{\Delta}\right)^n$$

Time scaling of the IRF with N_a. For this (i) Note that if multiply the parameters σ^2 and λ by a constant $k > 0$, leaving $\bar{y}$ unaltered, then $N_a' = k\, N_a$, where primes are used to denote the values that correspond to the scaled parameters. This follows directly from the expression we derive for $N_a = 1/T(0)$ in Proposition 4. (ii) By Proposition 5 with these changes the distribution of price changes implied by $(\sigma^2, \lambda, \bar{y})$ is exactly the same as the one implied by $(k\sigma^2, k\lambda, \bar{y})$. (iii) we change notation and write $(\sigma^2, \lambda, \bar{y})$ instead of $(\lambda, \sigma^2, \psi/B)$ and omit n. We establish that

$$\mathcal{P}_n\left(\frac{t}{k}, \delta; k\sigma^2, k\lambda, \bar{y}\right) = \mathcal{P}_n\left(t, \delta; \sigma^2, \lambda, \bar{y}\right)$$

We will do so by establishing this proposition for the discrete time version of the IRF. Yet the result is immediate, since λ and σ^2 are the only two parameters which are rates per unit of time (the other parameters are n and $\bar{y}$), so by multiplying them by k we just scale time. The details can be found in the discrete time formulation, whose notation we develop below. We show that

$$\mathcal{P}(t, \delta; k\sigma^2, k\lambda, \bar{y}, \Delta/k) = \mathcal{P}(t/k, \delta; \sigma^2, \lambda, \bar{y}, \Delta) \tag{19}$$

We will do so by establishing this proposition for the discrete time version of the IRF. Let $\Delta' = \Delta/k$, $\sigma'^2 = \sigma^2 k$ and $\lambda' = \lambda k$. Note that, by construction $\sigma' \sqrt{\Delta'} = \sigma \sqrt{\Delta}$ and $\lambda'/(\sigma')^2 = \lambda/(\sigma)^2$. To establish this we first note that, for a given shock δ, Θ depends only on n, $\bar{y}$, $\sigma\sqrt{\Delta}$, and λ/σ^2. This is because the invariant density g and the scaling factor to convert it into probabilities depends only on those parameters. Second we show that

$$\sum_{s=\Delta/k}^{t/k} \frac{\Delta}{k}\, \theta\left(s, \delta; k\sigma^2, k\lambda, \bar{y}, \frac{\Delta}{k}\right) = \sum_{s=\Delta}^{t} \Delta\, \theta\left(s, \delta; \sigma, \lambda, \bar{y}, \Delta\right)$$

This follows because for each s and $p(0)$

$$\begin{aligned} & \mathbb{E}\left[\frac{\sum_{j=0}^{n} \bar{p}_j\left(s,p\right)}{n}\, \mathbf{1}_{\{\tau(p)=s\}} \;\Big|\; p = p(0) - \iota\delta\,;\, \sigma, \lambda, \Delta\right] \\ = \;& \mathbb{E}\left[\frac{\sum_{j=0}^{n} \bar{p}_j\left(\frac{s}{k}, p\right)}{n}\;\; \mathbf{1}_{\left\{\tau(p)=\frac{s}{k}\right\}} \;\Big|\; p = p(0) - \iota\delta\,;\, \sigma', \lambda', \Delta'\right] \end{aligned}$$

where we include the parameters $(\lambda, \sigma^2, \Delta)$ as argument of the expected values. This itself follows because, using equation (17) and equation (18) then the processes for $\{\bar{p}_i\}$ are the same in the original time and in the time time scales by k since the probabilities of the counter to go up $\lambda'\Delta' = \lambda\Delta$ and the steps of the symmetric random walks $\sigma'\sqrt{\Delta'} = \sigma\sqrt{\Delta}$

are the same in the original time and the time scaled by k. In particular we have that

$$\bar{p}_j\left(\frac{s}{k}, p; \lambda', \sigma'^2, \Delta'\right) \equiv \bar{p}_j\left(\frac{s}{k}, p; k\lambda, k\sigma^2, \frac{\Delta}{k}\right) = \bar{p}_j\left(s, p; \lambda, \sigma^2, \Delta\right) = \hat{p}$$

with exactly the same probabilities for each price gap $\hat{p} \in \mathbb{R}$ and each time $s \geq 0$. Also, repeating the arguments used for Θ, we have $g\left(p(0); n, \frac{\lambda}{\sigma^2}, \bar{y}\right)\left(\sigma\sqrt{\Delta}\right)^n = g\left(p(0); n, \frac{\lambda'}{\sigma'^2}, \bar{y}\right)\left(\sigma'\sqrt{\Delta'}\right)^n$. Thus, since equation (19) holds for all $\Delta > 0$, taking limits

$$\mathcal{P}\left(\frac{t}{k}, \delta; k\sigma^2, k\lambda, \bar{y}\right) = \lim_{\Delta \downarrow 0} \mathcal{P}\left(\frac{t}{k}, \delta; k\sigma^2, k\lambda, \bar{y}, \frac{\Delta}{k}\right) = \lim_{\Delta \downarrow 0} \mathcal{P}\left(t, \delta; \sigma^2, \lambda, \bar{y}, \Delta\right) = \mathcal{P}\left(t, \delta; \sigma^2, \lambda, \bar{y}\right)$$

Scaling of the IRF in the monetary shock with $Std[\Delta p]$. For this we use properties of the invariant distribution f, which are then inherited by g. In particular, we will compare the IRF with parameters $(\lambda, \sigma^2, \bar{y})$ with one with parameters $(\lambda', \sigma'^2, \bar{y})$ where $\lambda' = \lambda$, $\sigma'^2 = k\,\sigma^2$ and $\bar{y}' = k\,\bar{y}$. With this choice we have $N'_a = N_a$ and thus $\lambda/N_a = \lambda'/N'_a$ since $\lambda\bar{y}/(n\sigma^2) = \lambda'\bar{y}'/(n\sigma'^2)$ (see Proposition 4). Then by Proposition 1 we have that the standard deviation of price changes scales up with k, i.e.: $Std[\Delta p]' = \sqrt{k}\,Std[\Delta p]$. The main idea is that the invariant distribution corresponding to the $'$ parameters is a radial expansion of the original, so that $\int_0^y f(x; \lambda, \sigma^2, \bar{y})dx = \int_0^{yk} f(x; \lambda', \sigma'^2, \bar{y}')dx$ and thus $f(y, \lambda, \sigma^2, \bar{y}) = kf(yk, \lambda', \sigma'^2, \bar{y}')$. Indeed using Lemma 2 we have:

$$f\left(y; \frac{\lambda}{\sigma^2}, \bar{y}\right) = k\,f\left(yk; \frac{\lambda}{k\sigma^2}, k\bar{y}\right) \equiv k\,f\left(yk; \frac{\lambda'}{\sigma'^2}, \bar{y}'\right) \quad . \tag{20}$$

Thus we have:

$$\begin{aligned}
g\left(p_1, ..., p_n, \frac{\lambda}{\sigma^2}, \bar{y}\right) &= f\left(p_1^2 + \cdots + p_n^2; n, \frac{\lambda}{\sigma^2}, \bar{y}\right) \frac{\Gamma(n/2)}{2\,\pi^{n/2}\ \left(p_1^2 + \cdots + p_n^2\right)^{(n-2)/2}} = \\
&= kf\left(k(p_1^2 + \cdots + p_n^2); n, \frac{\lambda'}{\sigma'^2}, \bar{y}'\right) \frac{\Gamma(n/2)\,k^{(n-1)/2}}{2\,\pi^{n/2}\ \left(k(p_1^2 + \cdots + p_n^2)\right)^{(n-2)/2}} \\
&= g\left(\sqrt{k}(p_1, ..., p_n); n, \frac{\lambda'}{\sigma'^2}, \bar{y}'\right) k^{(n-2)/2}\,k
\end{aligned}$$

Using this for the discrete time formulation we have:

$$\begin{aligned}
g\left(p; n, \frac{\lambda}{\sigma^2}, \bar{y}\right)\left(\sigma\sqrt{\Delta}\right)^n &= g\left(\sqrt{k}\,p; n, \frac{\lambda'}{\sigma'^2}, \bar{y}'\right)\left(\sigma'\sqrt{\Delta}\right)^n k^{(n-2)/2}\,k\,k^{-n/2} \\
&= g\left(\sqrt{k}\,p; n, \frac{\lambda'}{\sigma'^2}, \bar{y}'\right)\left(\sigma'\sqrt{\Delta}\right)^n
\end{aligned}$$

Note that $\{||p(0) - \iota\delta|| \geq \bar{y}\} = \{||\sqrt{k}\,p(0) - \iota\sqrt{k}\,\delta|| \geq \sqrt{k}\,\bar{y}\} = \{||\sqrt{k}\,p(0) - \iota\delta'|| \geq \bar{y}'\}$. Also

$$\left(\delta - \frac{\sum_{j=0}^n p_j(0)}{n}\right)\sqrt{k} = \left(\delta' - \frac{\sum_{j=0}^n \sqrt{k}\,p_j(0)}{n}\right)$$

Thus

$$\sqrt{k} \sum_{||p(0)-\iota\delta|| \geq \bar{y}} \left(\delta - \frac{\sum_{j=0}^{n} p_j(0)}{n} \right) g\left(p(0); n, \frac{\lambda}{\sigma^2}, \bar{y} \right) \left(\sigma\sqrt{\Delta} \right)^n$$
$$= \sum_{||\sqrt{k}p(0)-\iota\delta'|| \geq \bar{y}'} \left(\delta' - \frac{\sum_{j=0}^{n} \sqrt{k} p_j(0)}{n} \right) g\left(\sqrt{k}\, p(0); n, \frac{\lambda'}{\sigma'^2}, \bar{y}' \right) \left(\sigma'\sqrt{\Delta} \right)^n$$

Using the definition of $\Theta(\cdot, \Delta)$:

$$\sqrt{k}\, \Theta(\delta; \sigma, \lambda, \bar{y}, \Delta) = \Theta\left(\sqrt{k}\, \delta; k\sigma^2, \lambda, k\bar{y}, \Delta \right) \equiv \Theta\left(\delta'; \sigma'^2, \lambda', \bar{y}'\Delta \right) .$$

Since this holds for all Δ, by taking limits as $\Delta \downarrow 0$, we have shown the desired result for Θ. The result for θ follows the steps for g. We set $\Delta' = \Delta$ and note that for all $p(0) \in \mathbb{R}^n$, scaling factor $k > 0$ and time horizon $s > 0$:

$$\sqrt{k}\; \mathbb{E}\left[\frac{\sum_{j=0}^{n} \bar{p}_j\,(s, p)}{n}\; \mathbf{1}_{\{\tau(p)=s\}} \;\middle|\; p = p(0) - \iota\delta\,;\, \sigma, \lambda, \Delta \right]$$
$$= \mathbb{E}\left[\frac{\sum_{j=0}^{n} \bar{p}_j\,(s, p)}{n}\; \mathbf{1}_{\{\tau(p)=s\}} \;\middle|\; p = \sqrt{k}\, p(0) - \iota\delta'\,;\, \sigma', \lambda', \Delta \right] .$$

This follows because $\lambda' = \lambda$ and $\sigma'\sqrt{\Delta'} = \sqrt{k}\,\sigma\,\sqrt{\Delta}$, thus the each $p \in \mathbb{R}^n$ the paths $\sqrt{k}\,\bar{p}(s, p; \sigma, \lambda) = \bar{p}(s, \sqrt{k}p; \sigma', \lambda')$ occur with the same probabilities. $\square$

ADDITIONAL MATERIAL
(NOT FOR PUBLICATION)

Small and large price changes in menu cost models

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C Related literature

Models with random cost of adjustment have been introduced by Caballero and Engel (1999) and Dotsey, King, and Wolman (1999). Caballero and Engel (1999) study and solve numerically a model of investment with random fixed cost of adjustement. Two outcomes are (i) that the decision rule has the form of a "generalized (S,s) rule", thus time-varying inaction thresholds, and (ii) that higher moments of the cross-sectional distribution of firm disequilibria can predict aggregate investment. Dotsey, King, and Wolman (1999) introduce random menu cost in a price-setting context, to develop a tractable general equilibrium model of state-dependent pricing. While at the individual level, adjustment is discrete, the introduction of random menu costs makes the fraction of adjusting firms become a relevant state variable which reacts smoothly to shocks. The model can then be solved with standard linearization techniques, using the property that it is sufficient to keep track of vintages of firms each characterized by the same reset dates.[25] Dotsey, King, and Wolman (1999) and Dotsey and King (2005) use the model to investigate how the response to monetary policy shocks under state–dependent pricing differ from that time–dependent pricing.

Recently, a series of papers have used random menu cost models with the explicit aim of fitting of micro data on price changes.[26] Dotsey, King, and Wolman (2008) follow up on Dotsey, King, and Wolman (1999) by introducing idiosyncratic shocks and calibrating the model using inter alia the distribution of micro economic price changes in the US. Caballero and Engel (2007) apply the generalized hazard approach of Caballero and Engel (1999) to price dynamics and illustrate how introducing random free opportunity of price changes alters the response of the economy to a monetary shock. Midrigan (2011) show that economies of scales in price setting for a multiproduct firm, and random menu costs, are alternative mecanisms that generate small price changes at the individual level. He concludes that under either economies of scales in price setting, or random menu costs, monetary policy have more persistent effect than in the Golosov and Lucas (2007) menu cost model. Burstein and Hellwig (2006) reach the same conclusion when adding random menu cost in a model with pricing complementarity. Nakamura and Steinsson (2010) also examine, in a multisector menu cost model, to which extend monetary non-neutrality is increased in a variant of the model in which the menu cost can randomly receive a low or high value. Woodford (2009) develops a model of price-setting under information capacity constraint. Optimal policy gives rise to randomisation of the price review decision. Costain and Nakov (2011, 2012) develop a model in which the probability of adjustment is a function of the value of adjustment for firms. Both in Woodford (2009) and Costain and Nakov (2011, 2012), the model is calibrated using moments of the distribution of price changes from micro data, and the obtained decision rule is observationally equivalent to that derived under a random menu cost model. Overall, two common features of this series of recent models is that they are solved using numerical techniques, and they obtain that under random menu cost the degree of monetary policy non-neutrality is to some extent larger than in the fixed menu cost model of Golosov and

[25]The model is set-up under the assumptions of no idiosyncratic shocks and i.i.d. random cost, so all firms that reset price set the same price. To have a finite number of vintages, the model requires positive steady state inflation.

[26]Previous to the recent research, Willis (2000) had estimated a partial equilibrium model stochastic menu cost model on magazine data

Lucas (2007).

The present paper is related to this recent literature. A distinctive feature is that results are derived analytically, and the way the impact of monetary policy shock depends on "deep" parameters is studied in a systematic way. The model with random menu cost is also extended to incorporate economies of scales in price adjustment.

D Details of the solution for the model with $n = 1$

Integrating the bellman equation gives the following value function

$$v(p) = \frac{Bp^2 + \lambda v(0)}{\lambda + r} + \frac{B\sigma^2}{(\lambda + r)^2} + C\left(e^{p\sqrt{\frac{2(\lambda+r)}{\sigma^2}}} + e^{-p\sqrt{\frac{2(\lambda+r)}{\sigma^2}}} \right)$$

where we already used that $v(p) = v(-p)$. Notice that the value function has a minimum (and zero derivative) at $p = 0$, which is the optimal return point. The constant C and the threshold value $\bar{p}$ are the values that solve the 2 equation system given by the value matching condition and the smooth pasting conditions.

The expected time to adjustment, $T(p)$ obeys the differential equation $\lambda T(p) = 1 + \frac{\sigma^2}{2}T''(p)$ with boundary condition $T(\bar{p}) = 0$. Given the symmetry of the law of motion for p, the function is symmetric, i.e. $T(p) = T(-p)$. Integrating gives $T(p) = \frac{1}{\lambda}\left(1 - \frac{e^{\sqrt{\frac{2\lambda}{\sigma^2}}p} + e^{-\sqrt{\frac{2\lambda}{\sigma^2}}p}}{e^{\sqrt{\frac{2\lambda}{\sigma^2}}\bar{p}} + e^{-\sqrt{\frac{2\lambda}{\sigma^2}}\bar{p}}}\right)$.

The distribution of price gaps $h(p)$ satisfies the Kolmogorov forward equation $0 = -\frac{2\lambda}{\sigma^2}h(p) + h''(p) \quad \text{for} \quad 0 < |p| \leq \bar{p}$. The density is symmetric, $h(p) = h(-p)$, and satisfies the boundary conditions: $h(\bar{p}) = 0$ and it integrates to one i.e. $2\int_0^{\bar{p}} h(p)\ dp = 1$ where we used that it is symmetric.[27]

Now we compute some moments for price changes Δp_i which are illustrative of the mapping between the model and the data. The mean absolute value of price changes is

$$\mathbb{E}|\Delta p| \quad = \quad \frac{2\lambda}{N_a}\int_0^{\bar{p}} p\ h(p)\ \mathrm{d}p + \left(1 - \frac{\lambda}{N_a}\right)\bar{p} = \mathcal{H}\ \bar{p}$$

where it is to be noticed that the term $\mathcal{H}$ depends only on ϕ, namely

$$\mathcal{H} = \left[\frac{\lambda}{N_a}\left(\frac{e^{\sqrt{2\phi}} - e^{-\sqrt{2\phi}} - 2\sqrt{2\phi}}{\sqrt{2\phi}\ \left(e^{\frac{\sqrt{2\phi}}{2}} - e^{\frac{-\sqrt{2\phi}}{2}}\right)^2}\right) + 1 - \frac{\lambda}{N_a}\right]$$

The variance of price changes

$$Var\,(\Delta p) \quad = \quad \frac{2\lambda}{N_a}\int_0^{\bar{p}} p^2\ h(p)\ \mathrm{d}p + \left(1 - \frac{\lambda}{N_a}\right)\bar{p}^2 = \frac{\sigma^2}{N_a}$$

[27] The first boundary can be derived as the limit of the discrete time, discrete state, low of motion where each period is of length Δ and where p increases or decreases with probability 1/2, so that $h(p) = \frac{1}{2}h(p+\Delta) + \frac{1}{2}h(p-\Delta)$. At the boundary $\bar{p}$ this law of motion is $h(\bar{p}) = \frac{1}{2}h(\bar{p}-\Delta)$, which shows that $h(\bar{p}) \downarrow 0$ as $\Delta \downarrow 0$.

and a closed form expression for the kurtosis

$$kurt\left(\Delta p\right) = \frac{\mathbb{E}\left(\Delta p\right)^4}{\left[\mathbb{E}\left(\Delta p\right)^2\right]^2} = \frac{\frac{2\lambda}{N_a}\left(\frac{12}{(\sqrt{2\phi})^4} - \frac{12+(\sqrt{2\phi})^2}{(\sqrt{2\phi})^2\left(e^{\frac{\sqrt{2\phi}}{2}} - e^{\frac{-\sqrt{2\phi}}{2}}\right)^2}\right) + \left(1 - \frac{\lambda}{N_a}\right)}{\left(\frac{2\lambda}{N_a}\left(\frac{1}{(\sqrt{2\phi})^2} + \frac{1}{2-e^{-\sqrt{2\phi}}-e^{\sqrt{2\phi}}}\right) + \left(1 - \frac{\lambda}{N_a}\right)\right)^2}$$

Since λ/N_a is a function only of ϕ, then the equation shows that ϕ completely determines the value of the kurtosis. Likewise, the absolute value of the mean price change and the variance depend on this parameter (through the term in the square bracket) as well as on $\bar{p}$.

Next we compute the fraction of price adjustments below a given threshold $\kappa\,\mathbb{E}|\Delta p_i$, which we label $\mathcal{F}(\kappa)$. We can use this formula to quantify the fraction of price changes smaller than a proportion $\kappa \in (0, 1)$ of the mean absolute price change $\mathbb{E}|\Delta p| = \mathcal{H}\bar{p}$ and compare this to the data. This gives

$$\mathcal{F}(\kappa) = \frac{2\lambda}{N_a}\int_0^{\kappa\mathcal{H}\bar{p}} h(p)\;\mathrm{d}p = \frac{\lambda}{N_a}\left(\frac{e^{2\sqrt{2\phi}}\left(1 - e^{-\kappa\mathcal{H}\sqrt{2\phi}}\right) + 1 - e^{\kappa\mathcal{H}\sqrt{2\phi}}}{\left(e^{\sqrt{2\phi}} - 1\right)^2}\right)$$

which is an expression that depends only on two parameters: κ and ϕ. Simple algebra shows that in the limit as $\bar{p} \to \infty$ and adjustments occur *only* when the free opportunity arrives, as in the Calvo model, then the right hand side of the function is $1 - e^{-k}$.

E Note on Solutions of value function V, expected time to adjust $\mathcal{T}$ and invariant density of the squared price gap f.

The alert reader may have noticed that to solve for the invariant density f we have followed a standard procedure, i.e. set a 2nd order ordinary linear difference equation (the Kolmogorov forward equation) and find its solutions in terms of two constant, and using two boundary conditions to find the value of the constants. Instead to solve for V and $\mathcal{T}$ we have followed a different approach, we guess an infinite expansion around $y = 0$ and compute its coefficients. Additionally, it may have looked that we did not provide enough boundary conditions to be able to solve for $\mathcal{T}$ and V. For instance, for $\mathcal{T}$ we gave only one equation as boundary conditions, namely $\mathcal{T}(\bar{y}) = 0$. Here we explain that we could have followed the more standard route, which required an analysis of the behavior close to the $y = 0$ boundary, to set one constant to zero and also would have produced a less informative result, i.e. one in terms of modified Bessel functions. Nevertheless we include it here for completeness.

Note that $V(y)$, $\mathcal{T}(y)$ and $f(y)$ are solutions to a linear ODE on y whose homogeneous component, say $q(\cdot)$, solves :

$$y\,q''(y) + a\,q'(y) + b\,q(y) \;=\; 0 \tag{21}$$

for $y \in [0, \bar{y}]$, for (different) constants a and b, with different particular solution, and different boundary conditions. The general solution of the homogeneous equation (21) is given by:

$$q(y) = |b\,y|^{(1-a)/2} \left[C_1 I_\nu \left(2\sqrt{|b\,y|} \right) + C_2 K_\nu \left(2\sqrt{|b\,y|} \right) \right] \tag{22}$$

provided that $b\,y < 0$, i..e. that $b < 0$, where C_1 and C_2 are arbitrary constants, $\nu = |1-a|$ and where I_ν and K_v are the modified Bessel functions of the first and second kind respectively. The values of $b = -\lambda/(2\sigma^2)$ in the three cases. The value of $a = n/2$ for $\mathcal{T}$ and for V, which are the same Kolmogorov backward equation, and $a = -(n/2 - 2)$ for f, which is the Kolmogorov forward equation.

It is important to notice the behavior of $I_\nu(z)$ and $K_\nu(z)$ for values of $0 < z$ but very close to zero. We have:

$$I_\nu \backsim \frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2} \right)^\nu \tag{23}$$

and

$$K_\nu \backsim \begin{cases} \frac{\Gamma(\nu+1)}{2} \left(\frac{2}{z}\right)^\nu & \text{if } \nu > 0 \\ -\log(z/2) - \gamma & \text{if } \nu = 0 \end{cases} \tag{24}$$

We thus have that each of the solution will behave as:

$$\begin{aligned} I_{|1-a|}\left(y^{1/2}\right)\; y^{(1-a)/2} \quad &\backsim \quad \frac{1}{\Gamma(|1-a|+1)} \left(\frac{y^{1/2}}{2} \right)^{|1-a|} \quad y^{(1-a)/2} \\ &= \quad \frac{1}{\Gamma(|1-a|+1)} \left(\frac{1}{2} \right)^{|1-a|} \quad y^{(1-a)/2+|1-a|/2} \end{aligned}$$

So if $1 - a = -|1 - a|$, i.e. if $1 - a \leq 0$, the value of this product is finite at $y \downarrow 0$. Otherwise it diverges to ∞. Likewise for $\nu = |1 - a| > 0$:

$$\begin{aligned} K_{|1-a|}\left(y^{1/2}\right)\; y^{(1-a)/2} \quad &\backsim \quad \frac{\Gamma(|1-a|+1)}{2} \left(\frac{2}{y^{1/2}} \right)^{|1-a|} \quad y^{(1-a)/2} \\ &= \quad \frac{\Gamma(|1-a|+1)}{2} \left(\frac{2}{1} \right)^{|1-a|} \quad y^{(1-a)/2-|1-a|/2} \end{aligned}$$

So if $1 - a = |1 - a|$, i.e. if $1 - a \geq 0$, the value of this product is finite at $y \downarrow 0$. Otherwise it diverges to ∞. The case of $\nu = 0$ i.e. $a = 1$ is special, but $K_0(z)$ also diverges and $I_0(z)$ converges to a non-zero constant as $z \downarrow 0$.

Note that $V(0)$ and $\mathcal{T}(0)$ are both finite. For these two cases the Kolmogorov backward equation has $a = n/2$ so $1 - a \geq 0$ iff $n \geq 2$. In these cases we have that C_2, the constant associated with K_ν must be zero. We can use the constant C_1 to impose the boundary condition $\mathcal{T}(\bar{y}) = 0$ for $\mathcal{T}$ and to have a one dimensional representation of V in the range of inaction given $\bar{y}$. Then we can use smooth pasting and value matching, i.e. two boundary conditions, to find the constants C_1 and $\bar{y}$.

Note that for f we don't require that $f(0)$ be zero, since the density at zero gap can be infinite if the y mean reverts to zero fast enough. Thus in this case we will, in general, have both constants be non-zero.

F Proof that $\lim_{\bar{y}\to\infty} \xi(\sigma^2, r+\lambda, n, \bar{y}) = 0$

Note that, by examining the definition of κ_i and the sums in the expression for ξ we have that:

$$\lim_{\bar{y}\to\infty} \xi(\sigma^2, r+\lambda, n, \bar{y}) = \lim_{\bar{y}\to\infty} \xi\left(1, 1, n, \frac{(r+\lambda)\,\bar{y}}{\sigma^2}\right)$$

so this limit cannot depend on $r+\lambda$ or σ^2. Thus we denote it as:

$$\bar{\xi}(n) \equiv \lim_{\bar{y}\to\infty} \xi(1, 1, n, \bar{y})$$

So we have:

$$\bar{y} \approx \frac{\psi}{B}\,(r+\lambda)\,\left[1-\bar{\xi}(n)\right] \quad \text{for large } \psi\,.$$

Now we show that $\bar{\xi}(n) = 0$. First we notice that the power series:

$$g(x) = \sum_{i=1}^{\infty} \prod_{s=1}^{i} \frac{1}{(s+2)(n+2s+2)}\; x^i$$

converges for all values of x since its coefficients satisfy the Cauchy-Hadamard inequality. Then we can write:

$$\xi(1, 1, n, \bar{y}) \equiv \frac{\frac{2(n+2)}{\bar{y}} \frac{1}{g(\bar{y})} + \frac{1}{g(\bar{y})} + \frac{1}{\bar{y}^2}}{\frac{2(n+2)}{\bar{y}} \frac{1}{g(\bar{y})} + 2\frac{1}{g(\bar{y})} + \sum_{i=1}^{\infty} \omega(i, \bar{y})\;(2+i)}$$

where the weights $\omega(i, \bar{y})$ are given by:

$$\omega(i, x) = \frac{\frac{x^i}{\prod_{s=1}^{i}(s+2)(n+2s+2)}}{\sum_{j=1}^{\infty} \prod_{s=1}^{j} \frac{1}{(s+2)(n+2s+2)}\; x^j}$$

Note that for higher x the weights of smaller i decrease relative to the ones for higher i. Now since $g(\bar{y}) \to \infty$ as $\bar{y} \to \infty$, then:

$$\bar{\xi}(n) = \frac{1}{\lim_{\bar{y}\to\infty} \sum_{i=1}^{\infty} \omega(i, \bar{y})\;(2+i)}$$

To show that $\bar{\xi}(n) = 0$, suppose, by contradiction that is finite. Say, without loss of generality that equals $j+2$ for some integer j. Note that, by the form of the $\omega' s$ and because $g(\bar{y})$ diverges as $\bar{y}$ gets large enough, then by any j and $\epsilon > 0$ there exist a y^* large enough so that $\sum_{i=1}^{j} \omega(i, \bar{y}) < \epsilon$ for any $\bar{y} > y^*$. Thus, the expected value must be larger than $2+j$.

Finally, we consider the case of $n \to \infty$. In this case we have that, the value function divided by n gives:

$$v = \min_{T} B \int_0^T \sigma^2\, t\, e^{-(\lambda+r)}\, dt + e^{-(r+\lambda)T}(\Psi + v)$$

where $\Psi = \lim_{n\to\infty} \psi/n$. The first order condition for T gives, for a finite T:

$$0 = \big(B\,\sigma^2\,T - (r+\lambda)\Psi\big) - (r+\lambda)e^{-(r+\lambda)T}v \tag{25}$$

Now consider the case where $\Psi \to \infty$. Note that v is finite since $T = \infty$, a feasible strategy as a finite value. Also let $\bar{Y} = \sigma^2 T = \lim_{n\to\infty} \frac{\bar{y}(n)}{n}$. Note that as $\Psi \to \infty$ then $\bar{Y}$ must also diverge towards ∞. Dividing the previous expression by Ψ:

$$\frac{\bar{Y}}{\Psi} = \frac{(r+\lambda)}{B} + (r+\lambda)e^{-(r+\lambda)T}\frac{v}{\Psi}$$

and taking the limits:

$$\lim_{\Psi\to\infty} \frac{\bar{Y}}{\Psi} = \frac{r+\lambda}{B}\ .$$

□

G More model statistics

This appendix reports more model statistics that are functions only of n and λ/N_a. First we provide a formula to quantify the fraction of price changes that are smaller than a threshold $\kappa\mathbb{E}|\Delta p_i|$, which will prove useful to compare with the empirical evidence discussed above:

$$\mathcal{F}_n(\kappa) = 2\int_0^{\kappa\mathbb{E}|\Delta p_i|} w(x)\ \mathrm{d}x$$

where $w(x)$ is density of price changes in equation (7).

Table 8: Model statistic for the fraction of price changes smaller than $\frac{1}{4}\,\mathbb{E}|\Delta p_i|$

% of free adjustments:	number of products n					
λ/N_a	1	2	4	6	10	50
0 %	0.00	0.10	0.13	0.14	0.15	0.16
10%	0.04	0.12	0.15	0.15	0.16	0.16
20%	0.08	0.13	0.16	0.16	0.17	0.17
50%	0.17	0.18	0.19	0.19	0.19	0.19
70%	0.20	0.20	0.20	0.20	0.20	0.21
80%	0.21	0.21	0.21	0.21	0.21	0.21
90%	0.22	0.22	0.22	0.22	0.22	0.22
95%	0.22	0.22	0.22	0.22	0.22	0.22

Table 9: Model statistic for $\mathbb{E}|\Delta p_i|/Std(\Delta p_i)$

% of free adjustments:	number of products n					
λ/N_a	1	2	4	6	10	50
0 %	1.00	0.90	0.85	0.83	0.82	0.80
10%	0.98	0.87	0.84	0.82	0.81	0.80
20%	0.95	0.86	0.83	0.81	0.80	0.79
50%	0.87	0.81	0.79	0.78	0.77	0.76
70%	0.81	0.77	0.76	0.75	0.75	0.75
80%	0.78	0.75	0.74	0.74	0.74	0.73
90%	0.74	0.73	0.73	0.73	0.72	0.72
95%	0.73	0.72	0.72	0.72	0.72	0.71

H A model with random costly adjustment

This version of the model assumes that with probability λ per unit of time the menu cost is smaller than the regular adjustment, namely that it costs $b\psi$ with $b \in (0,1)$. The optimal policy now involves two thresholds: $\underline{p}$ and $\bar{p}$. For $p \in [0, \underline{p}]$ the firm optimally decides not to adjust the price, even if an opportunity for cheap adjustment occurs. For $p \in [\underline{p}, \bar{p})$ the firm adjusts the price only if a cheap adjustment opportunity arises. For $p \geq \bar{p}$ the firm adjusts the price. The value function then solves:

$$\begin{aligned} r\ v_0(p) &= Bp^2 + \frac{\sigma^2}{2}\ v_0''(p), \quad \text{for } p \in [0, \underline{p}]\ , \\ r\ v_1(p) &= Bp^2 + \lambda\left[v_0(0) + b\psi - v_1(p)\right] + \frac{\sigma^2}{2}\ v_1''(p), \quad \text{for } p \in [\underline{p}, \bar{p}] \end{aligned}$$

where we use that the optimal return point upon adjustment is $v_0(0)$. This gives

$$\begin{aligned} v_0(p) &= \frac{Bp^2}{r} + \frac{B\sigma^2}{r^2} + K_0\left(e^{p\sqrt{\frac{2r}{\sigma^2}}} + e^{-p\sqrt{\frac{2r}{\sigma^2}}}\right) \\ v_1(p) &= \frac{Bp^2 + \lambda\left(v_0(0) + b\psi\right)}{\lambda + r} + \frac{B\sigma^2}{(\lambda + r)^2} + K_1\left(e^{p\sqrt{\frac{2(\lambda+r)}{\sigma^2}}} + e^{-p\sqrt{\frac{2(\lambda+r)}{\sigma^2}}}\right) \end{aligned}$$

where we already used that $v_i(p) = v_i(-p)$.

To solve for the two constants K_0 and K_1 and the two parameters $0 < \underline{p}, \bar{p}$ we use that the value function must satisfy the boundary conditions $v_0(\underline{p}) = v_1(\underline{p})$ and $v_0(0) + \psi = v_1(\bar{p})$ and the smooth pasting conditions $v_0'(\underline{p}) = v_1'(\underline{p})$ and $0 = v_1'(\bar{p})$

The expected time to adjustment, $T(p)$ obeys the following ODE

$$0 = 1 + \frac{\sigma^2}{2} T_0''(p) \quad \text{for } \ 0 < |p| \leq \underline{p} \quad \text{and} \quad \lambda T_1(p) = 1 + \frac{\sigma^2}{2} T_1''(p) \quad \text{for } \underline{p} < |p| \leq \bar{p}$$

with $T_i(p) = T_i(-p)$ and boundary conditions $T_0(\underline{p}) = T_1(\underline{p})$ and $T_1(\bar{p}) = 0$. Thus

$$T_0(p) = \frac{1}{\lambda}\left(1 - \frac{e^{\phi \underline{p}} + e^{-\phi \underline{p}}}{e^{\phi \bar{p}} + e^{-\phi \bar{p}}}\right) - \frac{p^2 - \underline{p}^2}{\sigma^2} \quad \text{and} \quad T_1(p) = \frac{1}{\lambda}\left(1 - \frac{e^{\phi p} + e^{-\phi p}}{e^{\phi \bar{p}} + e^{-\phi \bar{p}}}\right) \quad \text{where} \quad \phi \equiv \sqrt{\frac{2\lambda}{\sigma^2}}$$

so that the average number of adjustment per period is

$$N_a = \frac{1}{T_0(0)} = \frac{1}{\frac{1}{\lambda}\left(1 - \frac{e^{\phi \underline{p}} + e^{-\phi \underline{p}}}{e^{\phi \bar{p}} + e^{-\phi \bar{p}}}\right) + \frac{\underline{p}^2}{\sigma^2}} \tag{26}$$

The density function for the price gaps $h(p) \in [0, \bar{p}]$ solves

$$\begin{aligned} 0 &= f_0''(p) \text{ for } 0 \leq |p| \leq \underline{p} \quad \text{and} \quad 0 = -\frac{2\lambda}{\sigma^2} f_1(p) + f_1''(p) \text{ for } \underline{p} < |p| \leq \bar{p} \qquad \text{or} \\ f_0(p) &= C_1 + C_2\, |p| \text{ for } 0 \leq |p| \leq \underline{p} \quad \text{and} \quad f_1(p) = C_3 e^{\phi |p|} + C_4 e^{-\phi |p|} \text{ for } \underline{p} \leq |p| \leq \bar{p} \end{aligned}$$

where the 4 constants solve the 4 equations $f_0(\underline{p}) = f_1(\underline{p})$, $f_0'(\underline{p}) = f_1'(\underline{p})$, $f_1(\bar{p}) = 0$ and $1/2 = \int_0^{\underline{p}} f_0(p)\, \mathrm{d}p + \int_{\underline{p}}^{\bar{p}} f_1(p)\, \mathrm{d}p$ which use that the density is differentiable (see the Appendix).

Then using that only the fraction $2\int_{\underline{p}}^{\bar{p}} f_1(x)\mathrm{d}x$ of cheap adjustment opportunities will trigger an actual price change, the distribution of (non-zero) price changes $x \in [-\bar{p}, -\underline{p}] \cup [\underline{p}, \bar{p}]$ is symmetric and is given by (we only report the formulas for $x > 0$)

$$\begin{cases} \text{density for a price change of size } x \in [\underline{p}, \bar{p}): & \frac{\lambda}{N_a} f_1(x) \\ \text{mass point at } \bar{p} & \frac{1}{2} - \frac{\lambda}{N_a}\int_{\underline{p}}^{\bar{p}} f_1(x)\mathrm{d}x \end{cases}$$

So the mean absolute price change is

$$\mathbb{E}|\Delta p| = \frac{\lambda}{N_a}\int_{\underline{p}}^{\bar{p}} x\ 2 f_1(x)\ \mathrm{d}x + \left(1 - \frac{\lambda \int_{\underline{p}}^{\bar{p}} 2 f_1(x)\mathrm{d}x}{N_a}\right)\bar{p}$$

and the $j-th$ moment of price changes for j even is

$$\mathbb{E}\left(\Delta p^j\right) = \frac{\lambda}{N_a}\int_{\underline{p}}^{\bar{p}} x^j\ 2 f_1(x)\ \mathrm{d}x + \left(1 - \frac{\lambda \int_{\underline{p}}^{\bar{p}} 2 f_1(x)\mathrm{d}x}{N_a}\right)\bar{p}^j$$

H.1 Density function

To determine the 4 unknowns of the density function, using $f_1(\bar{p}) = 0$ and $f_0'(\underline{p}) = f_1'(\underline{p})$, gives

$$C_3 = -C_4 e^{-2\phi \bar{p}} \quad \text{and} \quad C_2 = -C_4 \phi\left(e^{-2\phi \bar{p} + \phi \underline{p}} + e^{-\phi \underline{p}}\right)$$

Next, using $f_0(\underline{p}) = f_1(\underline{p})$ gives

$$C_1 = -C_2 \underline{p} - C_4\left(e^{-2\phi \bar{p} + \phi \underline{p}} - e^{-\phi \underline{p}}\right) = C_4\left[e^{-2\phi \bar{p} + \phi \underline{p}}\left(\phi \underline{p} - 1\right) + e^{-\phi \underline{p}}\left(\phi \underline{p} + 1\right)\right]$$

Finally we solve for C_4 by imposing $1/2 = \int_0^{\underline{p}} f_0(p)\, \mathrm{d}p + \int_{\underline{p}}^{\bar{p}} f_1(p)\, \mathrm{d}p$ i.e.

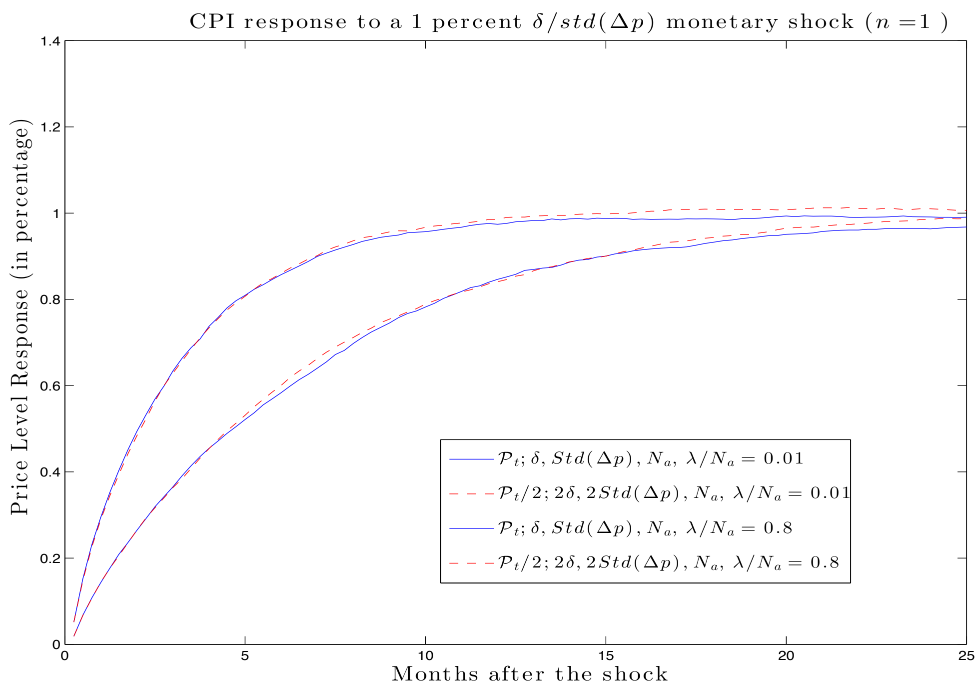
$$\frac{1}{2} = C_1\underline{p} + \frac{1}{2}C_2\underline{p}^2 + \frac{1}{\phi}\left[C_3\left(e^{\phi\bar{p}} - e^{\phi\underline{p}}\right) - C_4\left(e^{-\phi\bar{p}} - e^{-\phi\underline{p}}\right)\right]$$

or, substituting the expressions,

$$\begin{aligned}\frac{1}{2C_4} \quad = \quad & \left[e^{-2\phi\bar{p}+\phi\underline{p}}\left(\phi\underline{p} - 1\right) + e^{-\phi\underline{p}}\left(\phi\underline{p} + 1\right)\right]\underline{p} - \frac{1}{2}\phi\left(e^{-2\phi\bar{p}+\phi\underline{p}} + e^{-\phi\underline{p}}\right)\underline{p}^2 \\ & -\frac{1}{\phi}\left[e^{-2\phi\bar{p}}\left(e^{\phi\bar{p}} - e^{\phi\underline{p}}\right) + e^{-\phi\bar{p}} - e^{-\phi\underline{p}}\right]\end{aligned}$$

I Homogeneity of IRF: Example

Figure 8: Impulse response of the CPI to a monetary shock of size $\delta = 1\%$: homogeneity



Impulse response for economies with $N_a = 1.3$ and $std(\Delta p_i) = 0.10$. (see the text).

J More sectoral empirical results

Table 10: Statistics by type of goods and outlet category (un-standardized price changes)

Good type	**Outlet type**	**Freq**	**Avg** $\lvert\Delta p\rvert$	**Std** $\lvert\Delta p\rvert$	**Kurt** (Δp_i)	**Frac25**
Food	Hypermakets	27.56	8.89	11.88	10.25	30.79
Food	Supermarkets	18.84	9.84	13.48	10.57	30.36
Food	Traditional	7.52	7.84	8.48	11.68	15.63
Food	Services	7.14	9.45	9.41	7.52	12.06
Durable goods	Hypermakets	15.82	13.35	12.97	6.36	21.02
Durable goods	Supermarkets	19.11	14.96	12.97	5.52	16.38
Durable goods	Traditional	7.93	14.77	15.82	7.08	22.02
Durable goods	Services	8.02	23.45	20.95	3.36	20.14
Clothing	Hypermakets	8.09	45.13	27.42	1.89	17.41
Clothing	Supermarkets	9.55	43.23	25.42	2.20	10.79
Clothing	Traditional	12.68	41.85	24.23	2.24	7.31
Clothing	Services	10.86	41.20	21.76	1.87	12.53
Other manufactured goods	Hypermakets	15.69	9.40	12.92	11.25	32.71
Other manufactured goods	Supermarkets	12.14	11.87	14.79	7.94	33.99
Other manufactured goods	Traditional	8.22	11.51	16.40	8.16	34.59
Other manufactured goods	Services	11.25	6.59	10.55	12.91	32.85
Energy	Hypermakets	80.89	3.56	2.84	9.23	8.28
Energy	Supermarkets	76.43	3.56	2.81	8.50	8.60
Energy	Traditional	75.55	4.22	3.51	5.39	14.35
Energy	Services	71.93	3.35	2.56	4.69	8.99
Services	Hypermakets	5.13	13.84	14.32	7.71	22.64
Services	Supermarkets	9.99	9.70	10.99	10.33	26.22
Services	Traditional	6.34	7.74	10.13	19.97	19.54
Services	Services	6.41	7.65	10.20	18.30	20.86

Source is INSEE, monthly price records from French CPI, data from 2003:4 to 2011:4. Coverage is aroud 65% of CPI weight since rents, and prices of fresh food and centrally collected items (e.g. electricity, train and airplane tickets) are not included in the dataset. Freq. denotes monthly frequency of price change in percent. Size of price change Δp are the first-difference in the logarithm of price per unit, expressed in percent. Avg is average, Std standard devation, Frac25 the share of absolute price change that are inferior to 0.25 Avg[$\lvert\Delta p\rvert$], Kurt denotes Kurtosis. Observations with imputed prices or quality change are discarded. Moments are computed aggregating all prices changes using CPI weigths at the product level.