EIEF WORKING PAPER SERIES

EINAUDI INSTITUTE FOR ECONOMICS AND FINANCE

EIEF Working Paper 02/13
February 2013

# Price setting with menu cost for multi-product firms

by
Fernando Alvarez
(University of Chicago)
Francesco Lippi
(University of Sassari and EIEF )

# Price setting with menu cost for multi-product firms*

**Fernando Alvarez**
University of Chicago

**Francesco Lippi**
University of Sassari, EIEF

January 21, 2013

**Abstract**

We model the decisions of a multi-product firm that faces a fixed "menu" cost: once it is paid, the firm can adjust the price of *all* its products. We characterize analytically the steady state firm's decisions in terms of the structural parameters: the variability of the flexible prices, the curvature of the profit function, the size of the menu cost, and the number of products sold. We provide expressions for the steady state frequency of adjustment, the hazard rate of price adjustments, and the size distribution of price changes, all in terms of the structural parameters. We study analytically the impulse response of aggregate prices and output to a monetary shock. The size of the output response and its duration increase with the number of products, they more than double as the number of products goes from 1 to ten, quickly converging to the ones of Taylor's staggered price model.

*JEL Classification Numbers: E3 , E5*

*Key Words: menu cost, economies of scope in price changes, optimal control in multiple dimensions, fixed costs, monetary shocks, impulse responses.*

*First draft December 2010. We thank the Editor and 3 anonymous referees. We benefited from discussions with Andy Abel, Ricardo Caballero, Carlos Carvalho, John Leahy, Bob Lucas, Virgiliu Midrigan, Luigi Paciello, Ricardo Reis, Rafael Schoenle, Kevin Sheedy, Rob Shimer, Paolo Surico, Nancy Stokey, Harald Uhlig, Ivan Werning, as well as seminar participants at EIEF, the University of Chicago, NYU, Tinbergen Institute, the ASSA 2012, the Federal Reserve Banks of Chicago, Minneapolis, New York and Philadelphia, the Bank of Italy, the European Central Bank, the London Business School, the Bank of Norway, and the 2012 NBER Monetary Economics Meeting in NY for their comments. Alvarez thanks the ECB for the Wim Duisenberg fellowship. Lippi thanks the Italian Ministry of Education for sponsoring this project as part of the PRIN 2010-11. We are grateful to the Fondation Banque de France for sponsoring this project. Katka Borovickova provided excellent assistance.

# 1 Introduction and Overview

We develop an analytically tractable model of the optimal price setting decisions of a firm that faces a fixed cost of price adjustment common to $n \geq 1$ goods. This problem was proposed by Lach and Tsiddon (1996, 2007) as a way to generate the small price changes as well as the synchronized price adjustments within the firm that appear in the micro data, e.g. Cavallo (2010).[1] We solve the firm's decision problem, derive the steady state predictions for a cross-section of firms and study the response of the aggregate economy to a monetary shock. The challenges involved with modeling the propagation of monetary shocks in canonical menu cost problems have led many authors to resort to numerical methods. Our contribution is to present an approximate analytical solution to the general equilibrium of an economy where firms face a multidimensional and non-convex control problem.

There are two sets of results. The first one concerns the model's cross section predictions in a steady state. The model substantially improves the ability of state-of-the-art menu cost models to account for observed price setting behavior. As documented by several empirical studies and summarized by e.g. Klenow and Malin (2010) the data display a large mass of small price changes: the size distribution of price changes appears bell-shaped. Existing menu cost models cannot account for this fact: we show that when $n = 1$ or $n = 2$, as in the models of Golosov and Lucas (2007) and Midrigan (2011) respectively, the size distribution of price changes is bimodal and "U" shaped, featuring a minimal amount of small price changes. Our model produces a bell-shaped distribution provided $n \geq 6$, thus accounting for a robust feature of the data while retaining tractability (any $n$ can be studied). Simple expressions are derived to map the model fundamental parameters (the size of the menu cost, the variance of the shocks, the demand elasticity, the number of products sold) into observable statistics such as the frequency of price adjustment $N_a$, and the standard deviation of price changes $Std(\Delta p_i)$.

The second set of results concerns the analytical characterization of the response of the aggregate price level and of output to a monetary shock. This characterization extends the pioneering contributions of Caballero and Engel (1993, 2007) by going beyond their analysis of the impact effect, allowing for any number of goods ($n \geq 1$), and justifying their simplifying assumption of using the steady-state decision rules to analyze the transition dynamics. The last result gives a proof, and an intuitive explanation, for when the general equilibrium feedback on decision rules can be "neglected" in these models.[2] The analytical

[1] For more evidence on the synchronization of price changes see Lach and Tsiddon (1992), Baudry et al. (2007), Dhyne and Konieczny (2007), Dutta et al. (1999), and Midrigan (2009, 2011).

[2] This simplifying assumption was used by Caballero and Engel (2007)). Golosov and Lucas (2007) noticed in their quantitative analysis that decision rules were very close to the ones of the steady state. We replicate their findings and provide an explanation.

results highlight two key determinants of the *size* and the *duration* of the impulse response of output and prices (IRF for short) of a once and for all monetary shock. The size and duration of the IRF depend, for a given number of products $n$, on the steady state frequency of price changes $N_a$ as well as on the steady state standard deviation deviation of price changes, $Std(\Delta p_i)$. For given values of these steady state statistics the shape of the IRF depends only on $n$. Compared to the previous literature, which focussed almost exclusively on the frequency of price changes as a proxy of aggregate stickiness, our analysis suggests that the dispersion of price changes is an equally important determinant of the real effect of a monetary shock. We show that the flexibility of the aggregate price level is highest in the classic menu cost model with $n = 1$, due the strong "selection effect" of price changes discussed by Golosov and Lucas (2007). We show that for small monetary shocks the selection effect weakens as $n$ increases, and vanishes completely as $n \to \infty$. In this case the price level and the output response to shocks is linear, as in a Taylor's (1980) model, and the real effects of monetary policy are maximal, about two times those of a model where $n = 1$. Our analysis thus provides an upper bound to the real effect of monetary shocks which is still smaller (about half) than predicted by a Calvo pricing mechanism. We also analyze the effect of different size of monetary shocks on output, a hallmark of menu cost models. The effect is small either if the shock is small or if it is large, since in the latter case all firms change prices. We characterize the value of the monetary shock for which the cumulated effect on output (the area under the IRF) is maximized. Interestingly, for a given $n$, the monetary shock that maximizes the cumulative output effect is about one half of $Std(\Delta p_i)$. Moreover, the maximum value of the cumulated output effect is proportional to $Std(\Delta p_i)/N_a$. For example for economies with large steady state price stickiness the cumulated output effect ranges from 0.6% of annual output for $n = 1$ to 1.4% for a large $n$.

Our analytical solution rests on carefully chosen approximations, whose antecedents in the literature and whose precision is discussed in Section 5 and explored quantitatively in Appendix C. The simplifications needed preclude the analysis of the case of asymmetric demands for goods, although section Section 6 considers extensions such as correlation on the shocks across products, the presence of inflation, and different elasticities of substitution of products within and between firms. Directions for future research are discussed in Section 7.

### Overview of the analysis and main findings

In Section 2 we set-up the problem of a multi-product firm that can revise prices only after paying a fixed cost. The key assumption is that once the fixed menu cost $\psi$ is paid the firm can adjust the price of *all* its products. We assume that the static profit maximizing prices for each of the $n$ products, i.e. the prices that would be charged absent menu cost, follow $n$ independent random walks without drift and with volatility $\sigma$. We refer to the difference between the frictionless and the actual prices as to the (vector of) price gaps. The period return function is shown to be proportional to the sum of the squared price gaps. The proportionality constant $B$ measures the second order losses associated with charging a price different from the optimum, i.e. it is a measure of the curvature of the profit function. The firm minimizes the expected discounted cost, which includes the stream of lost profits from charging prices different from the frictionless as well as the fixed cost at the time of adjustments.

The solution of the firm's problem in Section 3 involves finding the set over which prices are adjusted, and its complement the "inaction" set. To our knowledge this is the first fixed cost adjustment problem in $n$-dimensions whose solution is analytically characterized. Somewhat surprisingly the solution to this complex problem turns out to have a simple form: the optimal decision is to control the price gaps as to remain in the interior of the $n$-dimensional ball centered at the origin. The economics of this is clear: the firm will adjust either if many of its price gaps have a medium size, or if a few gaps are very large. The size of this ball, whose squared radius is denoted by $\bar{y}$, is chosen optimally. We solve for the value function and completely characterize the size of the inaction set $\bar{y}$ as a function of the parameters of the problem. We show that the approximate solution $\bar{y} \approx [2(n+2)\sigma^2 B/\psi]^{1/2}$ gives an accurate approximation of the exact solution for a small cost $\psi$.[3]

In Section 4 we explore several steady-state implications of the model. First we show that the expected number of price adjustments per unit of time, denoted by $N_a$, is given by $n\sigma^2/\bar{y}$, which together with our result for $\bar{y}$ gives a complete characterization of the frequency of price adjustments. Second we solve in closed form for the hazard rate of the price changes as a function of the time elapsed since the last change. The scale is determined by the expected number of adjustment $N_a$. Fixing the scale the shape of this function depends *exclusively* on the number of products $n$. We show that the hazard rate gets steeper as $n$ increases. Third, while price changes occur simultaneously for the $n$ products, we characterize the marginal distribution of price changes, i.e. the statistic that is usually computed in actual data sets. A closed form expression for the density of the marginal distribution of price changes as a function of $\bar{y}$ and $n$ is derived and used to compute several statistics, such as the standard deviation of price changes $Std(\Delta p_i)$, and other moments which are only functions of $n$, such as the coefficient of variation and the excess kurtosis of the absolute value of price changes. As the number of products increases the size of the adjustments decreases monotonically, i.e. with more products the typical price adjustment of each product is smaller. These

[3] The special case of $n = 1$ gives the same *quartic root* as in Barro, Karlin-Taylor, and Dixit, since $\bar{y}$ is the *square* of the price threshold.

cross-section predictions could be used to identify the parameters of the model and test its implications. We show that, once the scale of price changes is controlled for, the shape of the size-distribution is *exclusively* a function of the number of products $n$. For $n = 2$ the distribution is bimodal, with modes at the absolute value of $\sqrt{\bar{y}}$, for $n = 3$ it is uniform, for $n = 4$ it peaks at zero and it is concave, and for $n \geq 6$ it is bell-shaped. As $n \to \infty$, the density of price changes converges to a Normal.

In Section 5 we use the firm's optimal decisions to characterize the response of the aggregate price level, and of output, to a monetary shock. In doing this we keep the decision rules of the firms constant, an approximation used in some of the calculations by Golosov and Lucas (2007) and Caballero and Engel (1991, 1993, 2007) among many others. Indeed, we justify this practice in our model by showing that the general equilibrium feedback effects have negligible consequences on the size of the inaction region –a result closely related to the one in Gertler and Leahy (2008). In particular, we characterize analytically the effect on aggregate prices of a permanent unexpected increase in money supply in an economy that starts at the cross sectional stationary distribution of price gaps under zero inflation.

The analytical IRF of prices to a monetary shock is made of two pieces: an impact effect (a jump in the price level), and the remaining part. The IRF depends only on three parameters: the number of products $n$, the frequency of price changes, $N_a$, and the standard deviation of price changes, $Std(\Delta p_i)$. More precisely, the IRF is homogenous of degree one in the size of the shock, $\delta$, and in $Std(\Delta p_i)$. Instead the duration of the impulse response is inversely proportional to the steady state frequency of price changes, $N_a$, i.e. time can be measured relative to the steady state average duration of prices. When monetary shocks are larger than twice $Std(\Delta p_i)$ the economy features complete price flexibility. Instead, for small monetary shocks the impact effect on prices is second order compared to the shock size −and hence the impact effect on output is of the order of the monetary shock. These results, together with the homogeneity, characterize the precise sense in which the size of the shocks matters. Fixing the two steady state parameters –$N_a$ and $Std(\Delta p_i)$– the whole shape of the impulse response depends only on the number of products $n$ and the normalized size of the shock $\delta/Std(\Delta p_i)$. As we move from $n = 1$ to a large number of products (say $n \geq 10$) the impact effect on prices, as well as the half life of a monetary shock, more than double. Indeed as $n \to \infty$ the IRF converges to the one corresponding to the staggered price setting of Taylor's (1980) model, or the inattentiveness model of Caballero (1989), Bonomo and Carvalho (2004) and Reis (2006). In this case there is absolutely no selection and the impulse response is linear in time, and has -for small shocks- a half-life of $1/(2\ N_a)$, i.e. half the average duration of steady state price changes. In the language of Golosov and Lucas (2007), economies with higher values of $n$ have a smaller amount of "selection".

Our analysis extends Midrigan's (2011) contribution to any number of goods $n$, and derives the implication for the shape of the distribution of price changes and hazard rates, not derived his paper. We show that the $n = 2$ case produces a size distribution of price changes that is "strongly" bimodal, very similar to the one in Golosov and Lucas. We show that a larger number of goods, in the order of $n = 10$, is necessary to replicate qualitatively the large mass of small price changes and the bell-shaped distribution of price changes that are seen in the data. Concerning the hazard rate Midrigan comments that "Economies of scope flatten the adjustment hazard and thus weaken the strength of the selection effect even further" (pp. 1167). We show that without fat tailed shocks the hazard rate steepens with $n$, and indeed the economy converges to Taylors' staggered adjustment model, not to Calvo's (flat hazard) random adjustment model. Concerning the optimal decision rule for adjustment after an aggregate shock Midrigan interprets the price adjustment decision his model with $n = 2$ using the techniques developed by Caballero and Engel for the case of $n = 1$ (Section 4.B, pages 1165-1168). We show in Section 5 that for the multi-product case ($n > 1$) the threshold condition for price adjustments involves *a vector of price gaps*, not just one. Finally, we clarify how the "multiproduct hypothesis" affects the consequences of monetary shocks in comparison to the seminal paper of Golosov and Lucas (2007). Midrigan (2011) tackled this question numerically in a model where $n = 2$ which, moreover, assumed the presence of infrequent - large shocks. A question then arises as to which of those features is responsible for the large real effects produced by his model, about 4 times larger, which resemble those of a "Calvo" model. Our analytical results show that without the infrequent-large shocks the multiproduct hypothesis for the $n = 2$ case produces real effects that are only 20% larger than in Golosov and Lucas (2007), and even smaller for the case of correlated shocks. We infer that the large effects of monetary shocks obtained by Midrigan are due to the presence of the infrequent-large shocks, which reintroduce an element of stochastic time dependence (a la Calvo) in the price setting decision.

# 2 The firm's problem: setup and interpretation

Let $n$ be the number of products sold by the firm. The mathematical model we use has an $n$-dimensional state $p$ that we refer to as the vector of price gaps, whose interpretation is discussed below. Each price gap $p_i$, while it is not controlled, evolves according to a random walk without drift, so that $\mathrm{d}p_i = \sigma \, \mathrm{d}\mathcal{W}_i$ where $\mathrm{d}\mathcal{W}_i$ is a standard Brownian Motion. The $n$ Brownian Motions (BM henceforth) are independent, so $\mathbb{E}\left[\mathcal{W}_i(t)\mathcal{W}_j(t')\right] = 0$ for all $t, t' \geq 0$ and $i, j = 1, ..., n$. The value function $V(p)$ is the minimum value of the function $\mathbf{V}$ defined over the processes $\{\boldsymbol{\tau}, \boldsymbol{\Delta p}\} \equiv \{\tau_j, \Delta p_i(\tau_j)\}_{j=1}^{\infty}$:

$$V(p) = \min_{\boldsymbol{\tau}, \boldsymbol{\Delta p}} \mathbf{V}(\boldsymbol{\tau}, \boldsymbol{\Delta p}; p) \equiv \mathbb{E}\left[ \sum_{j=1}^{\infty} e^{-r\tau_j}\psi + \int_0^{\infty} e^{-rt} B \left( \sum_{i=1}^{n} p_i^2(t) \right) dt \middle| \; p(0) = p \right] \tag{1}$$

where each element of the vector of price gaps $p$ follows

$$p_i(t) = \sigma \mathcal{W}_i(t) + \sum_{j:\tau_j < t} \Delta p_i(\tau_j) \text{ for all } t \geq 0 \text{ and } i = 1, 2, ..., n, \tag{2}$$

$\Delta p_i(\tau_j) \equiv \lim_{t \downarrow \tau_j} p_i(t) - \lim_{t \uparrow \tau_j} p_i(t)$ and $p(0) = p$.

The $\tau_j$ are the (stopping) times at which control is exercised. At these times, after paying the cost $\psi$, the state can be changed to any value in $\mathbb{R}^n$. We denote the vector of changes in the price gaps as $\Delta p(\tau_j) \in \mathbb{R}^n$. This is a standard adjustment cost problem subject to a fixed cost, with the exception that after paying the adjustment cost $\psi$ the decision maker can adjust the state in the $n$ dimensions.

Next we discuss three interpretations of the problem that can be summarized by saying that the firm "tracks" the prices that maximize instantaneous profits from the $n$ products. In each interpretation a monopolist sells $n$ goods with additively separable demands; in the first one subject to costs shocks, and in the second subject to demand shocks. For the first interpretation consider a system of $n$ independent demands, with constant elasticity $\eta$ for each product, random multiplicative shifts in each of the demand, and a time varying marginal (and average) cost $W\, Z_i(t)$. This is a stylized version of the problem introduced by Midrigan (2011) where the elasticity of substitution between the products sold *within* the firm is the same as the one of the bundle of goods sold *across* firms. The instantaneous profit maximizing price is proportional to the marginal cost, or in logs $p_i^*(t) = \log W + \log Z_i(t) + \log(\eta/(\eta-1))$. In this case we assume that the log of the marginal cost evolves as a random walk with drift so that $p_i^*(t)$ inherits this property. The period cost is a second order expansion of the profit function with respect to the vector of the log of prices, around the prices that maximize current profits (see Appendix B for a detailed presentation of this interpretation). The units of the objective function are loss profits relative to the the value of the current maximum profits for the $n$ goods. The first order price-gap terms in the expansion are zero because we are expanding around $p^*(t)$. There are no second order cross terms due to the separability of the demands. Thus we can write the problem in terms of the gap between the actual price and the profit maximizing price: $p(t) = \hat{p}(t) - p^*(t)$. Under this approximation the constant $B$ is given by $B = (1/2)\eta(\eta-1)/n$, where $n$ appears in the denominator since the cost of the deviations for the $n$ price gaps is divided by the *total* profits generated by the $n$ goods.[4] The fixed cost relative to the profit of the $n$ products is then $\psi/n$. Clearly all that matters to characterize the decision rules is the ratio of $B$ to $\psi$, thus in equation (1) we omit the terms that are common (such as $n$ and the expression for total profits) which only scales the units of the value function. For the second interpretation of the model consider a monopolist facing identical demands for each of the $n$ products that she sells. The demands are linear in its own price, and have zero cross partials with respect to the other prices. The marginal costs of producing each of the products are also identical, and assumed to be linear. The intercepts of each of the $n$ demands follow independent standard BMs. In this interpretation the firm's profits are the sum of the $n$ profit functions derived in the seminal work by Barro (1972), so that our $\psi$ is his $\gamma$ and our $B$ is his $\theta$, as defined in his equation (12). A third interpretation is in terms of an *optimal inattention* or *inattentiveness* problem, similar to the one studied by Reis (2006), and Alvarez, Lippi, and Paciello (2011). The firm has the same demand system for the $n$ products, and hence the same total period losses $B||p(t)||^2$, which are assumed to be continuously and freely observed. Furthermore, if the firm pays an observation cost $\psi$, it observes the determinants of the profits of each of the products separately, and is able to set prices based on this information. In this case $\psi$ represents the cost of gathering and processing the information, in addition to (or instead of) the menu cost of changing prices.

# 3 Characterization of the firm's decisions

We first note the following properties of the firm's problem:

1. Given the symmetry of the return function, of the law of motion and of target prices, it is immediate to see that after an adjustment the state is reset at the origin, i.e. $p(\tau_j^+) = 0$, or $\Delta p(\tau_j) = -p(\tau_j^-)$.

2. The state space $\mathbb{R}^n$ can be divided in two open sets: an inaction region $\mathcal{I} \subset \mathbb{R}^n$ and a control region $\mathcal{C} \subset \mathbb{R}^n$. We use $\partial\mathcal{I}$ for the boundary of the inaction region. We have that $\mathcal{C} \cap \mathcal{I} = \emptyset$, that inaction is strictly preferred in $\mathcal{I}$, that control is strictly preferred in $\mathcal{C}$, and that in $\partial\mathcal{I}$ the agent is indifferent between control and inaction.

3. The instantaneous return of the problem in equation (1) is a function of the scalar $y$,

[4]The terms with $\eta$ appear since the curvature of profits depends on the elasticity of demand, the term $1/2$ comes from the second order expansion.

the squared norm of the vector of price gaps:

$$y = \sum_{i=1}^{n} p_i^2 \tag{3}$$

4. The process for $y$ is a *one dimensional* diffusion given by:

$$\mathrm{d}y = n\sigma^2 \mathrm{dt} + 2\sigma\sqrt{y}\ \mathrm{d}\mathcal{W} \text{ for } y \in [0, \bar{y}]. \tag{4}$$

   To see why use Ito's Lemma on equation (3) to get $\mathrm{d}y = n\sigma^2\mathrm{dt} + 2\sigma\sum_{i=1}^{n} p_i(t)\mathrm{d}\mathcal{W}_i$ implying $\mathbb{E}(\mathrm{d}y)^2 = 4\sigma^2\left(\sum_{i=1}^{n} p_i^2(t)\right)\mathrm{d}t$, which gives the diffusion shown above.

5. Points 3 and 4 imply that the $n$ dimensional state of the original problem and decision rules can be summarized by a single scalar, namely $y$. The optimal policy for this problem is given by a threshold rule such that if $y < \bar{y}$, there is inaction. The first time that $y$ reaches $\bar{y}$, all prices are adjusted to the origin, so that $y = 0$. The one dimensional problem has the following value function

$$v(y) = \min_{\bar{y}} \mathbb{E}\left[\sum_{j=1}^{\infty} e^{-r\tau_j}\psi + \int_0^{\infty} e^{-rt} B\ y(t)\ dt \;\middle|\; y(0) = y\right] \tag{5}$$

   subject to equation (4) for $y \in [0, \bar{y}]$ and the $\tau_j$'s are the times at which $y(t)$ hits $\bar{y}$.

The function $v$ solves:

$$r\ v(y) = By + n\sigma^2\ v'(y) + 2\sigma^2 y\ v''(y), \quad \text{for } y \in (0, \bar{y})\ . \tag{6}$$

Since policy calls for adjustment at values higher than $\bar{y}$ we have: $v(y) = v(0) + \psi$ for all $y \geq \bar{y}$. If $v$ is differentiable at $\bar{y}$ we can write the two boundary conditions:

$$v(\bar{y}) = v(0) + \psi \quad \text{and} \quad v'(\bar{y}) = 0\ . \tag{7}$$

These conditions are typically referred to as value matching and smooth pasting. For $y = 0$ to be the optimal return point, it must be a global minimum, and thus we require that:

$$v'(0) \geq 0\ . \tag{8}$$

Note the weak inequality, since $y$ is non-negative.

The next proposition gives an analytical solution for $v$ in the range of inaction.

Proposition 1. Let $\sigma > 0$. The ODE given by equation (6) is solved by the following analytical function:

$$v(y) = \sum_{i=0}^{\infty} \beta_i \, y^i \ , \qquad \text{for } y \in [0, \bar{y}] \ . \tag{9}$$

where for any $\beta_0$ the coefficients $\{\beta_i\}_{i=1}^{\infty}$ solve:

$$\beta_0 = \frac{n\sigma^2}{r}\beta_1 \quad , \quad \beta_2 = \frac{r\beta_1 - B}{2\sigma^2(n+2)} \ , \quad \beta_{i+1} = \frac{r}{(i+1)\ \sigma^2\ (n+2i)}\ \beta_i \ , \quad \text{for } i \geq 2 \ . \tag{10}$$

The proof follows by replacing the function in equation (9) into the ODE (6) and matching the coefficients for the powers of $y^i$. By the Cauchy-Hadamard theorem, the power series converges absolutely for all $y > 0$ since $\lim_{i\to\infty} \beta_{i+1}/\beta_i = 0$. The next proposition shows that there exists a unique solution of the ODE (6) satisfying the relevant boundary conditions (see Appendix A for the proof).

Proposition 2. Assume $r > 0, \sigma > 0, n \geq 1$. There exists $\bar{y}$ and a unique function $v(\cdot)$ solving the ODE (6) satisfying the boundary conditions in equations (7). Moreover: i) $v(y)$ is minimized at $y = 0$, ii) $v(y)$ is strictly increasing in $(0, \bar{y})$, and iii) $\bar{y}$ is a local maximum, i.e. $\lim_{y \uparrow \bar{y}} v''(y) < 0$.

We conclude by noting that a slightly modified version of a verification theorem in Øksendal (2000) can be used to prove that value function $v$ and threshold policy $\bar{y}$ that we found in Proposition 2 for the one-dimensional representation indeed characterize the inaction $\mathcal{I} = \{p : ||p||^2 < \bar{y}\}$ and control sets $\mathcal{C}$, as well as the value function $V(p)$ for the original $n$-dimensional problem (see Appendix C in Alvarez and Lippi (2012) for more details and references to related results in the applied math literature).

We note that the solution for the $n = 1$ case and the expression for the approximation for $\bar{y}$ are the same ones derived in Karlin and Taylor (1981) Section 3.F in chapter 15, and in Dixit (1991) expression (11). We finish this section by characterizing the optimal policy $\bar{y}$ in terms of the structural parameters of the model $(\frac{\psi}{B}, \sigma^2, n, r)$.

Proposition 3. The optimal threshold is given by a function $\bar{y} = \frac{\sigma^2}{r} Q(\frac{\psi}{B}\frac{r^2}{\sigma^2}, n)$ so that

(i) $\bar{y}$ is strictly increasing in $\frac{\psi}{B}$ with $\bar{y} = 0$ if $\frac{\psi}{B} = 0$ and $\bar{y} \to \infty$ as $\frac{\psi}{B} \to \infty$,

(ii) $\bar{y}$ is strictly increasing in $n$ and $\bar{y} \to \infty$ as $n \to \infty$,

(iii) $\bar{y}$ is bounded below by $\sqrt{2(n+2)\sigma^2 \frac{\psi}{B}}$ and as $\frac{\psi}{B}\frac{r^2}{\sigma^2} \to 0$ then $\frac{\bar{y}}{\sqrt{2(n+2)\sigma^2\frac{\psi}{B}}} \to 1$,

(iv) the elasticity of $\bar{y}$ with respect to $r$ and $\sigma^2$ satisfy:

$$\frac{r}{\bar{y}}\frac{\partial \bar{y}}{\partial r} = 2\frac{(\psi/B)}{\bar{y}}\frac{\partial \bar{y}}{\partial(\psi/B)} - 1 \quad \text{and} \quad \frac{\sigma^2}{\bar{y}}\frac{\partial \bar{y}}{\partial \sigma^2} = 1 - \frac{(\psi/B)}{\bar{y}}\frac{\partial \bar{y}}{\partial(\psi/B)} \ .$$

See Appendix A for the proof. For the case of $n = 1$ the formula for the threshold is the same one derived by Barro (1972), Karlin and Taylor (1981), and in Dixit (1991), though our characterization is a bit more general and, more importantly, holds for any number of products $n \geq 1$.[5] That $\bar{y}$ is only a function of the ratio $\psi/B$ is apparent from the definition of the sequence problem. That, as stated in part (i), $\bar{y}$ is strictly increasing in the ratio of the fixed cost to the benefit of adjustment $\psi/B$ is quite intuitive. Item (ii) says that the threshold is increasing in the number of products $n$. This is because as $n$ increases, equation (4) shows that the drift of $y = ||p||^2$ increases, thus if $\bar{y}$ would stay constant there will be more adjustments per unit of time, and hence higher menu cost will be paid. Additionally, if $\bar{y}$ remains unchanged, the average cost per unit of time also increases. One can show that the second effect is smaller, and hence an increase in $n$ makes it optimal to increase $\bar{y}$. Part (iii) gives an expression for a lower bound for $\bar{y}$, which becomes arbitrary accurate for either a small value of the cost $\psi/B$, so that the range of inaction is small, or a small value of the interest rate $r$, so that the problem is equivalent to minimizing the steady state average net cost. We note that in the approximation:

$$\bar{y} = \sqrt{\frac{\psi \ \sigma^2 \ 2 \ (n+2)}{B}} \ , \tag{11}$$

the effect of $\psi\sigma^2/B$ is exactly the same as in the case of one product. Note that the approximation in part (iii) implies that the elasticity of $\bar{y}$ with respect to $\psi/B$ is $1/2$ for small values of the $\psi/B$ ratio. Then, using part (iv), we obtain that $\bar{y}$ has elasticity $1/2$ with respect to $\sigma^2$ and also that it is independent of $r$. Moreover, for small normalized adjustment cost, i.e. as $\psi/(B\sigma^2) \downarrow 0$, (iii) and (iv) imply that $\partial\bar{y}/\partial r \to 0$, so that interest rates have only second order effects on the range of inaction. Finally we found that the quadratic approximation to $v(\cdot)$, which amounts to a quartic approximation to $V(\cdot)$, gives very accurate values for $\bar{y}$ across a very large range of parameters, as documented in Appendix C.1. What happens is that for a realistic application the values of $r$ and $\psi$ are small relative to $B\sigma^2$, hence the approximation of part (iii) applies.

[5] See expression (19) in Barro (1972), Chapter 15, Section 3.F of Karlin and Taylor (1981) for the case of undiscounted returns and expression (11) in Dixit (1991) for an approximation to the threshold for the discounted case.

# 4 Implications for frequency and size of price changes

In this section we explore the implications for the frequency and distribution of price changes. The expected time for $y(t)$ to hit the barrier $\bar{y}$ starting at $y$ is given by the function $\mathcal{T}(y)$ satisfying: $0 = 1 + \; n\ \sigma^2\ \mathcal{T}'(y) + \; 2\ y\ \sigma^2\ \mathcal{T}''(y)$ for $y \in (0, \bar{y})$ with a boundary condition $\mathcal{T}(\bar{y}) = 0$, which gives $\mathcal{T}(y) = \frac{\bar{y}-y}{n\ \sigma^2}$ for $y \in [0, \bar{y}]$. Thus $\mathcal{T}(0)$ gives the expected time between successive price adjustments, so that the average number of adjustments, denoted by $N_a$ is $\frac{1}{\mathcal{T}(0)}$. We summarize this result in:

Proposition 4. Let $N_a$ be the expected number of price changes for a multi-product firm with $n$ goods. It is given by

$$N_a = \frac{n\sigma^2}{\bar{y}} = \frac{nr}{Q\left(\frac{\psi r^2}{B\ \sigma^2},\ n\right)} \cong \sqrt{\frac{B\ \sigma^2}{2\ \psi} \frac{n^2}{(n+2)}}\ . \tag{12}$$

The second equality in equation (12) uses the function $Q(\cdot)$ derived in Proposition 3, while in the last equality we use the approximation of $\bar{y}$ for small $\psi r^2/(B\sigma^2)$ (see Appendix C.1 for more documentation on the accuracy of the approximation). It is interesting that this expression extends the well known expression for the case of $n = 1$, simply by adjusting the value of the variance from $\sigma^2$ to $n\sigma^2$. The number of products $n$ affects $N_a$ through two opposing forces. One is that with more products the variance of the deviations of the price gaps increases, and thus a given value of $\bar{y}$ is hit sooner in expected value. This is the "direct effect". On the other hand, with more products, the optimal value of $\bar{y}$ is higher. Expression equation (12) shows that the direct effect dominates, and the frequency of adjustment increases with $n$.

Next we characterize the hazard rate of price adjustments (see Appendix A for the proof)

Proposition 5. Let $t$ denote the time elapsed since the last price change. Let $J_\nu(\cdot)$ be the Bessel function of the first kind. The hazard rate for price changes is given by

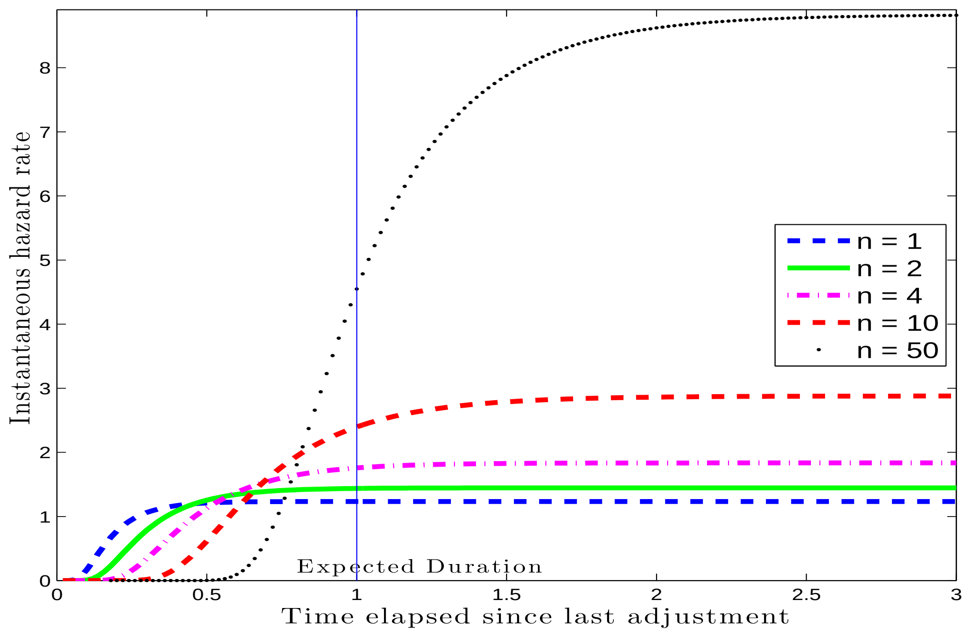
$$h(t) = \sum_{k=1}^{\infty} \frac{\xi_{n,k}}{\sum_{s=1}^{\infty} \xi_{n,s} \exp\left(-\frac{q_{n,s}^2 \sigma^2}{2\bar{y}}\ t\right)} \frac{q_{n,k}^2 \sigma^2}{2\bar{y}} \exp\left(-\frac{q_{n,k}^2 \sigma^2}{2\bar{y}}\ t\right)\ , \text{ where } \nu = \frac{n}{2} - 1\ ,$$

$$\xi_{n,k} = \frac{1}{2^{\nu-1}\Gamma(\nu+1)} \frac{q_{n,k}^{\nu-1}}{J_{\nu+1}(q_{n,k})}\ , \quad \text{and } q_{n,k} \text{ are the positive zeros of } J_\nu(\cdot),$$

which asymptotes to $\mathcal{T}(0) \cdot \lim_{t\to\infty} h_n(t) = \frac{q_{n,1}^2}{2n} > \max\left\{1\ ,\ \frac{(n-1)^2}{2n}\right\}$.

Proposition 5 compares the asymptote of the hazard rate with the expected time until adjustment, which equals $\mathcal{T}(0) = \bar{y}/(n\sigma^2)$, as derived above. Notice that for a model with a constant hazard these two quantities are the reciprocal of each other, i.e. the expected duration is the reciprocal of the hazard rate. We use the product $\mathcal{T}(0) \cdot \lim_{t\to\infty} h_n(t)$, which is larger than one, as a measure of how close the model is to have a constant hazard rate, or equivalently as a summary measure of how increasing the hazard function is. Also notice that the expression in Proposition 5 immediately shows that, keeping the expected time until adjustment $\mathcal{T}(0)$ fixed, the hazard rate is only a function of $n$. Thus the shape of the hazard function depends *only* on the number of products $n$. Changes in $\sigma^2, B, \psi$ only stretch the horizontal axis linearly.

Figure 1: Hazard rate of Price Adjustments for various choices of $n$




For each $n$ the value of $\sigma^2/\bar{y}$ is chosen so that the expected time elapsed between adjustments is one.

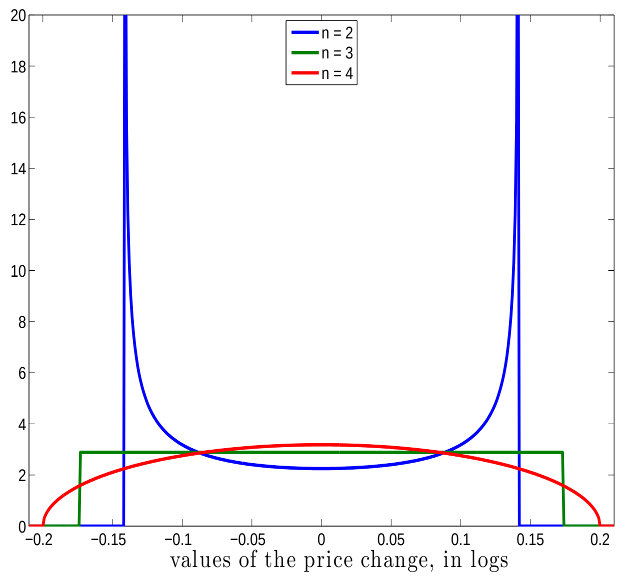
Figure 1 plots the hazard rate function $h$ for different choices of $n$ keeping the expected time between price adjustment fixed at one. As Proposition 5 shows the function $h$ has an asymptote, which is increasing in the number of products $n$. Moreover, since the asymptote diverges to $\infty$ as $n$ increases with no bound, the hazard rate converges to a an inverted L shape, as the one for a model where adjustment are done exactly every $\mathcal{T}(0) = 1$ periods, like in Taylor's (1980) model. To see this note that, defining $\tilde{y} \equiv y/\bar{y}$ and fixing the ratio $\mathcal{T}(0) = \bar{y}\,/(n\,\sigma^2)$ so that for any $n$ the expected time elapsed between price changes is $\mathcal{T}(0)$, we have:

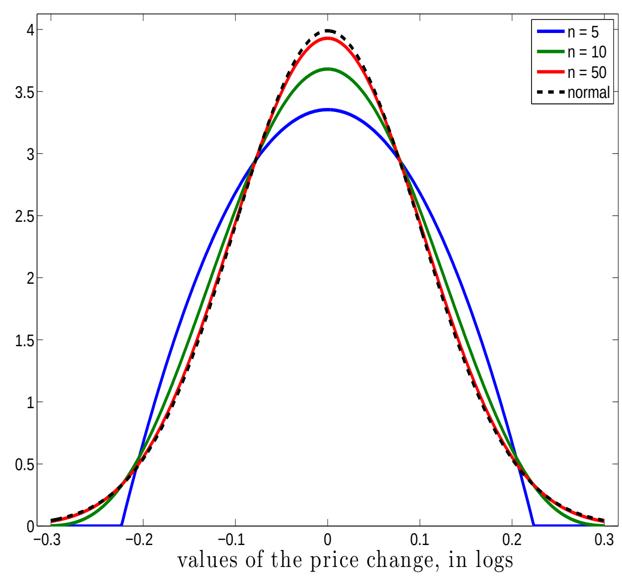
$$d\tilde{y} = \frac{1}{\mathcal{T}(0)}dt + 2\sqrt{\tilde{y}\,\frac{1}{n\,\mathcal{T}(0)}}\,d\mathcal{W} \quad \text{for} \quad \tilde{y} \in [0,1]. \tag{13}$$

As $n \to \infty$ the process for the normalized size of the price gap $\tilde{y}$ described in equation (13) converges to the deterministic one, in which case the hazard rate is zero between times 0 and below $\mathcal{T}(0)$ and $\infty$ precisely at $\mathcal{T}(0)$.

The shape of estimated hazard rates varies across studies, but many have found flat or decreasing ones, and some have found hump-shape ones. As can be seen from Figure 1 the hazard rate for the case of $n = 1$ is increasing but rapidly reaches its asymptote. As $n$ is increased, the shape of the hazard rate becomes closer to the inverted L shape of its limit as $n \to \infty$. For instance, when $n = 10$ the level of the hazard rate evaluated at the expected duration is about twice as large as the one for $n = 2$. This is a prediction that can be tested in the cross section using the data set in Bhattarai and Schoenle (2010) or Wulfsberg (2010).

Figure 2: Density of the price changes for various choices of $n$: $w(\Delta p)$




All distributions have the same standard deviation of price changes: $Std(\Delta p) = 0.10$.

Next we characterize the *marginal* distribution of price changes. The reason to focus on the marginal distribution is that it corresponds to what is measured in the data, where no record is kept of the joint distribution of price changes. This distribution is characterized by two parameters: the number of goods $n$, and the optimal boundary of the inaction set $\bar{y}$. The value of $\bar{y}$, as discussed above, depends on all the parameters. Let $\tau$ be a time when $y$

hits the boundary of the range of inaction: since after an adjustment all price gaps are reset to zero, the price changes coincide with $\Delta p(\tau) = -p(\tau)$ where $p(\tau) \in \partial\mathcal{I} \subset \mathbb{R}^n$, i.e. the price vector belongs to the surface of an $n$-dimensional sphere of radius $\sqrt{\bar{y}}$. Given that each of the (uncontrolled) $p_i(t)$ is independently and identically normally distributed, the price changes $\Delta p(\tau) = -p(\tau)$ are uniformly distributed on the $n$-dimensional surface of the sphere.[6] We can now state the following result:

Proposition 6. Let $p \in \partial\mathcal{I} \subset \mathbb{R}^n$ denote a vector of price gaps on the boundary of the inaction region, triggering price changes $\Delta p = -p$. The distribution of the price change of an individual good $i$, i.e. the marginal distribution of $\Delta p_i \in [-\sqrt{\bar{y}},\ \sqrt{\bar{y}}\ ]$, has density:

$$w(\Delta p_i) \;=\; \frac{1}{Beta\left(\frac{n-1}{2},\frac{1}{2}\right)\ \sqrt{\bar{y}}} \left(1-\left(\frac{\Delta p_i}{\sqrt{\bar{y}}}\right)^2\right)^{(n-3)/2} \tag{14}$$

where $Beta(\cdot,\cdot)$ denotes the Beta function. The standard deviation and kurtosis of the price changes, the expected value of the absolute value of price changes and its coefficient of variation are given by:

$$\begin{aligned} Std\,(\ \Delta p_i\ ) &= \sqrt{\bar{y}/n}\ , \qquad Kurt\,(\ \Delta p_i\ ) = \frac{3\ n}{n+2}\ , \\ \mathbb{E}\,[\ |\Delta p_i|\ ] &= \frac{\sqrt{\bar{y}}}{\frac{n-1}{2}\ Beta\left(\frac{n-1}{2},\frac{1}{2}\right)}\ ,\ \frac{Std\,(\ |\Delta p_i|\ )}{\mathbb{E}\,(\ |\Delta p_i|\ )} = \sqrt{\left[\frac{n-1}{2} Beta\left(\frac{n-1}{2},\frac{1}{2}\right)\right]^2 \frac{1}{n} - 1}\ . \end{aligned}$$

As $n \to \infty$ the distribution of $\Delta p_i / Std(\Delta p_i)$ converges point-wise to a standard normal.

The proof uses results from the characterization of spherical distributions by Song and Gupta (1997), see Appendix A. Using the previous proposition and the approximation for $\bar{y}$ we obtain the following expression: $Std\,(\ \Delta p_i\ ) = \left(\frac{\sigma^2\psi}{B}\frac{2(n+2)}{n^2}\right)^{1/4}$ which shows that the standard deviation of price changes is decreasing in $n$, while the kurtosis of the price changes is increasing in $n$.[7]

The proposition establishes how the shape of the distribution of price changes $w(\Delta p)$ varies substantially with $n$, as shown in Figure 2. For $n = 2$ the distribution is U-shaped, for $n = 3$ it is uniform, for $n = 4$ it has the shape of a half circle, and for $n \geq 6$ it has a bell shape. The proposition also establishes that as $n \to \infty$ the distribution converges to a

[6]To see this notice that the p.d.f. of a jointly normally distributed vector of $n$ identical and independent normals is given by a constant times the exponential of the *square radius of the sphere*, divided by half of the common variance.

[7] The approximations $\mathbb{E}\,[\ |\Delta p_i|\ ] \approx Std\,(\,\Delta p_i\,)\sqrt{\frac{2}{\pi}\left(1+\frac{1}{2\ n}\right)}$ and $\frac{Std(\ |\Delta p_i|\ )}{\mathbb{E}(\ |\Delta p_i|\ )} \approx \sqrt{\frac{\pi}{2}\left(\frac{2n}{1+2n}\right)-1}$ are useful to see how these statistics vary with $n$.

normal: this can be seen in Figure 2 by the comparison of the distribution for $n = 50$ and the p.d.f of a standard normal distribution with standard deviation equal the one obtained when $n = 50$. Interestingly the expressions in Proposition 6 show that $w(\Delta p)$ and $|\Delta p|$ depend only on $n$ *and* on the scale of the distribution: $\sqrt{\bar{y}}$. Thus, any normalized statistic, such as ratio of moments (kurtosis, skewness, etc) or a ratio of points in the c.d.f., depends exclusively on $n$. This property can be used to parametrize or estimate the model.

Table 1: Statistics for price changes as function of number of products: Model economy

| statistics \ number of products $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 10 | 20 | 50 |
|---|---|---|---|---|---|---|---|---|---|
| $Std(\|\Delta p_i\|)\ /\ \mathbb{E}(\|\Delta p_i\|)$ | 0 | 0.48 | 0.58 | 0.62 | 0.65 | 0.67 | 0.70 | 0.74 | 0.75 |
| Kurtosis$(\Delta p_i)$ | 1.0 | 1.5 | 1.8 | 2.0 | 2.1 | 2.3 | 2.5 | 2.8 | 2.9 |
| Fraction: $\|\Delta p_i\| < \frac{1}{2}\mathbb{E}(\|\Delta p_i\|)$ | 0 | 0.21 | 0.25 | 0.27 | 0.28 | 0.28 | 0.30 | 0.31 | 0.31 |
| Fraction: $\|\Delta p_i\| < \frac{1}{4}\mathbb{E}(\|\Delta p_i\|)$ | 0 | 0.10 | 0.12 | 0.13 | 0.14 | 0.14 | 0.15 | 0.16 | 0.16 |

$\Delta p_i$ denotes the log of the price change, and $|\Delta p_i|$ the absolute value of the log of price changes. They are computed using the results in Proposition 6. All statistics in the table depend exclusively on $n$. Kurtosis defined as the fourth moment relative to the square of the second.

We conclude this section by summarizing the most interesting results of the model concerning the cross-section predictions in comparison with the data and with the previous literature. To this end Table 1 uses the the model to compute several moments of interest that depend only on $n$. Table 2 reports the empirical counterparts to those moments as estimated by Midrigan (2011) (using two scanner data sets) and by Bhattarai and Schoenle (2010) (using BLS producer data).

First, in comparison with the menu cost models of Golosov and Lucas (2007) or Midrigan (2011) the model ability to account for the shape of the distribution of price changes improves dramatically. These models predict a bimodal distribution of price changes with a nil, or very small, mass of small price changes, as can be seen by the $n = 1$ and the $n = 2$ cases in Figure 2. In contrast, as documented by e.g. Klenow and Malin (2010) as well as Midrigan (2011), the size distribution of price changes has a bell-shape and displays a large mass of small price changes, as shown in Table 2. We showed that the number of goods that is necessary to produce the bell-shaped distribution is at least 6. Therefore, compared to the US data in Table 2, the models with $n = 1, 2$ generate too few small price changes. Much bigger values of $n$ are necessary to reproduce the patterns that are seen in the data.

Table 2: Statistics for price changes as function of the number of products: US data

| Statistics | *Bhattarai and Schoenle* | | | | *Midrigan* | | | |
|---|---|---|---|---|---|---|---|---|
| | Number of products $n$ | | | | AC Nielsen | | Dominick's | |
| | 2 | 4 | 6 | 10 | All | No Sales | All | No Sales |
| $Std(\lvert\Delta p_i\rvert)$ / $\mathbb{E}(\lvert\Delta p_i\rvert)$ | 1.02 | 1.15 | 1.30 | 1.55 | 0.68 | 0.72 | 0.84 | 0.81 |
| Kurtosis($\Delta p_i$) | 5.5 | 7.0 | 11 | 17 | 3.0 | 3.6 | 4.1 | 4.5 |
| Fraction: $\lvert\Delta p_i\rvert < \frac{1}{2}\mathbb{E}(\lvert\Delta p_i\rvert)$ | 0.39 | 0.45 | 0.47 | 0.50 | 0.24 | 0.25 | 0.34 | 0.31 |
| Fraction: $\lvert\Delta p_i\rvert < \frac{1}{4}\mathbb{E}(\lvert\Delta p_i\rvert)$ | 0.27 | 0.32 | 0.35 | 0.38 | 0.10 | 0.10 | 0.17 | 0.14 |

Sources: For the Bhattarai and Schoenle (2010) data: the number of product $n$ is the mean of the categories considered based on the information in Table 1, the ratio $Std(|\Delta p_i|)$ / $\mathbb{E}(|\Delta p_i|)$ is from Table 2 (Firm-Based), the fraction of $|\Delta p_i|$ which are small is from Table 14, the Kurtosis is from Figure 7. The data from Midrigan (2009) are taken from distribution of standardized prices in Table 2a.

Another novelty of the model is that, in addition to producing a larger mass of small price changes compared to existing models, it also produces a greater mass of "large" price changes, so that its Kurtosis is higher than the one obtained in models with small $n$. This is seen immediately by noting that the kurtosis is $3n/(2+n)$, an increasing function of $n$. The multi product hypothesis is thus able to account for more kurtosis than the canonical menu cost model. We notice however that despite this improvement the model is not yet able to match the very high level of Kurtosis that are measured by some datasets. We see this as a challenge, both theoretical and empirical, for future work.[8]

The model provides a simple explanation for the empirical regularity, documented by Goldberg and Hellerstein (2009) and Bhattarai and Schoenle (2010) using BLS producer prices, that firms selling more goods (or larger firms) tend to adjust prices more frequently and by smaller amounts. This is precisely the model predictions as from equation (12) and Proposition 6. Notice that this prediction holds even if, in doing the comparative statics with respect to $n$, one assumes that the fixed cost $\psi$ increases linearly with $n$.[9]

Finally, the model predicts that price changes are synchronized within the store, a feature of the data that is found in many empirical analyses of the cross section of price changes as in the seminal paper by Lach and Tsiddon (1996) who showed that price changes are

[8]The level of kurtosis appears imprecisely measured in the data: the estimates vary widely from around 3 to 20, depending on the data source, industry, sample selection criteria and measurement error as discussed by Eichenbaum et al. (2012) and Alvarez, Le Bihan, and Lippi (2012).

[9]See the NBER version of the paper for a discussion of this point and some empirical evidence.

synchronized within stores but staggered across stores, and in the paper by Cavallo's (2010) who finds that price changes of similar goods (in online supermarket chains) are synchronized within a store for goods of similar type (see footnote 1 for more references).

# 5 The response to a monetary shock

In this section we study the response of the aggregate price level to an unexpected permanent monetary shock. Understanding this impulse response is useful to quantify the real effects of monetary shocks in the presence of menu costs, to identify its determinants, and to characterize how the effects vary with the number of products $n$ sold by the firm. We will show how the determinants of the real effects of monetary shocks map into simple observable statistics about the size and frequency of price changes that are available for many economies.

A main novelty of this paper is to solve for the whole impulse response analytically using an approximation to characterize the firm's decision problem. In particular, besides the second order approximation of the profit function and the assumption of no drift in the price gaps used above, we assume that after an aggregate monetary shock the firm uses the same decision rule $\bar{y}$ used in the steady state, i.e. we ignore the feedback effect on the firm's decision that arises in a general equilibrium problem. Interestingly we show that the approximation provides a very accurate benchmark to the exact solution of the original problem. The explanation for this result, formally given in Proposition 7 below and documented quantitatively in Appendix C, is that in the class of models we considered the general equilibrium feedback only has second order effects on the decision rules.

The general equilibrium set up where we embed our price setting problem is an adaptation of the one in Golosov and Lucas (2007) to multiproduct firms (see Appendix B for details). The representative household has preferences given by

$$\int_0^\infty e^{-rt}\left(u\left(c(t)\right) - \alpha\ell(t) + \log\frac{M(t)}{P(t)}\right)dt \quad , \quad \text{and} \quad c(t) = \left(\int_0^1 \sum_{i=1}^n \left(Z_{k,i}(t)\, c_{k,i}(t)\right)^{\frac{\eta-1}{\eta}} dk\right)^{\frac{\eta}{\eta-1}}$$

where $u(c) = (c^{1-\epsilon}-1)/(1-\epsilon)$, $c_{k,i}(t)$ is the consumption of product $i$ produced by firm $k$, $\ell(t)$ are labor services, $M(t)$ is the nominal quantity of money, $P(t)$ is the nominal ideal price index of one unit of aggregate consumption, and $r > 0, \epsilon \geq 1, \alpha > 0, \eta > 1$ are parameters. The elasticity of substitution between any two products $\eta$ is the same, regardless of the firms that produced them.[10] The production function for good $i$ in firm $k$ at time $t$ is linear in labor –the

[10] If the elasticity of substitution across firms is different from the one across goods then the second order approximation of the profit function will feature the cross products of the different price gaps. See Appendix F.5 for a discussion of this point and an illustration of its solution.

only input in the economy– with productivity $1/Z_{ki}(t)$, so the marginal cost of that product is $W(t)Z_{ik}(t)$, where $W(t)$ is the nominal wage. We assume that the idiosyncratic productivity and demand shocks are perfectly correlated, and that $Z_{ik}(t) = \exp(\sigma \mathcal{W}_{ki}(t))$ where $\mathcal{W}_{ki}$ are standard BM independent across all $i, k$. This assumption ensures that consumer expenditure is the same across goods so that the ideal price index is well defined and has equal weights. Firm $k$ can adjust one or more of its $n$ nominal prices paying a fixed menu cost equal to a number of labor service units, which we express as $\psi$ times the steady state profits from producing $n$ goods evaluated at the profit maximizing price. Markets are complete, and all firms are owned by the representative household. We use $R(t), W(t)$ and $P_{ik}(t)$ for the time $t$ nominal interest rate, nominal wage, and nominal price of firm $k$ product $i$ respectively. As before we use $p_{ki}(t)$ for the price gap: the log of the ratio of the nominal price of firm $k$ on product $i$ to the frictionless optimal price, or

$$p_{ki}(t) = \log P_{ki}(t) - \log(W Z_{ki}(t)) - \log\left(\eta/(\eta-1)\right) \quad . \tag{15}$$

We consider an economy that starts at the invariant distribution of firm's prices that correspond to a steady state with constant money supply equal to $\bar{M}$. We assume that at time $t = 0$ there is an unanticipated permanent increase in the level of the money supply by $\delta$ log points, so $\log M(0) = \log \bar{M} + \delta$, where bars denote the steady state values. As in the general equilibrium sticky price model of Danziger's (1999) or Golosov and Lucas's (2007) we have that, for all $t \geq 0$, the interest rate is constant and wages and consumption follow

$$R(t) \;=\; r \;,\; \log \frac{W(t)}{\bar{W}} = \delta \;,\; \log \frac{c(t)}{\bar{c}} = \frac{1}{\epsilon}\left(\delta - \log \frac{P(t)}{\bar{P}}\right) \quad . \tag{16}$$

The equation shows that the shock induces an immediate permanent increase in (the log of) nominal wages, and hence marginal cost, by $\delta$. The effect on output on the other hand is gradual, and at each $t$ it depends on how much the aggregate price level $P(t)$.

The next proposition illustrates why the approximate decision rule $\bar{y}$ derived in a partial equilibrium from the quadratic loss function $\mathbf{V}(\boldsymbol{\tau}, \boldsymbol{\Delta p}; p)$ in equation (1) provides an accurate approximation of the effect of a monetary shock $\delta$ in a general equilibrium. To this end recall that $\{\boldsymbol{\tau}, \boldsymbol{\Delta p}\} \equiv \{\tau_j, \Delta p_i(\tau_j)\}_{j=1}^{\infty}$ denote the stochastic processes for the stopping times and the $n$ price gaps, and let $\mathbf{c} \equiv \{c(t)/\bar{c} - 1\}_{t \geq 0}$ denote the path of aggregate output deviations from the steady state. Let the value function $-\mathcal{V}(\boldsymbol{\tau}, \boldsymbol{\Delta p}, \mathbf{c}; p)$ measure the expected profits in the general equilibrium of a generic firm with a vector of price gaps $p$. We have:

Proposition 7. Let $\delta \geq 0$. For all $t \geq 0$ a general equilibrium satisfies:

$$(i) \qquad \log \frac{P(t)}{\bar{P}} \;=\; \delta + \int_0^1 \left( \sum_{i=1}^{n} (p_{ki}(t) - \bar{p}_{ki}) \right) dk + \int_0^1 \left( \sum_{i=1}^{n} o(||p_{ki}(t) - \bar{p}_{ki}||) \right) dk$$

$(ii)$ A Taylor expansion of $\mathcal{V}(\boldsymbol{\tau}, \boldsymbol{\Delta p}, \mathbf{c}; p)$ around $p_{i,t} = 0$ for all $i = 1, ..., n$ and around $c_t = \bar{c}$ is proportional to the quadratic loss function $\mathbf{V}(\boldsymbol{\tau}, \boldsymbol{\Delta p}; p)$; moreover the terms including the cross products of the price gap $p_{it}$ and the aggregate consumption $c_t$ are of third order or higher.

Part $(i)$ states that the effect of the shock on $P(t)$ can be approximated by analyzing the dynamic response of the price gaps, each of which falls by a constant $\delta$ before any adjustment takes place, i.e. $p_{ki}(0) = \bar{p}_{ki} - \delta$. Part $(ii)$ states that the objective function $\mathbf{V}$ in the partial equilibrium set-up of equation (1) is proportional to the objective function in the general equilibrium setup $\mathcal{V}$. The difference between these functions in terms of the price gaps (the relevant object for the firm's decision) only involves third and higher order terms, so that the general equilibrium effect on the boundary of the inaction set, $\bar{y}$, is second order. The result provides a foundation to Caballero and Engel (1991, 1993, 2007) who pioneered the analytical study of the impulse response in $Ss$ models while ignoring the general equilibrium feedback effects on the decision rules. To see why the result holds note that, as Golosov and Lucas (2007) remark, the general equilibrium feedback on the firms' decision rules is completely captured, at each time $t$, by the effect of the aggregate output $c(t)$ on profits. Inspection of the firm's profit function shows that the discounted period $t$ profits from good $i$ are the *product* of a term including the GE effect – a function of $c(t)$–, and a term whose maximum is achieved by the frictionless price. Hence the discounted time $t$ profits have a zero first derivative with respect to the (log of the) price gap $p_i(t)$, and a zero second cross derivative with respect to the (log of the) price gap $p_i(t)$ and $c(t)$. Thus for small shocks $\delta$ and small adjustment cost $\psi$, the general equilibrium effect on decisions are negligible (see Appendix B for a proof and Appendix C for a quantitative exploration of this result).

Next we use the results of Proposition 7 to study the effect of an aggregate monetary shock of size $\delta$ on the aggregate price level $P(t)$ at $t \geq 0$ periods after the shock, which we denote by $\mathcal{P}_n(\delta, t)$. As commonly done in the sticky price literature, we characterize the first order approximation to the price index, so in particular we study $\mathcal{P}_n(\delta, t) \equiv \delta + \int_0^1 \left( \sum_{i=1}^{n} p_{ki}(t) \right) dk \approx \log P(t)/\bar{P}$. Once we characterize the effect on the price level, we describe the effect on employment and output. The impulse response is made of two parts: an instantaneous impact adjustment (a jump) of the aggregate price level which occurs at the time of the shock, denoted by $\Theta_n(\delta)$, and a continuous flow of adjustments from $t > 0$ on, denoted by

$\theta_n(\delta, t)$. The cumulative effect of the price level $t \geq 0$ periods after the shock is

$$\mathcal{P}_n(\delta, t) \;\; = \;\; \Theta_n(\delta) + \int_0^t \theta_n(\delta, s) ds \; . \tag{17}$$

We also study the impact effect on the fraction of firms that change prices, denoted by $\Phi_n(\delta)$.

We focus on the cumulative price response because its difference with the monetary shock, $\delta - \mathcal{P}_n(t, \delta)$, is proportional to the aggregate output at time $t$, as discussed in Section 5.5. Next we present our main results on $\mathcal{P}_n(\delta, t)$ and $\Phi_n(\delta)$ following an aggregate shock:

Proposition 8. Fix $n$, the number of goods sold by each firm.

1. Parameters. The impulse response $\mathcal{P}_n(\delta, t)$ depends only on two parameters: $\sqrt{\bar{y}}$ and $\sigma$, which we re-parameterize as functions of two steady state statistics: the *standard deviation of price changes* $Std[\Delta p]$ and the *frequency of price changes* $N_a$.

2. Scaling and Stretching: The IRF of an economy with steady state $Std[\Delta p], N_a$ and a shock $\delta$ at horizon $t \geq 0$ is a scaled version of the one of an economy with *unit* steady state parameters, *normalized monetary shock* $\delta / Std[\Delta p]$, and a *stretched horizon* $N_a\, t$:

$$\mathcal{P}_n\left(\delta,\, t;\, N_a,\, Std[\Delta p]\,\right) \;\; = \;\; Std[\Delta p] \;\; \mathcal{P}_n\left(\frac{\delta}{Std[\Delta p]},\, N_a\, t;\, 1,\, 1\right)$$

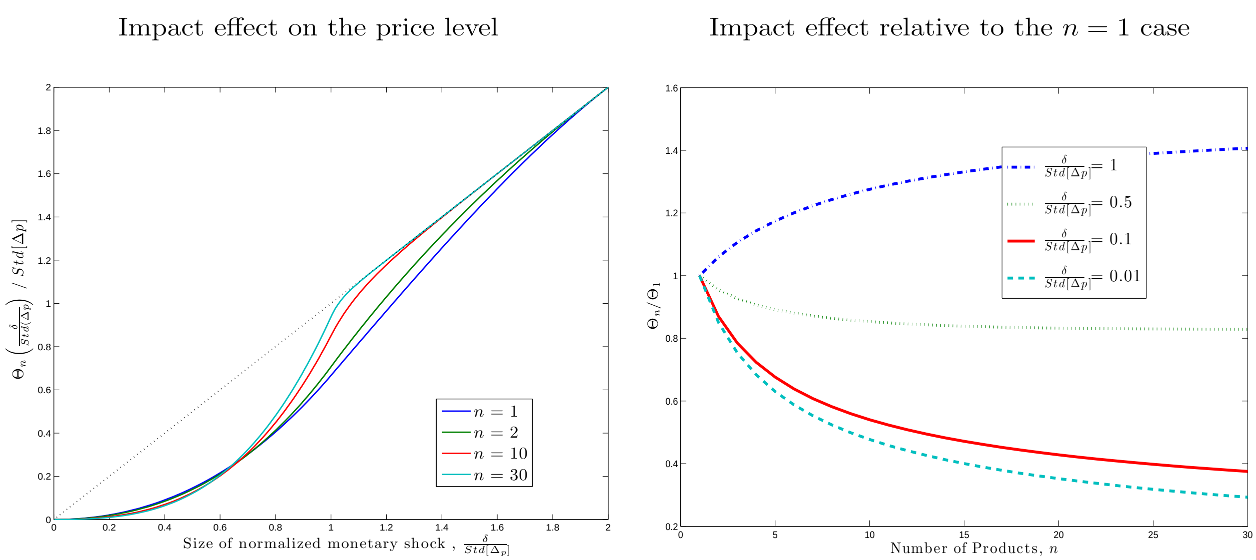
3. Impact Effects: the impact effects $\mathcal{P}(\delta, 0) = \Theta_n(\delta)$ and $\Phi_n(\delta)$ are strictly increasing in $\delta$, they are respectively strictly below $\delta$ and 1, in the interval $(0,\, 2\, Std[\Delta p])$ and achieve these values outside this interval. Moreover, impact effects are second order on the monetary shock: $\Theta'_n(0) = \Phi'_n(0) = 0$.

Part 1 of Proposition 8 provides a re-parameterization of the impulse response that is interesting for three reasons: (i) the steady state statistics $Std[\Delta p]$ and $N_a$ are readily available for actual economies, (ii) the results of Section 4 imply that, even fixing $n$, one can always choose two parameters values of $\psi/B$ and $\sigma^2$ to match these two statistics, and (iii) keeping fixed these two observable statistics and just changing $n$ we can isolate completely the role of the number of products $n$.

Part 2 of Proposition 8 states a useful "scaling" property of the impulse response function. First notice that at $t = 0$, the impact effect of a monetary shock $\Theta_n$ is the same for *any* two economies with the same steady state average size of price changes $Std[\Delta p]$, and is independent of the value of the steady state frequency of price adjustment $N_a$. Moreover, for all times following the impact ($t > 0$) the effect of a monetary shock $\delta$, in an economy characterized by steady state statistics $Std[\Delta p]$ and $N_a$ depend only on $n$. This means that for a fixed $n$, the whole profile of the impulse response functions in economies with different values of $Std[\Delta p]$ and $N_a$ are simply scaled version of each other. For instance, fixing $n$, $\delta$ and $Std[\Delta p]$, the impulse response functions in two economies that differ in the frequency of price adjustments, say $N_a$ vs 2 $N_a$, will have exactly the same values of $\mathcal{P}_n$ but will reach these values at different times, respectively 2 $t$ vs $t$, i.e. an economy with twice more flexible prices in steady state has an impulse response that reaches each value in half of the time. Furthermore, keeping $N_a$ fixed, the height of the whole impulse response function $\mathcal{P}_n$ is proportional to the scaled value of the monetary shock. We find this characterization interesting in itself, i.e. even interesting for the $n = 1$ case, but more importantly it will allow us to compare the impulse response for economies that feature different values of $n$.

Part 3 of Proposition 8 shows that the size of the monetary shock matters, so for large shocks there is instantaneous full price flexibility ($\Theta_n = \delta$), but for small shocks the size of the initial jump in price is second order compared to the shock. This, together with part 2, implies that whether a monetary shock is large or not is completely characterized by comparing it with the typical price change in steady state, i.e. it is a function of $\delta/Std[\Delta p]$.

Figure 3: The impact effect of an aggregate shock on the price level: $\Theta_n$




Normalized impact response of the aggregate price level to a permanent shock in the level of money of size $\delta/Std[\Delta p]$. The normalization in the left panel consists of dividing the impact response of the price level by $Std[\Delta p]$, the steady state standard deviation of price changes. See the text for more details.

For the reader who is not interested in the derivation of the impulse responses, and an explanation of the different effects behind it, we include two figures that summarize the quantitative conclusions of our analysis. Before getting to these figures, we note that its computation for large value of $n$ would have been extremely costly without the characterization given in the sections below. Figure 3 has two panels that illustrate the impact effect on prices of monetary shocks of different size for economies with different values of $n$. The left panel shows the normalized impact on the aggregate price level, $\Theta_n/Std[\Delta p]$ (on the vertical axis), of a normalized monetary shock, i.e. a shock $\delta/Std[\Delta p]$ (on the horizontal axis). Each line plotted in this panel corresponds to a different number of products $n$. Recall that if $\Theta_n(\delta) = \delta$ the shock is neutral, and that instead when $\Theta_n(\delta) < \delta$ the shock implies an increase in real output. As stated in Proposition 8, if $\delta \geq 2\sqrt{\bar{y}/n} = 2\,Std[\Delta p]$, then all firms adjust prices, and hence the shock is neutral. This explains the range of the normalized shock, between 0 and 2. For the quantification of this figure it is helpful to notice that on the one hand a typical estimate of the standard deviation of price changes for US or European countries is 10% or higher, i.e. $Std[\Delta p] \approx 0.1$. On the other hand to quantify $\delta$ note that in a short interval -say a month- changes of the money supply or prices in the order 1% are very rare.[11] This figure also shows that for small $\delta$ the aggregate price effects are of order $\delta^2$, as stated in Proposition 8. Interestingly, the impact response of a monetary shock changes order with respect to $n$ as the value of $\delta$ increases, as can be seen for shocks smaller than $\delta/Std[\Delta p] \approx 0.7$. Note that using $Std[\Delta p] = 0.1$ this means that shock for which they reverse order is higher than 7%, a very large value. The right panel of Figure 3 displays four lines, each corresponding to a different value of $\delta$. Each line shows the aggregate effect on prices as $n$ changes, relative to the $n = 1$ case. From these two panels it can be seen that, for monetary shocks in the order of those experienced by economies with inflation close to zero, i.e. for increases in money $\delta/Std[\Delta p]$ smaller than a 0.5 (or for the benchmark value, for $\delta$ smaller than 5%), economies with more products are more sticky than those with fewer.

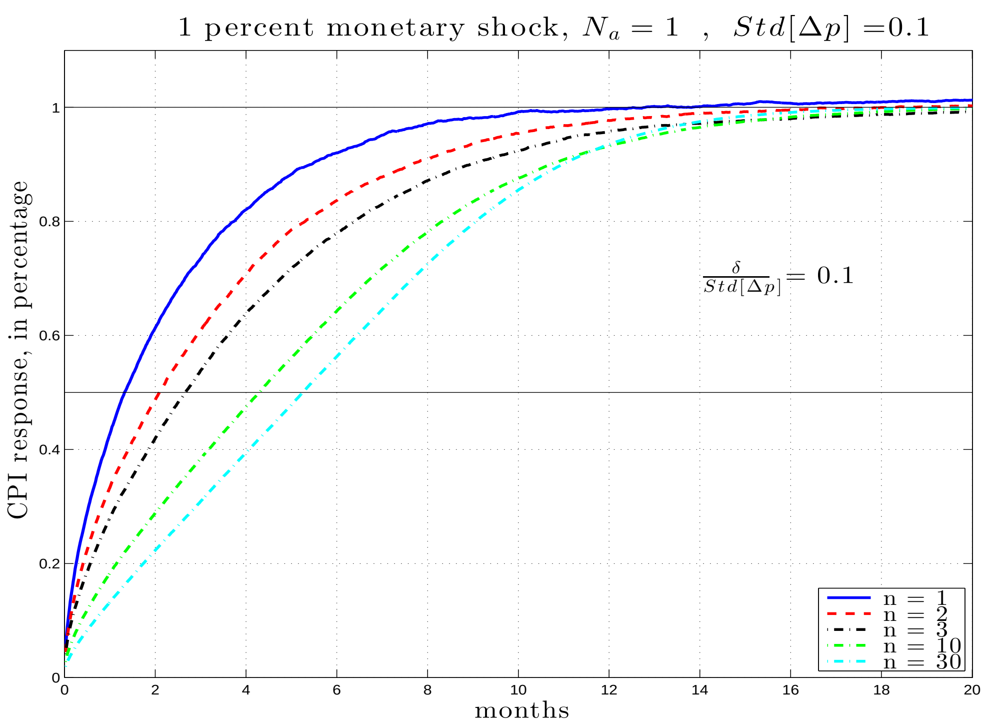
Figure 4 plots the impulse response function $\mathcal{P}_n(\delta, t)$ for economies with different $n$ keeping fixed the steady state deviation of price changes to 10%, i.e. $Std[\Delta p] = 0.1$ and an average of one price change per year, i.e. $N_a = 1$. The size of the monetary shock is 1%, so that $\delta/Std[\Delta p] = 0.1$. In this figure we have time aggregated the effect on the aggregate price level up to daily periods. As required, all impulse responses display impact effect on the first period, and a monotone convergence to the full adjustment of the shock. The impact effect of the monetary shock during the first periods is to increase prices about 5% of the long run value (i.e. 5 basis points) for $n = 1$. This effect is smaller in economies where firms produce more products, i.e. the impact at $t = 0$ is decreasing in $n$. This difference is small between one and two products, but the effect is almost halved for firms with 10 products, as shown

[11] For instance, the monthly innovations on a time series representation of M1 are on the order of 50 basis points, and that is without any conditioning. Presumably exogenous monetary shocks are much smaller.

in Figure 3 for a monetary shock of the same size.

Likewise, the shape and duration of the shocks depend on $n$. The half-life of the shock more than doubles as the number of products goes from 1 to 10. The shape of the impulse responses for $n = 1$ is quite concave, but for large $n$ it becomes almost linear, up to a value of $t$ of about $1/N_a$. This pattern of the shape is consistent with the result of Proposition 5 which shows that for large $n$ the model becomes a version of either Taylor (1980) staggering price model or of Caballero (1989), Bonomo and Carvalho (2004) and Reis (2006) inattentiveness model, where the staggering lasts for $\mathcal{T}(0) = 1/N_a$ periods. Indeed, in Proposition 12 below we show that as $n \to \infty$ the impulse response becomes linear up to time $1/N_a$ because there is no "selection effect". Summarizing, we find that extending the analysis from $n = 1$ to a larger number of product (say $n \approx 10$) almost halves the impact effect on the aggregate price level and doubles the half-life of the shock, for empirically reasonable monetary shocks.[12]

Figure 4: Impulse response of the aggregate price level




The rest of this section is organized as follows. First, we obtain a closed form solution for the IRF in the case of $n = 1$. This result, which is novel and interesting in its own right, is also helpful to better understand the derivation for the $n \geq 2$ case as well as to compare the results. After that, we develop the analytical expressions for the $n \geq 2$ case, concentrating

[12]The results are very similar for shocks of 1/2 and 2 percent, as reported in the Appendix E.3.

first on the impact effects and then on the remaining part of the impulse response. The proof of Proposition 8, as well as explicit expressions for the impulse responses, are presented as separate propositions in the next subsections. We conclude the section by discussing the real effect of the monetary shocks.

## 5.1 Impulse response for the $n = 1$ case

In the $n = 1$ case, which we refer to (abusing a bit the analogy) as the Golosov and Lucas (GL) case, the firms controls the price gap between two symmetric thresholds, $\pm\bar{p}$, and when the price gap hits either of them it returns it to zero. Hence the invariant distribution of price gaps is triangular: the density function has a maximum at the price gap $p = 0$ and decreases linearly on both sides to reach a value of zero at the thresholds $\bar{p}$ and $-\bar{p}$, since firms that reach the thresholds will adjust upon a further shock. An example of such a distribution is depicted by the solid line in the left panel Figure 5. A straightforward computation gives that the slope of this density is $\pm(1/\bar{p})^2$. Consider an aggregate shock that displaces the distribution by reducing all price gaps by $\delta$. If the value of $\delta > 2\bar{p}$ then all the firms will adjust their price, so that $\Phi = 1$, and after a simple calculation one can see that the aggregate price level is increased by $\delta$. Instead, if the value of $\delta$ is smaller than $2\bar{p}$, only the firms with a sufficiently small price gap will adjust. Denoting the price gap right after the shock by $p_0$, these are the firms that end up with $p_0 < -\bar{p}$. The density of the distribution of the price gaps immediately after the shock, denoted by $\lambda$, is depicted by the dotted line of Figure 5 and it is given by:

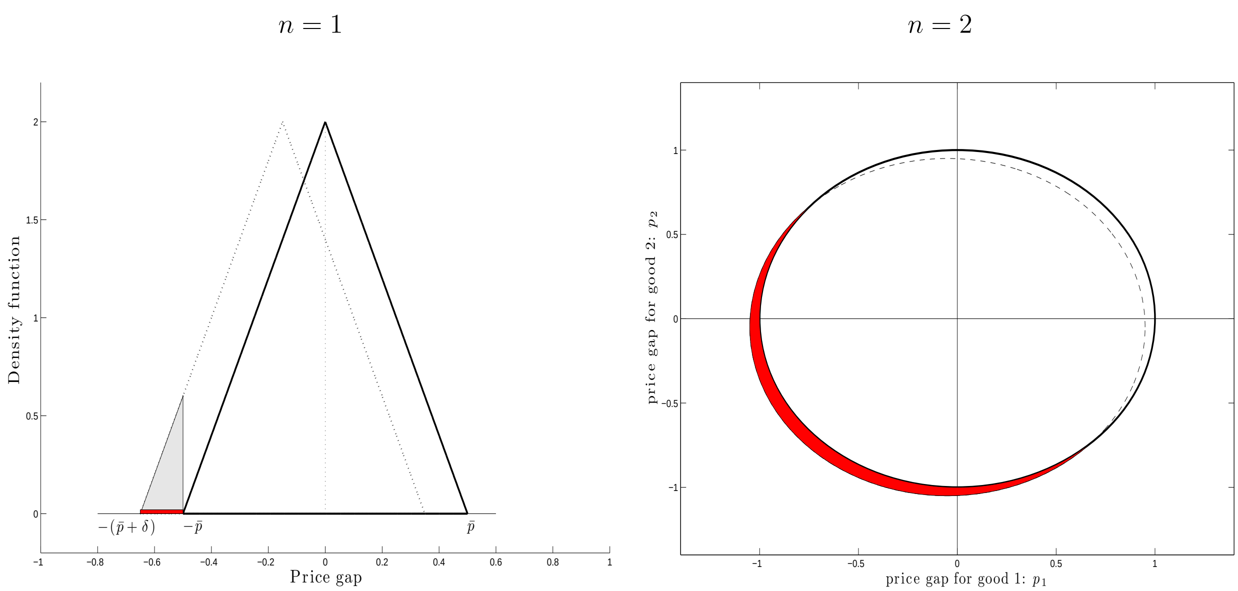
$$\lambda(p_0, \delta; \bar{p}) = \begin{cases} \frac{1}{\bar{p}}\left(1 + \frac{\delta}{\bar{p}} + \frac{p_0}{\bar{p}}\right) & \text{if } \frac{p_0}{\bar{p}} \in [-1 - \frac{\delta}{\bar{p}}, -\frac{\delta}{\bar{p}}] \\ \frac{1}{\bar{p}}\left(1 - \frac{\delta}{\bar{p}} - \frac{p_0}{\bar{p}}\right) & \text{if } \frac{p_0}{\bar{p}} \in (-\frac{\delta}{\bar{p}}, 1 - \frac{\delta}{\bar{p}}] \end{cases} \tag{18}$$

For a shock of size $\delta$ the mass of such firms is $\Phi = (1/2)(\delta/\bar{p})^2$, which uses the slope of the density given above (to simplify notation we suppress the $n = 1$ subindex). Note that the magnitude of this fraction is proportional to the square of the shock, a feature that is due to the fact that there are a few firms close to the boundary of the inaction set. This case is depicted by the dotted line in the left panel of Figure 5. Firms that change prices "close the price gap" completely, so that price increase will be $\delta + \bar{p}$ for the firm that prior the shock had price gap $-\bar{p}$ and it will be equal to $\bar{p}$ for the firm with pre-shock price gap equal to $-\bar{p} + \delta$. Using the triangular distribution of price gaps we have that the average price increase among those that adjust prices equals $\bar{p} + \delta/3$. Let us denote by $\Theta$ the impact effect on aggregate prices of a monetary shock of size $\delta$, the product of the number of firms that adjust times the average adjustment among them. Note that in steady state the average size of price changes, as measured by the standard deviation of price changes $Std[\Delta p]$, is given by $\bar{p}$. Thus for $\delta \leq Std[\Delta p] = \bar{p}$ we can write

$$\Theta = Std[\Delta p]\ \frac{1}{2}\left(\frac{\delta}{Std[\Delta p]}\right)^2\left(1 + \frac{1}{3}\frac{\delta}{Std[\Delta p]}\right)$$

so that for an economy with one good the impact effect on prices, normalized by the steady state average price change, depends on the normalized monetary shock, and it is locally quadratic, at least for a small shock. Note that the degree of aggregate stickiness is independent of the steady state fraction of price changes.

Figure 5: The selection effect on impact for the $n = 1$ and $n = 2$ case




We now develop expression for the impulse response at horizons $t > 0$. The density of the price gaps $p_0$ right after the monetary shock $\delta$ is the displaced triangular distribution $\lambda$ plotted in Figure 5 and described in equation (18), and hence it has $\bar{p}$ as a parameter. It peaks at $-\delta$, it has support $[-\bar{p} - \delta, \bar{p} - \delta]$. Note that the impact adjustment is concentrated on the firms whose price gap is smaller than $-\bar{p}$. Now consider the contribution to the change in aggregate prices of the firms whose price gap is $p_0 \in [-\bar{p}, \bar{p} - \delta]$, so they have not adjusted on impact, and of which there are $\lambda(p_0; \delta)\ dp_0$. Let $G^-(t; p_0)$ be the probability that a firm with price gap $p_0$ at time zero will increase price before time $t$, i.e. the probability that its price gap will hit $-\bar{p}$ before time $t$ without first hitting $\bar{p}$. Likewise define $G^+(t; p_0, )$ as the corresponding probability of a price decrease, let $G(t; p_0) = G^-(t; p_0) - G^+(t; p_0)$ be the difference between these probabilities, and let $g$ be its density. We note that these functions have $(\bar{p}, \sigma^2)$ as parameters. We can now define the contribution to the change in the price level of the adjustments that take place between $t$ and $t + dt$ as:[13]

$$\theta(\delta, t) = \bar{p} \int_{-\bar{p}}^{\bar{p}-\delta} g(t; p_0) \; \lambda(p_0; \delta) dp_0 \quad . \tag{19}$$

The integral excludes the initial price gaps $p_0$ that are below $-\bar{p}$. These correspond to firms that adjusted on impact. Note that $\theta(\delta, t)$ have $(\bar{p}, \sigma^2)$ as parameters. Expressions for the densities $g^+$ and $g^-$ can be found in equations (15)-(16) of Kolkiewicz (2002). This gives

$$g(t; p_0) = \frac{\sigma^2}{2 \; \bar{p}^2} \sum_{k=1}^{\infty} e^{-\frac{k^2 \pi^2}{2 \bar{p}^2} \sigma^2 t} \, k\pi \left[ \sin\left( k\pi \left( 1 + \frac{p_0}{\bar{p}} \right) \right) - \sin\left( k\pi \left( 1 - \frac{p_0}{\bar{p}} \right) \right) \right]$$

Four remarks are in order. First, by substituting our expressions for $g$ and $\lambda$ we have a closed form solution for each expression in equation (19). Second, note that we did not need to compute the evolution of the whole cross section distribution. Instead, we just follow each firm until the first time that it adjusted its price. This is because the subsequent adjustments have a zero net contribution to aggregate prices, since after the adjustment every firm price gap returns to zero, and the subsequent adjustments are as likely to be increases as decreases. Third, note that the role of the monetary shock is just to displace the initial distribution, i.e. $\delta$ is not an argument of $g$. Fourth, note that this function has two interesting forms of homogeneity. The first type of homogeneity is that it is homogenous of degree one in $\sigma, \bar{p}$ and $\delta$. This follows because scaling $\bar{p}$ and $\delta$ will just scale proportionally the distribution $\lambda$ of the initial price gaps. Furthermore, scaling $\bar{p}$ and $\sigma$ keeps the probabilities of hitting any two scaled up values in the same elapsed time to be the same. The second type of homogeneity uses that a standard Brownian Motion at time $t$ started at time zero has a normal distribution with variance $t$. So scaling the variance of the shock, the price gaps will hit any given value in a scaled time. These two homogeneity properties can be seen by integrating the previous expression gives an IRF which satisfies the properties stated in Proposition 8:

$$\begin{aligned} \mathcal{P}(\delta, t) \quad &= \quad \Theta(\delta) \; + \; Std[\Delta p] \sum_{k=1}^{\infty} \frac{1 - e^{-\frac{k^2 \pi^2}{2} N_a t}}{k\pi} \times \\ & \int_{-1}^{1-\frac{\delta}{Std(\Delta p)}} \left[ \sin\left( k\pi \left( 1 + x \right) \right) - \sin\left( k\pi \left( 1 - x \right) \right) \right] \lambda\left( x, \frac{\delta}{Std[\Delta p]}; 1 \right) dx \end{aligned}$$

[13]Alternatively one could compute the impulse response for the $n = 1$ case by adapting ideas from Bertola and Caballero (1994). They study the evolution of the whole cross section distribution following a shock for an irreversible investment problem with a reflecting barrier. Their formulas should be adapted to our fixed cost problem whose optimal return point implies a jump (not a reflection) of the state.

## 5.2 Invariant distribution of $y = ||p||^2$

Here we study the invariant distribution of the sum of the squares of the price gaps $y \equiv ||p||^2 = \sum_{i=1}^{n} p_i^2(t)$ under the optimal policy. This will be used to describe the starting point of the economy before the monetary shock. We will denote the density of the invariant distribution by $f(y)$ for $y \in [0, \bar{y}]$. This is interesting to study the response of firms that are in the steady state to an unexpected shock to their target that displaces the price gaps uniformly. The density of the invariant distribution for $y$ is found by solving the corresponding forward Kolmogorov equation, and the relevant boundary conditions (see Appendix A for the proof).

**Proposition 9.** *The density $f(\cdot)$ of the invariant distribution of the sum of the squares of the price gaps $y$, for a given thresholds $\bar{y}$ in the case of $n \geq 1$ products is for all $y \in [0, \bar{y}]$*

$$\begin{aligned} f(y) &= \frac{1}{\bar{y}} [\log(\bar{y}) - \log(y)] \text{ if } n = 2, \text{ and} \\ f(y) &= (\bar{y})^{-\frac{n}{2}} \left(\frac{n}{n-2}\right) \left[(\bar{y})^{\frac{n}{2}-1} - (y)^{\frac{n}{2}-1}\right] \text{ otherwise.} \end{aligned} \tag{20}$$

The density has a peak at $y = 0$, decreases in $y$, and reaches zero at $\bar{y}$. The shape depends on $n$. The density is convex in $y$ for $n = 1, 2, 3$, linear for $n = 4$, and concave for $n \geq 5$. This is intuitive, since the drift of the process for $y$ increases linearly with $n$, hence the mass accumulates closer to the upper bound $\bar{y}$ as $n$ increases. Indeed as $n \to \infty$ the distribution converges to a uniform in $[0, \bar{y}]$. Proposition 9 makes clear also that the shape of the invariant density depends exclusively on $n$, the value of the other parameters, $\psi, B, \sigma^2$ only enters in determining $\bar{y}$, which only stretches the horizontal axis proportionally.

## 5.3 Impact response in the $n \geq 2$ case

Now we turn to studying the economy-wide impact effect of the aggregate shock. To find out what is the fraction of firms that will adjust prices under the invariant we need to characterize some features of the invariant distribution of $p \in \mathbb{R}^n$. We assume that the aggregate shock happens once and for all, so that the price gap process remains the same and the firms solve the problem stated above. First we find out which firms choose to change prices and, averaging among their $n$ products, by how much. A firm with price gap $p \in \mathbb{R}^n$ and state $||p||^2 = y \leq \bar{y}$ before the shock, will have its price gaps displaced down by $\delta$ in each of its $n$ goods, i.e. its state immediately after the shock is $||p - 1_n \delta||$, where $1_n$ is a vector of ones. This firm will change its prices if and only if the state will fall outside the range of inaction, i.e. $||p - 1_n\delta|| \geq \bar{y}$, or equivalently if and only if:

$$||p||^2 - 2\delta\left(\sum_{i=1}^{n} p_i\right) + n\delta^2 \geq \bar{y} \quad \text{or} \quad \frac{\sum_{i=1}^{n} p_i}{\sqrt{y}} \leq \nu(y,\delta) \equiv \frac{y-\bar{y}}{2\delta\sqrt{y}} + n\frac{\delta}{2\sqrt{y}}\ . \tag{21}$$

Thus $\nu(y,\delta)$ gives the highest value for the sum of the $n$ price gaps for which a firm with state $y$ will adjust the price. The normalized sum of price gaps $\sum_{i=1}^{n} p_i/\sqrt{y}$ takes values on $[-\sqrt{n},\sqrt{n}]$. The right panel of Figure 5 shows the $n=2$ case by plotting a circle centered at zero that contains all the pre-shock price gap, and showing the "displaced" price gaps right after the $\delta$ shock, which are given by a circle centered at $(-\delta,\delta)$. The shaded area contains all the price gaps of the firms that, after the shock, will find it optimal to adjust their prices, i.e. firms for which equation (21) holds.

A firm whose price gap $p$ satisfies equation (21), i.e. one with $(1/\sqrt{y})\sum_{i=1}^{n} p_i \leq \nu(y,\delta)$, will change all its prices. The mean price change, averaging across its $n$ products, is $\delta - (1/n)\sum_{i=1}^{n} p_i$ .[14] Thus we can determine the fraction of firms that change its prices, and the amount by which they change them, analyzing the invariant distribution of the squared price gaps, $f(y)$. Let $S(z)$ denote the cumulative distribution function of the sum of the coordinates of the vectors distributed uniformly in the $n$ dimensional unit sphere. Formally we define $S:\mathbb{R}\to[0,1]$ as

$$S(z) = \frac{1}{L(\mathbb{S}^n)}\int_{x\in\mathbb{R}^n,||x||=1} \mathbf{I}\left\{x_1+x_2+...+x_n \leq z\right\} L(dx)\ .$$

where $\mathbb{S}^n$ is the $n-$dimensional sphere and where $L$ denotes its $n-1$ Lebesgue measure. Note that $S(\cdot)$ is weakly increasing, that $0 = S\left(-\sqrt{n}\right)$, $S(0)=1/2$, $S\left(\sqrt{n}\right)=1$ and that it is strictly increasing for $z\in(-\sqrt{n},\sqrt{n})$. Remarkably, as shown in Proposition 10, the distribution of the sum of the coordinates of a uniform random variable in the unit $n$-dimensional sphere is the same, up to a scale, than the marginal distribution of any of the coordinates of a uniform random variable in the unit $n$-dimensional sphere (which we discussed in Proposition 6), i.e. the c.d.f. satisfies:

$$S'(z) \equiv s(z) = \frac{1}{Beta\left(\frac{n-1}{2},\frac{1}{2}\right)\ \sqrt{n}}\left(1-\left(\frac{z}{\sqrt{n}}\right)^2\right)^{(n-3)/2} \quad \text{for} \quad z\in\left(-\sqrt{n},\sqrt{n}\right) \tag{22}$$

for $n\geq 2$, and for $n=1$ the c.d.f. $S$ has two points with mass $1/2$ at $-1$ and at $+1$. Now we are ready to give expressions for the effect of an aggregate shock $\delta$. First consider $\Phi_n$,

[14] Recall that $p_i$ are the price gaps, thus in order to set them to zero the price changes must take the opposite sign. Moreover, since $\delta$ has the interpretation of a cost increase, it decreases the price gap, and hence its correction requires a price increase.

the fraction of firms that adjust prices. There are $f(y)dy$ firms with state $y$ in the invariant distribution; among them the fraction $S(\nu(y,\delta))$ adjusts. Integrating across all the values of $y$ we obtain the desired expression. Second, consider $\Theta_n$, the change in the price level across all firms. There are $f(y)dy$ firms with state $y$ in the invariant distribution; among them we consider all the firms with normalized sum of price gaps less than $\nu(y,\delta)$, for which the fraction $s(z)dz$ adjust prices by $\delta - \sqrt{y}\ z/n$. Considering all the values of $y$ we obtain the relevant expression. This gives:

Proposition 10. Consider an aggregate shock of size $\delta$. The fraction of price changes on impact, $\Phi_n$, and the average price change across the $n$ goods among all the firms in the economy, $\Theta_n$, are given by:

$$\Phi_n(\delta) = \int_0^{\bar{y}} f(y)\ S\left(\nu(y,\delta)\right)\ dy \tag{23}$$

$$\Theta_n(\delta) = \delta\Phi_n(\delta) - \int_0^{\bar{y}} f(y)\frac{\sqrt{y}}{n}\left[\int_{-\sqrt{n}}^{\nu(y,\delta)} z\ s(z)dz\right] dy \tag{24}$$

where $s(\cdot)$ is given by equation (22) which depends on $n$, and where $f(\cdot)$ and $\nu(\cdot)$, which are also functions of $\bar{y}$ and $n$, are given in equation (20) and equation (21) respectively.

See Appendix A for the proof. Appendix E gives a closed form solution and the numerical evaluation of equation (23) and equation (24), and a lemma with the analytical characterization of $\Theta_n$ and $\Phi_n$ stated in part 3 of Proposition 8.

## 5.4 Impulse response at horizons $t > 0$ in the $n \geq 2$ case

We develop an expression for the impulse responses at horizon $t > 0$ for the general case of $n \geq 1$, in particular we derive an expression for the flow impact on the price level at horizon $t$ which we denote as $\theta_n(\delta, t)$. As in the case of one good, we start by describing the distribution of firms indexed by their price gaps, immediately after the monetary shock $\delta$ but before any adjustment takes place. The cdf $\Lambda_n(p_0)$ gives the fraction of price gaps smaller or equal to $p_0$ at time zero right after the impact adjustment caused by the monetary shock $\delta$. Note that $\Lambda_n(p_0) \leq \Theta_n(\delta)$ for all $p_0$. To understand this expression, let $\tilde{p}_0 \in \mathbb{R}^n$ be the price gap before the monetary shock, which has $y = ||\tilde{p}_0||^2$ distributed according to the density $f(y)$ described in equation (20). The price gaps with a given value of $||\tilde{p}_0||^2 \equiv y \leq \bar{y}$ have a uniform distribution on the sphere, so its density depends only of $||\tilde{p}_0||^2$, and integrates to the area of the sphere with square radius $y$. The surface area of this sphere is given by $2\pi^{n/2}y^{(n-1)/2}\Gamma(n/2)$. Right after the monetary shock these price gaps become $p_0 = \tilde{p}_0 - \mathbf{1}_n\delta$, where $\mathbf{1}_n$ is an $n$ dimensional vector of ones. So we have that the density of the distribution of the price gaps immediately after the monetary shock, but before any adjustment is

$$\lambda(p_0, \delta) = f\left(||p_0 + \mathbf{1}_n \delta||^2\right) \frac{\Gamma\left(n/2\right)}{2\,\pi^{n/2}\; ||p_0 + \mathbf{1}_n \delta||^{n-1}} \tag{25}$$

and recall that $f(y) = 0$ for any $y > \bar{y}$. We note that $\lambda$ is a function of $\bar{y}$ and $\delta$, but it is independent of $\sigma^2$.

The next step is to find the contribution of those firms with price gap $p_0$ to the change in aggregate prices at horizon $t$. As in the case of one good, it suffices to consider the contribution of those firms that have the first price change exactly at $t$. This is because all the subsequent adjustment have a zero net contribution to prices, since after the adjustment the firm start with a zero price gap. Since firms will adjust a price when the square radius of the vector of price gaps first reaches $\bar{y}$ at time $t$, we use the distribution of the corresponding hitting times and place in the sphere. In particular let $G(p; t, p_0)$, the probability that if a firm has a price gap $p_0$ at time zero, it will hit the surface of a sphere of radius $\sqrt{\bar{y}}$ at time $t$ or before, with a price gap smaller or equal than $p$. Note that $G$ is a function of $\sigma^2$ and $\bar{y}$ but it is independent of $\delta$. Explicit expressions for the joint density $g$ of the hitting time $t$ and place $p$ can be found in Wendel (1980) and Yin and Wang (2009). When the price gap of the firm hits the sphere of radius $\sqrt{\bar{y}}$ with a price gap $p$, the average change of its $n$ prices is given by "closing" each of the $n$ price gaps, i.e. the average price change is given by $-(p_1 + \ldots + p_n)/n$. Thus the contribution to the change in aggregate prices at time $t$ after a shock $\delta$ at time zero is given by

$$\theta_n(\delta, t) \;=\; \int_{||p_0||^2 \leq \bar{y}} \left[ \int_{||p||^2 = \bar{y}} \frac{-\left(p_1 + p_2 + \ldots + p_n\right)}{n}\, g(p, t, p_0)\, dp \right] \lambda(p_0, \delta)\, dp_0 \tag{26}$$

Note that the outer integral is computed only for the firms that have not adjusted on impact, i.e. for the price gaps $||p_0||^2 \leq \bar{y}$. Given the knowledge of the closed form expressions for both $\lambda$ and $g$ we can compute the multidimensional integrals in $\theta_n(\delta, t)$ by Monte Carlo.

We adapt the expression for the density $g$ of hitting times and places in Theorem 3.1 of Yin and Wang (2009) to the case of a BM with variance $\sigma^2$. Using the expression for the surface area of an $n$ dimensional sphere into equation (26) we obtain:

**Proposition 11.** *Fix $n \geq 2$, then the impulse response can be written as*

$$\mathcal{P}_n(\delta, t) - \Theta_n(\delta) \;=\; \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} e_{m,k}\left(\delta, \sqrt{\bar{y}}, n\right) \left[1 - \exp\left(-\frac{q_{m,k}^2}{2\,n} \frac{n\,\sigma^2}{\bar{y}} t\right)\right] \tag{27}$$

where the coefficients $q_{m,k}$ are the ordered (positive) zeroes of the Bessel function $J_{m+\frac{n}{2}-1}(\cdot)$. The coefficients $e_{m,k}(\cdot,\cdot,n)$ are functions homogeneous of degree one in $(\delta, \sqrt{\bar{y}})$ and do not depend on $\sigma^2$. Furthermore $\sum_{m=0}^{\infty}\sum_{k=1}^{\infty} e_{m,k}(\delta,\cdot,\cdot) = \delta - \Theta_n(\delta) \le \delta$.

See Appendix A for the proof. Using the properties of $\Theta_n$ from Proposition 10, and the homogeneity property of $e_{m,k}$ in equation (27) one verifies part 1 and part 2 of Proposition 8.

We end this section with a full characterization of the impulse response function in the limit case in which $n \to \infty$. The next proposition shows that when $n$ is large the impulse response is linear, identical to the one in the exogenous staggering model of Taylor (1980) and the model of Reis (2006), where the staggering emerges from the optimal choice of adjustment subject to costly information gathering.

Proposition 12. Assume that $\delta < Std(\Delta p)$. We will let $n \to \infty$ adjusting $\bar{y}$ and $\sigma^2$ to keep $Std(\Delta p) = \sqrt{\bar{y}/n}$ and $N_a$ fixed. Then the fraction of immediate adjusters $\Phi_n(\delta) \to (\delta/Std(\Delta p))^2$, the immediate impact in the price level $\Theta_n(\delta) \to \delta\Phi_n(\delta)$, and the impulse response becomes linear, i.e. $\mathcal{P}_n(\delta,t) \to \Theta_n(\delta) + \delta\, N_a\, t$ for $0 < t < T \equiv (1-\Phi_n(\delta))\,/\,N_a$ and $\mathcal{P}_n(\delta,t) \to \delta$ for $t \ge T$. Furthermore, the average price change across firms at every $0 \le t \le T$ is equal to $\delta$.

See Appendix A for the proof. The impact effect on the price level is of the order $\delta^3$, and hence for small values of $\delta$ it is negligible compared to the the impact for the $n = 1$ case, i.e. $\Theta_n/\Theta_1 \downarrow 0$ for $\delta \downarrow 0$, as shown in Figure 3. Moreover, Figure 4 shows that the half life of the shock is $\left(1/2 - (\delta/Std(\Delta p))^2\right)/N_a$, which converges to $1/(2N_a)$ for small shocks. In Figure 4, describing a 1% shock in the money supply, the half life is three times greater than the one produced by the $n = 1$ case. A main consequence of the large $n$ is that there is no selection effect. This is to be compared with the case of $n = 1$ where the selection effect is strongest and where, in the periods right after the shock (small $t$), all price adjustments are price increases. The reason for the lack of selection when $n$ is large is that for a firm selling many products there are, upon adjustment, many cancellations since some prices will be increased and others decreased, so that the average price change across the firms' good is simply $\delta$.

## 5.5 On the output effect of monetary shocks

This section discusses how the impulse response for prices are informative about the interpretation, time-profile and size of the output effect of a monetary shock. In the general equilibrium set-ups discussed at the beginning of this section the deviation of output from its steady state value is proportional to the deviation of the real balances, $\delta - \mathcal{P}_n(\delta,t)$, as shown in equation (16) and common to the models of Golosov and Lucas (2007); Caplin and Leahy (1997); Danziger (1999). From now on we refer to $\delta - \mathcal{P}_n(\delta, t)$ as to the impulse response of output, which is the expression predicted by our model in the case of log preferences ($\epsilon = 1$).[15]

The half life of the output response is identical to the half life of the price level only in the case in which $\Theta_n = 0$, i.e. when there is no jump of the price level on impact, a condition that holds for infinitesimal shocks. When the price level jumps on impact ($\Theta_n > 0$), the half-life of the output response is longer than the half-life of the price level. The reason is that the jump shortens the time required for the price level to reach its half-life value (i.e. $\delta/2$), whereas the half-life target of the real output effect, given by $(\,\delta - \Theta_n(\delta)\,)/2$ shifts, and so its half life is longer. To picture this effect in Figure 4, notice that different impact levels (corresponding to e.g. different values of $n$) do not shift the half-life line (whose position is at 0.5%), but will shift the half life line of the real-output effect (this line is not drawn in the figure, it is above 0.5 and shifts up as $\Theta_n$ increases). The impact effect on output also depends on the size of the shock: on one hand for very large shocks there is full price flexibility and hence no effect on output regardless of $n$; on the other hand for small monetary shocks the impact effect on prices is of order smaller than $\delta$, and hence the impact effect on output is approximately $\delta$ for all values of $n$.

As a summary statistic of the real effect of monetary shock we use the area under the impulse response for output, i.e.

$$\mathcal{M}_n(\delta) = \int_0^\infty \left(\delta - \mathcal{P}_n(\delta, t)\right)\, dt$$

which can be interpreted as the cumulative effect on output following the shock. This measure combines the size of the output deviations from the steady state with the duration of these deviations. Since $\mathcal{P}_n(\delta, t)$ depends only on the parameters $Std(\Delta p)$ and $N_a$, so does $\mathcal{M}_n(\delta)$. Because of the homogeneity of $\mathcal{P}_n(\delta, t)$ discussed in part 2 of Proposition 8, and the way time $(N_a\ t)$ enters $\mathcal{P}_n(\delta, t)$ shown in Proposition 11, we can thus write

$$\mathcal{M}_n\,(\,\delta\,;\,N_a\,,\,Std(\Delta p)\,) = \frac{Std(\Delta p)}{N_a}\,\mathcal{M}_n\left(\frac{\delta}{Std(\Delta p)}\,;\,1\,,\,1\right) \tag{28}$$

so that the effect of a shock of size $\delta$ in an economy characterized by parameters $Std(\Delta p)$ and $N_a$ can be readily computed using the "normalized" effect for an economy with unit parameters and a standardized shock.

The determinants of the real effects of monetary shocks identified by equation (28) offer a new insight to measure the degree of aggregate price stickiness in menu cost models. The

[15] If $\epsilon \neq 1$ then the effect on output should be divided by $\epsilon$, as shown in equation (16).

previous literature has focused almost exclusively on the frequency of price changes, $N_a$, as a measure of stickiness, and hence of the effect of monetary policy. But equation (28) shows that the dispersion of price changes, $Std(\Delta)$, is an equally important determinant. Indeed the area under the impulse response of output is proportional to the ratio of these two quantities, where the constant of proportionality depends on the (normalized) size of the monetary shock, $\delta / Std(\Delta p)$ and, in our set-up, on the number of products $n$.

Figure 6: Cumulative Output effect $\mathcal{M}(\delta)$ ; Parameters: $N_a = 1$ , $Std(\Delta p) = 0.10$

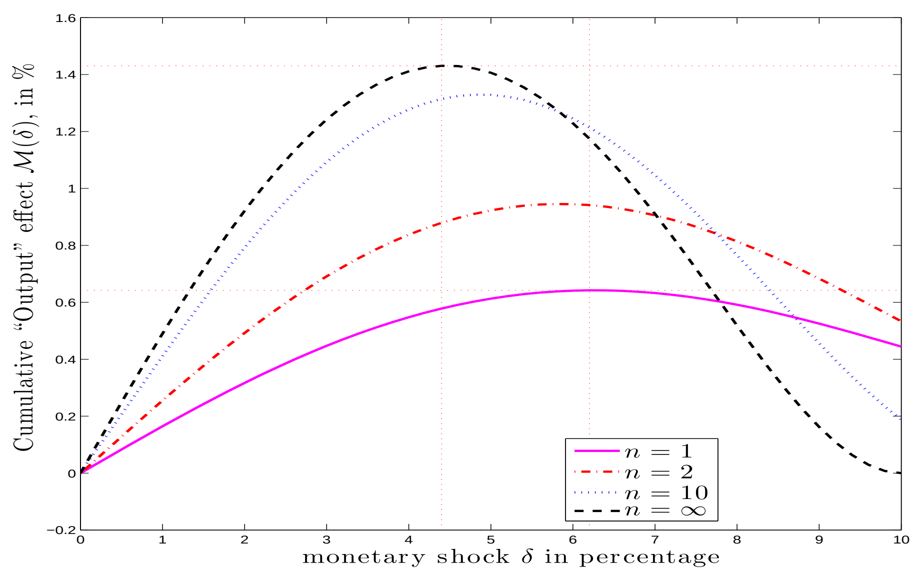



Figure 6 illustrates how the real output effect of a monetary shock varies with the size of the shock ($\delta$) and the number of goods sold by the firm ($n$). The figure plots the summary impact measure as a function of $\delta$ for an economy with $Std(\Delta p) = 0.10$ and $N_a = 1$, for four values of $n$. It is shown that for each value of $n$ the cumulative real effect of a monetary shock is hump-shaped in the size of the shock ($\delta$). The effect is nil at extremes, i.e. at $\delta = 0$ and at $2\ Std(\Delta p)$ (not shown), as a reflection of the fact that large shocks induce full price flexibility (see part 3 of Proposition 8). We characterize the value of $\delta$ for which $\mathcal{M}$ is maximized using equation (28) and the Figure 6. For a given $n$ the monetary shock that maximizes the cumulative output effect is about one half of $Std(\Delta p_i)$. Moreover, the maximum value of the cumulated output effect is proportional to $Std(\Delta p_i)/N_a$. More interestingly, for the purpose of this paper, the size of the real effects varies with the number of goods $n$. Larger values of $n$, i.e. firms selling more goods, produce larger cumulative effects for small values of the shock and also larger maximum values of the effect. In this sense the stickiness of the economy is increasing in $n$. The maximum cumulative effect on output, in the order of 1.4% output points, is obtained as $n \to \infty$, a similar value though obtains already for $n = 10$. On the other hand, smaller effects are produced in models with $n = 1$ or $n = 2$.

# 6 Correlation, inflation, and differential elasticities

This section extends the baseline model to the case of price gaps $p_i, p_j$ that feature: $(i)$ cross-correlated innovations $(ii)$ a common drift (to model inflation) $(iii)$ cross products $p_i\, p_j$ in the period return function of the firm (to model a demand system where the elasticity of substitution of products within the firm differs from the substitution elasticity of a product across firms). Surprisingly, despite the apparent complexity of these extensions, the modified problem remains tractable: instead of the scalar state variable $y$, the state of the problem with either drift, correlation, and/or cross products, includes *only* one additional variable measuring the sum of the coordinates of the vector, namely $z = \sum_{i=1}^{n} p_i$ for any $n$. For ease of exposition, and because it turns out to be the one with more substantial effects, we focus here on the problem with correlated price gaps but without drift and cross products. We formulate the problem, derive its cross section implications and characterize the impulse response to a monetary shock. A fuller derivation of the results for correlation and its computations, as well as a derivation and analysis of the differential elasticity case is given in Appendix F.

The problem solved by the firm is, as before, the minimization of the value function in equation (1), subject to a law of motion for the $p_i$ that allows for correlation but no drift. The diffusions for the price gaps satisfy: $\mathbb{E}\left[\mathrm{d}p_i^2(t)\right] = \hat{\sigma}^2\,\mathrm{d}t$ and $\mathbb{E}\left[\mathrm{d}p_i(t)\mathrm{d}p_j(t)\right] = \rho\,\hat{\sigma}^2\,\mathrm{d}t$ for all $i = 1, ..., n$ and $j \neq i$ and for two positive constants $\hat{\sigma}^2$ and $\rho$. Then we can write that each price gap follows $\mathrm{d}p_i(t) = \bar{\sigma}\,\mathrm{d}\bar{\mathcal{W}}(t) \; + \sigma \mathrm{d}\mathcal{W}_i(t)$ for all $i = 1, ..., n$ where $\bar{\mathcal{W}}, \mathcal{W}_i(t)$ are independent standard BMs so that $\hat{\sigma}^2 = \bar{\sigma}^2 + \sigma^2$ and the correlation parameter is $\rho = \frac{\bar{\sigma}^2}{\bar{\sigma}^2+\sigma^2}$. Define: $y(t) = \sum_{i=1}^{n} p_i^2(t)$ and $z(t) = \sum_{i=1}^{n} p_i(t)$, which by Ito's lemma obey the diffusions

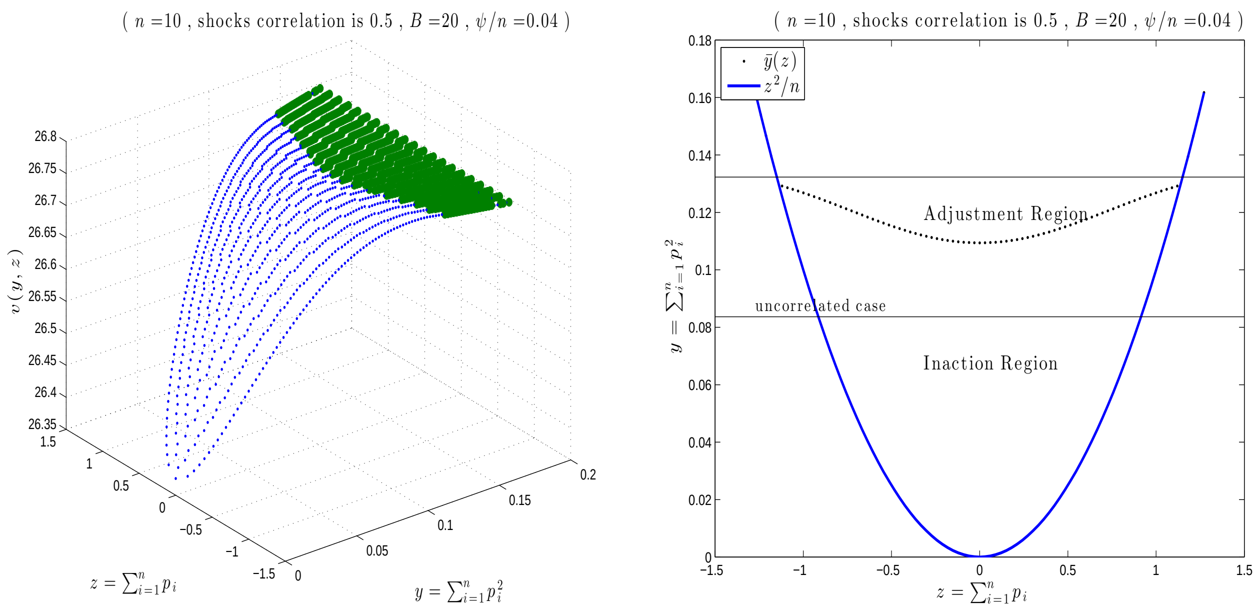
$$
\begin{aligned}
\mathrm{d}y(t) &= n\left[\sigma^2 + \bar{\sigma}^2\right]\mathrm{dt} + 2\sigma\sqrt{y(t)}\;\mathrm{d}\mathcal{W}^a(t) + 2\bar{\sigma}z(t)\;\mathrm{d}\mathcal{W}^c(t) \\
\mathrm{d}z(t) &= n\,\bar{\sigma}\;\mathrm{d}\mathcal{W}^c(t) + \sqrt{n}\sigma\left[\frac{z(t)}{\sqrt{n\,y(t)}}\;\mathrm{d}\mathcal{W}^a(t) + \sqrt{1 - \left(\frac{z(t)}{\sqrt{n\,y(t)}}\right)^2}\;\mathrm{d}\mathcal{W}^b(t)\right]
\end{aligned}
$$

where $(\mathcal{W}^a, \mathcal{W}^b, \mathcal{W}^c)$ are three standard (univariate) independent BM's. Notice that the introduction of correlation makes the variance of $y$ depend on the level of $z$.

In the case where $\sigma$ and $\bar{\sigma}$ are both positive the state of this problem will be the pair $(y, z)$ and the value function, denoted by $v(y, z)$, is symmetric in $z$ around zero, so $v(y, z) =$

$v(y,-z)$. The optimal policy is to have an inaction region $\mathcal{I} = \{(y,z) : 0 \leq \bar{y}(z)\}$ for some function $\bar{y}(z)$ satisfying $\bar{y}(z) = \bar{y}(-z) > 0$ for al $z > 0$. We solve $v(y,z)$ numerically for a problem with $r = 0.05$ per year, $B = 20$, and a volatility of each price gap of 13% with a pair-wise correlation of 1/2, so $\sigma = \bar{\sigma} = 0.13/\sqrt{2}$. The menu cost is 4% of friction-less profits per good, so $\psi/n = 0.04$. We display the results for the case of $n = 10$ products per firm.

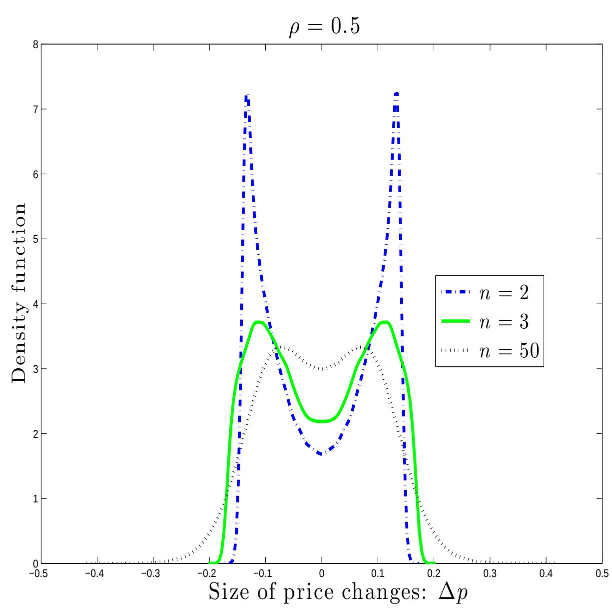
Figure 7: Value function and decision rules with correlated shocks: $\rho = 0.5$

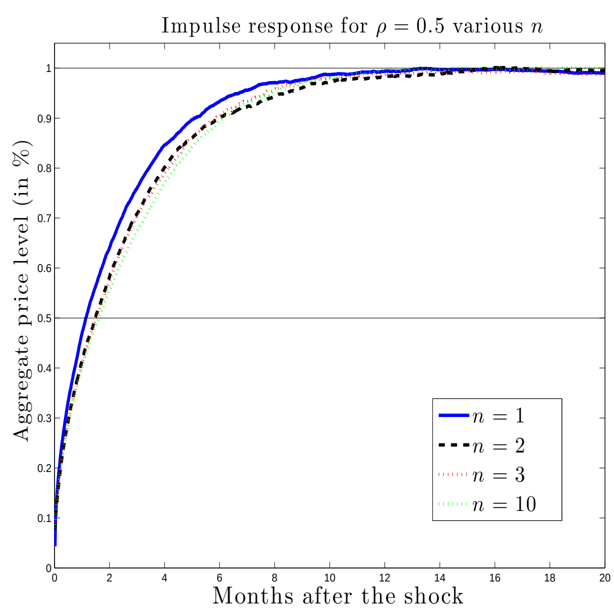



The left panel of Figure 7 plots the value function over its $(y,z)$ domain. The value function region where control (i.e. price adjustment) is optimal is marked by green stars. The feasible state space for the firm is the $y,z$ region inside the parabola in the right panel of the figure. For each $z$ the shape of the value function is similar to the case with no correlation. Fixing $y$, the value function is decreasing in $|z|$. This is because a higher $|z|$ implies a higher conditional variance of $y$, and hence a higher option value. Because of the higher option value the threshold $\bar{y}(z)$ is increasing in $|z|$. While the inaction set is two-dimensional we emphasize that the state of the problem is $n$ dimensional: for instance in the figure $n = 10$.

We use the decision rule described above to produce the invariant distribution of a cross section of firms using simulations. The model parameterization is close to the one used in the main body of the paper, i.e. it produces a frequency of adjustments per year that is $N_a = 1.3$ and a standard deviation $Std(\Delta p_i) = 0.11$. The left panel of Figure 8 plots the standardized distribution of price changes $w(\Delta p_i)$ for $n = 2, 3, 50$ when the correlation between the shocks is $\rho = 0.5$. The key effect of correlation is to increase the mass of price changes with similar sign, i.e. to move mass from the center of the distribution towards both sides. Not surprisingly, adding correlation makes the model closer to the $n = 1$ case, a feature that is important for both its empirical plausibility (i.e. the comparison with empirical distribution of price changes) and for the predicted effect of monetary shocks. The case of $n = 3$ is particularly revealing since for zero correlation the distribution is uniform, but as the correlation is positive the density becomes U shaped, with a minimum at zero and two maxima at a high values of the absolute value, as in the case of $n = 1$. The case of $n = 50$ is also informative because with zero correlation this distribution is essentially normal. However with positive correlation the distribution of price changes becomes bimodal, with a local minimum of its density at zero. Interestingly the simultaneous near normality and bimodality (i.e. a small dip of the density around the center of the support) which is displayed by the $n = 50$ case with correlation, is apparent in several data sets such as Midrigan (2009) (see his Figure 1, bottom two panels), Wulfsberg (2010) (see his Figure 4), and has been explicitly tested and estimated by Cavallo and Rigobon (2010).

Figure 8: The aggregate economy with correlated shocks: $\rho = 0.5$




We conclude with the analysis of the price level response to a once and for all shock to the money supply in the presence of correlated shocks. We stress that to solve for the IRF for any $n$ we only need to keep track of a two-dimensional object. This makes the procedure computationally feasible. We assume a correlation between shocks equal to $\rho = 1/2$ for four economies with $n = 1, 2, 3, 10$. These economies are observationally equivalent in the steady state in terms of the price adjustment frequency $N_a$ and standard deviation $Std(\Delta p)$.[16] The impulse responses for $n = 2, 3, 10$, displayed in Figure 8, show that introducing correlation significantly increases the price flexibility at all horizons: all impulse responses are now very close the the one produced by the model with $n = 1$. Contrast this outcome to the one that was obtained with no correlation in Figure 4. The intuition for this result is simple: introducing correlation increases the mass of "large" price changes, as was explained above. This effect brings back the "selection" effect that was being muted as $n$ got large in models with uncorrelated shocks.

# 7 Concluding remarks

This paper presented a stylized model of price setting that substantially improves the cross-sectional predictions of menu cost models in comparison to the patterns that characterize the micro data. For instance, the model is able to produce a substantial mass of small price changes and a bell-shaped size distribution of price changes. The analytical tractability of the model allowed us to derive a full characterization of the steady state predictions as well as of the economy's aggregate response to a once and for all unexpected monetary shock which we summarized in the Introduction.

Several simplifying assumptions were key in obtaining analytical results. In particular, our solution for the firm's decision problem used a second order approximation of the profit function and assumed no drift in the price gaps.[17] Moreover the impulse response functions were computed using the steady state decision rules i.e. ignoring the general equilibrium feedback effect. The paper discussed several extensions of the basic model allowing for drift (in e.g. inflation or aggregate productivity) as well as correlated shocks among the different goods, showing that the model retains a great deal of tractability.[18] We showed in Section 6 that correlation among the shocks tends to reinforce the selection effect, so that the real effect of monetary policy becomes smaller as correlation increases. Moreover we extensively explored the robustness of our analytical results compared to the ones produced by models that feature an asymmetric profit function, the presence of drift, and that account

[16]Motivated by the scaling and stretching results of Proposition 8 we normalize the parameters so that the expected number of price changes per year is 1 ($N_a = 1$) and consider a shock of 10% of the size of the steady state standard deviation of price changes (i.e. say $\delta = 0.01$ and $Std(\Delta p) = 0.1$, i.e. a one percent change in money supply and a 10% steady state standard deviation of price changes).

[17]In Alvarez, Lippi, and Paciello (2011) we proved that the zero inflation assumption provide a good approximation to the true rules for inflation rates that are small relative to the variance of idiosyncratic shocks, an assumption that seems appropriate for developed economies (see Gagnon (2009); Alvarez et al. (2011) for related evidence).

[18]We thank Ricardo Caballero, and especially Carlos Carvalho, for suggesting a two-dimensional state space representation that allowed a tractable analysis of the problem with drift.

for the general equilibrium feedback on decision rules following the aggregate shock. These investigations, reported in Appendix C, show that the approximate results obtained in the paper provide very accurate predictions of the exact numerical solution produced by models.

We think that several extensions are interesting for future research. One feature of the data that our model misses concerns the kurtosis of price changes. In the model the maximum level of kurtosis for the distribution of price changes predicted by the model is 3, as in the Normal distribution. This value is larger than the prediction of the classical Barro's (1972) or Dixit's (1991) menu cost models (where kurtosis is 1), but it is still small compared to the large excess kurtosis detected in micro datasets. Larger values of the kurtosis can be obtained by introducing the possibility of random adjustment opportunities, as in models where the size of the menu cost is stochastic. We explore this problem in Alvarez, Le Bihan, and Lippi (2012) and show that this assumption improves the empirical fit of the model cross section to the micro data and that it increases the real effect of a monetary shock by reducing the "selection" effect. Another interesting extension concerns the role of the linear production function (and no capital). The precision of our approximate solution benefited from this assumption since the firm's "optimal prices" did not depend on the level of the aggregate consumption. Instead, if production features decreasing returns to scale, the strength of the "pricing complementarities" increases, i.e. the optimal price depends on the aggregate consumption. We leave it for the future to explore the quantitative importance of this alternative assumption.

# References


Alvarez, Fernando, Martin Gonzalez-Rozada, Andres Neumeyer, and Martin Beraja. 2011. "From Hyperinflation to Stable Prices: Argentina's evidence on menu cost models." manuscript, University of Chicago.

Alvarez, Fernando E., Herve Le Bihan, and Francesco Lippi. 2012. "Sticky prices, menu costs and the effect of monetary shocks: theory and micro-evidence." in preparation, EIEF.

Alvarez, Fernando E. and Francesco Lippi. 2012. "Price setting with menu cost for multi-product firms." NBER Working Papers 17923, National Bureau of Economic Research, Inc.

Alvarez, Fernando E., Francesco Lippi, and Luigi Paciello. 2011. "Optimal price setting with observation and menu costs." *The Quarterly Journal of Economics* 126 (4):1909–1960.

———. 2012. "Monetary Shocks with Observation and Menu Costs." Tech. rep., University of Chicago mimeo.

Barro, Robert J. 1972. “A Theory of Monopolistic Price Adjustment.” *Review of Economic Studies* 39 (1):17–26.

Baudry, L., H. Le Bihan, P. Sevestre, and S. Tarrieu. 2007. “What do Thirteen Million Price Records have to Say about Consumer Price Rigidity?” *Oxford Bulletin of Economics and Statistics* 69 (2):139–183.

Bertola, Giuseppe and Ricardo J Caballero. 1994. “Irreversibility and Aggregate Investment.” *Review of Economic Studies* 61 (2):223–46.

Bhattarai, Saroj and Raphael Schoenle. 2010. “Multiproduct Firms and Price-Setting: Theory and Evidence from U.S. Producer Prices.” Working papers, Princeton University.

Bonomo, Marco and Carlos Carvalho. 2004. “Endogenous Time-Dependent Rules and Inflation Inertia.” *Journal of Money, Credit and Banking* 36 (6):1015–41.

Caballero, Ricardo J. 1989. “Time Dependent Rules, Aggregate Stickiness And Information Externalities.” Discussion Papers 198911, Columbia University.

Caballero, Ricardo J. and Eduardo M.R.A. Engel. 1991. “Dynamic (S, s) Economies.” *Econometrica* 59 (6):1659–86.

———. 1993. “Heterogeneity and output fluctuations in a dynamic menu-cost economy.” *The Review of Economic Studies* 60 (1):95.

———. 2007. “Price stickiness in Ss models: New interpretations of old results.” *Journal of Monetary Economics* 54 (Supplement):100–121.

Caplin, Andrew and John Leahy. 1997. “Aggregation and Optimization with State-Dependent Pricing.” *Econometrica* 65 (3):601–626.

Cavallo, Alberto. 2010. “Scraped Data and Sticky Prices.” Tech. rep., MIT Sloan.

Cavallo, Alberto and Roberto Rigobon. 2010. “The distribution of the Size of Price Changes.” Working papers, MIT.

Ciesielski, Z. and SJ Taylor. 1962. “First passage times and sojourn times for Brownian motion in space and the exact Hausdorff measure of the sample path.” *Transactions of the American Mathematical Society* 103 (3):434–450.

Danziger, Leif. 1999. “A Dynamic Economy with Costly Price Adjustments.” *American Economic Review* 89 (4):878–901.

Dhyne, Emmanuel and Jerzy Konieczny. 2007. “Temporal Distribution of Price Changes: Staggering in the Large and Synchronization in the Small.” Working Paper Series 01-07, Rimini Centre for Economic Analysis.

Dixit, Avinash. 1991. “Analytical Approximations in Models of Hysteresis.” *Review of Economic Studies* 58 (1):141–51.

Dutta, Shantanu, Mark Bergen, Daniel Levy, and Robert Venable. 1999. "Menu Costs, posted prices and multiproduct retailers." *The Journal of Money, Credit and Banking* 31 (4):683–703.

Eichenbaum, Martin S., Nir Jaimovich, Sergio Rebelo, and Josephine Smith. 2012. "How Frequent Are Small Price Changes?" NBER Working Paper 17956, National Bureau of Economic Research.

Gagnon, Etienne. 2009. "Price Setting During Low and High Inflation: Evidence from Mexico." *Quarterly Journal of Economics* 124 (3):1221–1263.

Gertler, Mark and John Leahy. 2008. "A Phillips Curve with an Ss Foundation." *Journal of Political Economy* 116 (3):533–572.

Goldberg, Pinelopi and Rebecca Hellerstein. 2009. "How Rigid Are Producer Prices?" Working Papers 1184, Princeton University, Department of Economics, Center for Economic Policy Studies.

Golosov, Mikhail and Robert E. Jr. Lucas. 2007. "Menu Costs and Phillips Curves." *Journal of Political Economy* 115:171–199.

Hethcote, H.W. 1970. "Bounds for zeros of some special functions." *Proceedings of the American Mathematical Society* :72–74.

Karlin, S. and H.M. Taylor. 1981. *A second course in stochastic processes*, vol. 2. Academic Pr.

Khokhlov, VI. 2006. "The Uniform Distribution on a Sphere in $R^S$. Properties of Projections. I." *Theory of Probability and its Applications* 50:386.

Klenow, Peter J. and Benjamin A. Malin. 2010. "Microeconomic Evidence on Price-Setting." In *Handbook of Monetary Economics*, *Handbook of Monetary Economics*, vol. 3, edited by Benjamin M. Friedman and Michael Woodford, chap. 6. Elsevier, 231–284.

Kolkiewicz, A.W. 2002. "Pricing and hedging more general double-barrier options." *Journal of Computational Finance* 5 (3):1–26.

Lach, Saul and Daniel Tsiddon. 1992. "The Behavior of Prices and Inflation: An Empirical Analysis of Disaggregated Price Data." *The Journal of Political Economy* 100 (2):349–389.

———. 1996. "Staggering and Synchronization in Price-Setting: Evidence from Multiproduct Firms." *The American Economic Review* 86 (5):1175–1196.

———. 2007. "Small price changes and menu costs." *Managerial and Decision Economics* 28 (7):649–656.

Midrigan, Virgiliu. 2009. "Menu Costs, Multi-Product Firms, and Aggregate Fluctuations." Working paper, NYU.

———. 2011. "Menu Costs, Multi-Product Firms, and Aggregate Fluctuations." *Econometrica,* 79 (4):1139–1180.

Øksendal, Bernt K. 2000. *Stochastic differential equations: an introduction with applications.* Sixth Edition, Springer Verlag.

Reis, Ricardo. 2006. "Inattentive producers." *Review of Economic Studies* 73 (3):793–821.

Sheshinski, Eytan and Yoram Weiss. 1992. "Staggered and Synchronized Price Policies under Inflation: The Multiproduct Monopoly Case." *Review of Economic Studies* 59 (2):331–59.

Song, D. and A. K. Gupta. 1997. "Lp-Norm Uniform Distribution." *Proceedings of the American Mathematical Society* 125 (2):595–601.

Stokey, Nancy L. 2008. *Economics of Inaction: Stochastic Control Models with Fixed Costs.* Princeton University Press.

Taylor, John B. 1980. "Aggregate Dynamics and Staggered Contracts." *Journal of Political Economy* 88 (1):1–23.

Tsiddon, D. 1993. "The (mis) behaviour of the aggregate price level." *The Review of Economic Studies* 60 (4):889.

Wendel, J.G. 1980. "Hitting spheres with Brownian motion." *The annals of probability* 8 (1):164–169.

Wulfsberg, F. 2010. "Price adjustments and inflation-evidence from Norwegian consumer price data 1975-2004." Working paper, Bank of Norway.

Yin, C. and C. Wang. 2009. "Hitting Time and Place of Brownian Motion with Drift." *The Open Statistics and Probability Journal* 1:38–42.

# A Proofs

**Proof.** (of Proposition 2 ) Notice that $v'(0) = \beta_1$ and that $v(0) = \beta_0$, so that we require $\beta_1 > 0$, which implies $\beta_0 > 0$. Moreover, if $\beta_1 > B/r$ then $v$ is strictly increasing and strictly convex. If $\beta_1 = B/r$ then $v$ is linear in $y$. If $0 < \beta_1 < B/r$, then $v$ is strictly increasing at the origin, strictly concave, and it reaches its unique maximum at a finite value of $y$. Thus, a solution that satisfies smooth pasting requires that $0 < \beta_1 < B/r$, and the maximizer is $\bar{y}$. In this case, $y = 0$ achieves the minimum in $[0, \bar{y}]$. Thus we have verified i), ii) and iii).

Next we prove uniqueness. Let $\beta_i(\beta_1)$ be the solution of equation (10), as a function of $\beta_1$. Note that for $0 < \beta_1 < B/r$, all the $\beta_i(\beta_1) < 0$ for $i \geq 2$ and are increasing in $\beta_1$, converging to zero as $\beta_1$ goes to $B/r$. Smooth pasting can be written as $0 = v'(\bar{y}; \beta_1) \equiv \sum_{i=1}^{\infty} \; i \; \beta_i(\beta_1) \; \bar{y}^{i-1}$ where the notation emphasizes that all the $\beta_i$ can be written as a function of $\beta_1$. From the properties of the $\beta_i(\cdot)$ discussed above it follows that we can write the unique solution of $0 = v'(\bar{\rho}(\beta_1); \beta_1)$ as a strictly increasing function of $\beta_1$, i.e. $\bar{\rho}'(\beta_1) > 0$. The value matching condition at $\bar{y}$ gives: $\psi = v(\bar{y}, \beta_1) - v(0, \beta_1) = v(\bar{y}, \beta_1) - \beta_0(\beta_1) = \sum_{i=1}^{\infty} \; \beta_i(\beta_1) \; \bar{y}^i$. We note that, given the properties of $\beta_i(\cdot)$ discussed above, for *any given* $y > 0$ we have that $v(y, \beta_1) - \beta_0(\beta_1)$ is strictly increasing in $\beta_1$, as long as $0 < \beta_1 < B/r$. Thus, define

$$\Psi(\beta_1) \quad = \quad v(\bar{\rho}(\beta_1), \beta_1) - v(0, \beta_1) = \sum_{i=1}^{\infty} \beta_i(\beta_1)\ \bar{\rho}(\beta_1)^i\ .$$

From the properties discussed above we have that $\Psi(\beta_1)$ is strictly increasing in $\beta_1$ and that it ranges from 0 to $\infty$ as $\beta_1$ ranges from 0 to $B/r$. Thus $\Psi$ is invertible. The solution of the problem is given by setting: $\beta_1(\psi) = \Psi^{-1}(\psi)$ and $\bar{y}(\psi) = \bar{\rho}(\beta_1(\psi))$. □

**Proof.** (of Proposition 3 ) Using the expression for $\{\beta_i\}$ obtained in Proposition 1, the value matching and smooth pasting conditions can be written as two equations in $\beta_2$ and $\bar{y}$:

$$\frac{\psi}{\bar{y}^2} \quad = \quad \frac{B}{r\bar{y}} + \beta_2 \left[ \frac{2\sigma^2(n+2)}{r\bar{y}} + 1 + \sum_{i=1}^{\infty} \kappa_i\ r^i\ \bar{y}^i \right]\ ,\ 0 = \frac{B}{r\bar{y}} + \beta_2 \left[ \frac{2\sigma^2(n+2)}{r\bar{y}} + 2 + \sum_{i=1}^{\infty} \kappa_i\ (i+2)\ r^i\ \bar{y}^i \right]$$

where $\kappa_i = r^{-i} \frac{\beta_{2+i}}{\beta_2} = \prod_{s=1}^{i} \frac{1}{\sigma^2(s+2)(n+2s+2)}$. This gives an implicit equation for $\bar{y}$:

$$\psi \quad = \quad \frac{B}{r}\bar{y} \left[ 1 - \frac{\frac{2\sigma^2(n+2)}{r\bar{y}} + 1 + \sum_{i=1}^{\infty} \kappa_i\ r^i\ \bar{y}^i}{\frac{2\sigma^2(n+2)}{r\bar{y}} + 2 + \sum_{i=1}^{\infty} \kappa_i\ (i+2)\ r^i\ \bar{y}^i} \right]$$

Since the right hand side is strictly increasing in $\bar{y}$, and goes from zero to infinity, then we obtain Part (i). Since the right hand side is strictly decreasing in $n$, and goes to zero as $n \to \infty$, then we obtain Part (ii). Rearranging this equation and defining $z = \bar{y}\ r/\sigma^2$

$$\frac{\psi\ 2(n+2)}{B\sigma^2}\ r^2 = z^2 + z^3 \left[ \frac{2(n+2) \sum_{i=1}^{\infty} \omega_i\ (i+1)\ z^{i-1} - 2 - \sum_{i=1}^{\infty} \omega_i\ (i+2)\ z^i}{2(n+2) + 2z + z \sum_{i=1}^{\infty} \omega_i\ (i+2)\ z^i} \right] \tag{29}$$

where $\omega_i = \prod_{s=1}^{i} \frac{1}{(s+2)(n+2s+2)}$. Using the expression for $\omega_i$ and collecting terms on $z^i$ one can show that the square bracket of equation (29) that multiplies $z^3$ is negative, and hence $\bar{y} > \sqrt{\psi\ 2(n+2)\sigma^2/B}$. Letting $b = \psi r^2 2(n+2)/(B\sigma^2)$ we can write equation (29) as:

$$1 = \frac{z^2}{b} \left( 1 + z \left[ \frac{2(n+2) \sum_{i=1}^{\infty} \omega_i\ (i+1)\ z^{i-1} - 2 - \sum_{i=1}^{\infty} \omega_i\ (i+2)\ z^i}{2(n+2) + 2z + z \sum_{i=1}^{\infty} \omega_i\ (i+2)\ z^i} \right] \right)$$

Since $z \downarrow 0$ as $b \downarrow 0$, then $z^2/b \downarrow 1$ as $b \downarrow 0$, establishing Part (iii). From equation (29) it is clear that the optimal threshold satisfies $\bar{y} = \frac{\sigma^2}{r} Q\left(\frac{\psi}{B\sigma^2}\ r^2, n\right)$. Differentiating this expression we obtain Part (iv). □

**Proof.** (of Proposition 5 ) The proof uses probability theory results on the first passage time of an $n$-dimensional brownian motion. Let $\tau$ be the stopping time defined by the first time when $||p(\tau)||^2$ reaches the critical value $\bar{y}$, starting at $||p(0)|| = 0$ at time zero. Let $S_n(t, \bar{y})$ be the probability distribution for times $t \geq \tau$, alternatively let $S_n(\cdot, \bar{y})$ be the survival function.

Theorem 2 in Ciesielski and Taylor (1962) shows that for $n \geq 1$:

$$S_n(t, \bar{y}) = \sum_{k=1}^{\infty} \xi_{n,k} \exp\left(-\frac{q_{n,k}^2}{2\bar{y}}\ \sigma^2\ t\right)\ , \text{ where } \xi_{n,k} = \frac{1}{2^{\nu-1}\Gamma(\nu+1)} \frac{q_{n,k}^{\nu-1}}{J_{\nu+1}(q_{n,k})}$$

where $J_\nu(z)$ is the Bessel function of the first kind, $\nu = (n-2)/2$ and $q_{n,k}$ are the positive zeros of $J_\nu(z)$, indexed in ascending order according to $k$, and where $\Gamma$ is the gamma function. The hazard rate is then given by:

$$h_n(t, \bar{y}) = -\frac{1}{S_n(t,\bar{y})} \frac{\partial S_n(t,\bar{y})}{\partial t}, \text{ with asymptote } \lim_{t\to\infty} h_n(t,\bar{y}) = \frac{q_{n,1}^2\ \sigma^2}{2\ \bar{y}}\ .$$

For $n > 2$ Hethcote (1970) provides the lower bound: $q_{n,k}^2 > \left(k - \frac{1}{4}\right)^2 \pi^2 + \left(\frac{n}{2} - 1\right)^2$ . □

**Proof.** (of Proposition 6 ) We first establish the following Lemma.

Lemma 1. Let $z$ be distributed uniformly on the surface of the $n$-dimensional sphere of radius one. We use $x$ for the projection of $z$ in any of the dimension, so $z_i = x \in [-1, 1]$. The marginal distribution of $x = z_i$ has density:

$$f_n(x) = \int_0^\infty \frac{s^{(n-3)/2} e^{-s/2}}{2^{(n-1)/2}\ \Gamma[(n-1)/2]} \frac{e^{-sx^2/[2(1-x^2)]}}{\sqrt{2\pi}} \frac{s^{1/2}}{(1-x^2)^{3/2}}\ ds = \frac{\Gamma(n/2)}{\Gamma(1/2)\ \Gamma[(n-1)/2]} (1-x^2)^{(n-3)/2}$$

where the $\Gamma$ function makes the density integrate to one.

The lemma applies Theorem 2.1, part 1 in Song and Gupta (1997) using $p = 2$ so that the norm is Euclidian and $k = 1$ so that we have the marginal of one dimension. Now consider the case where the sphere has radius different from one. Let $p \in \partial\mathcal{I}$, then $p = \frac{p}{\sum_{i=1}^n p_i^2} \bar{y} = \frac{p}{\sqrt{\sum_{i=1}^n p_i^2}}\ \sqrt{\bar{y}} = z\ \sqrt{\bar{y}}$ where $z$ is uniformly distributed in the $n$ dimensional sphere of radius one. Thus each $p_i$ has the same distribution than $x\ \sqrt{\bar{y}}$. Using the change of variable formula we obtain the required result. Some algebra using equation (14) for the density $w(\cdot)$, gives the expressions for the standard deviation, kurtosis, and the other moments in the proposition.

For the convergence of $\Delta p_i / Std(\Delta p_i)$ to a Normal, we show that $y = x^2\ n$ converges to a chi-square distribution with 1 d.o.f., where $x$ is the marginal of a uniform distribution in the surface of the $n$-dimensional sphere. The p.d.f of $y \in [0, n]$, the square of the standardized $x$, is $\frac{\Gamma\left(\frac{n}{2}\right)}{n\ \Gamma\left(\frac{n-1}{2}\right)\ \Gamma\left(\frac{1}{2}\right)} \left(1 - \left(\frac{y}{n}\right)\right)^{(n-3)/2} \left(\frac{y}{n}\right)^{-1/2}$, and the p.d.f. of a chi-square with 1 d.o.f. is $\frac{\exp(-y/2)\ y^{-1/2}}{\sqrt{2}\ \Gamma\left(\frac{1}{2}\right)}$ . Then, fixing $y$, taking logs in the ratio of the two p.d.f.'s, and taking the limit as $n \to \infty$, using that $\frac{\Gamma\left(\frac{n}{2}\right)\ \sqrt{2}}{\Gamma\left(\frac{n-1}{2}\right)\ \sqrt{n}} \to 1$ as $n \to \infty$ we obtain the desired convergence result. □

**Proof.** (of Proposition 9 ) The forward Kolmogorov equation is:

$$0 = \frac{1}{2} \frac{\partial^2}{\partial y^2} \left( [2\sigma\sqrt{y}]^2\ f(y) \right) - \frac{\partial}{\partial y} \left( n\sigma^2 f(y) \right) \quad \text{for } y \in (0, \bar{y})\ , \tag{30}$$

with boundary conditions: $1 = \int_0^{\bar{y}} f(y)\ dy$ and $f(\bar{y}) = 0$ . The first boundary ensures that

$f$ is a density. The second is due to the fact that when the process hits $\bar{y}$ it is returned to the origin, so the mass escapes from this point. Equation (30) implies the second order ODE: $f'(y)(\frac{n}{2}-2) = yf''(y)$. The solution of this ODE for $n \neq 2$ is $f(y) = A_1 y^{n/2-1} + A_0$ for two constants $A_0, A_1$ to be determined using the boundary conditions: $0 = A_1(\bar{y})^{n/2-1} + A_0$ and $1 = \frac{A_1}{n/2}(\bar{y})^{n/2} + A_0\bar{y}$. For $n = 2$ the solution is $f(y) = -A_1 \log(y) + A_0$ subject to the analogous boundary conditions. Solving for $A_0, A_1$ gives the desired expressions. $\square$

**Proof.** (of Proposition 10) The only result to be established is that the distribution of the sum of the coordinates of a vector uniformly distributed in the $n-$dimensional sphere has density given by equation (22). Using the result in page 387 of Khokhlov (2006), let $c : \mathbb{R} \to \mathbb{R}$ be measurable, and let $L$ be the Lebesgue measure in $n$ dimensional sphere, then

$$\int_{x\in\mathbb{R}^n, ||x||=1} c(x_1 + ... + x_n) dL(x) = \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})} \int_{-1}^{1} c\left(\sqrt{n}u\right)(1-u^2)^{(n-3)/2} du$$

$$= \frac{2\pi^{n/2}}{\sqrt{n}\,\Gamma\left(\frac{1}{2}\right)\Gamma(\frac{n-1}{2})} \int_{-\sqrt{n}}^{\sqrt{n}} c\left(\sqrt{n}u\right)\left(1-\left(\frac{\sqrt{n}u}{\sqrt{n}}\right)^2\right)^{(n-3)/2} d\left(\sqrt{n}u\right)$$

Consider a function $c(x_1 + \cdots + x_n) = 1$ if $\alpha \leq x_1 + \cdots + x_n \leq \beta$. Dividing by the surface area of the $n-$dimensional sphere we obtain equation (22). $\square$

**Proof.** (of Proposition 11) We use the expression (3.1) in Theorem 3.1 of Yin and Wang (2009) into equation (26) to obtain for $n \geq 2$:

$$\theta_n(\delta, t) = \sum_{m=0}^{\infty}\sum_{k=1}^{\infty} \varrho_{m,k}(\delta, \sqrt{\bar{y}}, n)\, \sigma^2\, e^{-\frac{q_{m,k}^2}{2n}\frac{n\,\sigma^2}{\bar{y}}t} \text{ where} \tag{31}$$

$$\varrho_{m,k}(\delta, \sqrt{\bar{y}}, n) = \int_{||p_0||^2\leq\bar{y}} \left[\int_{||p||^2=\bar{y}} \frac{(p_1+p_2+...+p_n)}{n}\, \varpi_{m,k}(p, p_0, \sqrt{\bar{y}}, n)\, dp\right] \lambda(p_0,\delta)\, dp_0$$

where $\varpi_{m,k}$ are given by

$$\varpi_{m,k}(p, p_0, \sqrt{\bar{y}}, n) = \frac{\Gamma(\frac{n}{2}-1)\,(m+\frac{n}{2}-1)\, Z_m^{\frac{n}{2}-1}\left(\cos\left(\angle p0p_0\right)\right)\, q_{m,k}\, J_{m+\frac{n}{2}-1}\left(\frac{||p||}{||p_0||}\, q_{m,k}\right)}{2\,\pi^{\frac{n}{2}}\, ||p||^{\frac{n}{2}+2}\, ||p_0||^{\frac{n}{2}-1}\, J'_{m+\frac{n}{2}-1}(q_{m,k})}$$

where $Z_m^{\nu}(x)$ are Gegenbauer polynomials of degree $m$ and $\nu$, $\angle p0p_0 \equiv (p\cdot p_0)\,/\,(||p||\,||p_0||)$ is the angle between $p_0$ and $p$, $q_{m,k}$ is the $k$-th (ordered) zero of the Bessel function $J_{m+\frac{n}{2}-1}(\cdot)$, and $J'_{m+\frac{n}{2}-1}(\cdot)$ is the derivative of the Bessel function. The expression in Proposition 11 follows by integrating the right hand side of equation (31) with respect to $t$, thus the coefficients $e_{m,k}$ are given by $e_{m,k}(\delta, \sqrt{\bar{y}}, n) = \varrho_{m,k}(\delta, \sqrt{\bar{y}}, n)\, \bar{y}\; 2/\, q_{m,k}^2$ . It is immediate that the homogeneity of degree one of $e_{m,k}(\cdot, n)$ is equivalent to the homogeneity of degree $-1$ of $\varrho_{m,k}(\cdot, n)$. To establish the homogeneity we prove two properties: i) Write $\lambda\left(p_0, \delta, \sqrt{\bar{y}}\right)$ which include $\bar{y}$ as an argument, since it is an argument of $f$, see equation (20). Direct computation on equation (25) gives $\lambda\left(p_0, a\delta, a\sqrt{\bar{y}}\right) = \lambda\left(p_0/a, \delta, \sqrt{\bar{y}}\right)/a^{n+1}$ for any $a > 0$. ii) Direct computation on $\varpi_{m,k}$ gives the following: $\varpi_{m,k}(p, p_0, a\sqrt{\bar{y}}, n) = \varpi_{m,k}(p/a, p_0/a, \sqrt{\bar{y}}, n)/a^{n+1}$ for any $a > 0$. Using i) and ii) into the expression for $\varrho_{m,k}$ and the change of variables $p_0' = p_0/a$

and $p' = p/a$, and that the determinant of the Jacobian in each of the two integrals is $a^n$ proves the homogeneity of degree $-1$ of $\varrho_{m,k}$.

$\square$

**Proof.** (of Proposition 12). Let $\tilde{y} \equiv y/\bar{y}$ be the values under the invariant distribution $f$ in equation (20), and let $\tilde{y}(\delta)$ denote the values of the same price gaps right after the monetary shock but before adjustment. Let $p$ be a vector of price gaps satisfying $\tilde{y} = ||p||^2/\bar{y}$ so that, for this $p$, $\tilde{y}(\delta) = ||p - \delta \mathbf{1}_n||^2/\bar{y}$. Taking $y \in (0, \bar{y})$, developing the square in the expression for the corresponding value of $\tilde{y}(\delta)$, multiplying and dividing the cross-product term by $\sqrt{y}$, and using the definition of $\tilde{y}$ and $Std(\Delta p) = \sqrt{\bar{y}/n}$ we have:

$$\tilde{y}(\delta) = \tilde{y} - 2\delta\sqrt{\frac{y}{\bar{y}}}\,\frac{\sum_{i=1}^n p_i}{\sqrt{y}}\frac{1}{\sqrt{\bar{y}}} + \frac{n}{\bar{y}}\delta^2 = \tilde{y} - 2\delta\sqrt{\tilde{y}}\left(\frac{\sum_{i=1}^n p_i}{\sqrt{y}}\frac{1}{\sqrt{n}}\right)\frac{1}{Std(\Delta p)} + \left(\frac{\delta}{Std(\Delta p)}\right)^2$$

Conditional on $\tilde{y}$, we can regard $\tilde{y}(\delta)$ as a random variable, whose realizations correspond to each of the price gaps with $||p||^2/\bar{y} = \tilde{y}$, and where the price gaps $p$ are uniformly distributed on the sphere with square radius $y$. Proposition 10 gives the density of the random variable $\frac{\sum_{i=1}^n p_i}{\sqrt{y}}$, and using Proposition 6 it follows that for all $n$ its standard deviation is equal to one and its expected value equal to zero. Thus $\frac{\sum_{i=1}^n p_i}{\sqrt{y}}\frac{1}{\sqrt{n}}$ has an expected value equal to zero and variance $1/n$. Hence $\lim_{n\to\infty} \tilde{y}(\delta) = \tilde{y} + \left(\frac{\delta}{Std(\Delta p)}\right)^2$ where the convergence to a (degenerate) random variable is in distribution. Combining this result for each $\tilde{y} \in [0, 1]$ with Proposition 9 for $n \to \infty$ we obtain that the distribution of $\tilde{y}(\delta)$ converges to a uniform distribution in the interval $\left[\left(\frac{\delta}{Std(\Delta p)}\right)^2, 1 + \left(\frac{\delta}{Std(\Delta p)}\right)^2\right]$. Immediately after the monetary shock any firm with $y > \bar{y}$, or equivalently any firm with $\tilde{y}(\delta) > 1$, adjust its prices. From here we see that the fraction of firms that adjusts immediately after the shock, denoted by $\Phi_n$, converges to $(\delta/Std(\Delta p))^2$.

To characterize $\mathcal{P}_n$ for $t \geq 0$ we establish three properties: i) the expected price change conditional on adjusting at time $t = 0$ is equal to $\delta$, ii) the fraction of firms that adjusts for the first time after the shock between 0 and $t < [\delta - \Theta_n(\delta)] / [\delta\, N_a]$ equals $N_a\, t$, and iii) the expected price change conditional on adjusting at time $0 \leq t < [\delta - \Theta_n(\delta)] / [\delta\, N_a]$ is equal to $\delta$. To establish i), note that, as argued above, as $n \to \infty$ firms adjust its price if and only if they have a price gap $p$ before the monetary shock with square radius larger than $1 - (\delta/Std(\Delta p))^2$. Since in the invariant distribution price gaps are uniformly distributed on each of the spheres, the expected price change across the firms with the same value of $y$ equals to $\delta$. To establish ii) note that, keeping constant $N_a$ as $n$ becomes large, the law of motion for $\tilde{y}$ in equation (13) converges to a deterministic one, namely $\tilde{y}_t = \tilde{y}_0 + N_a\, t$. This, together with the uniform distribution for $\tilde{y}_0$ implies the desired result. Finally, iii) follows from combining i) and ii). $\square$

# ONLINE APPENDICES

# Price setting with menu cost for multi-product firms

*Fernando Alvarez and Francesco Lippi*

January 21, 2013

# B General Equilibrium Set-Up

This appendix outlines the general equilibrium set-up that underlies our approximation. The preferences of the representative agents are given by:

$$\int_0^\infty e^{-rt} \left[ U\left(c(t)\right) - \alpha\, \ell(t) + \log\left(\frac{M(t)}{P(t)}\right)\right] dt \tag{32}$$

where $c(t)$ is an aggregate of the goods produced by all firms, $\ell(t)$ is the labor supply, $M(t)$ the nominal quantity of money, and $P(t)$ the nominal price of one unit of consumption, formally defined below (all variables at time $t$). We will use $U(c) = (c^{1-\epsilon} - 1)/(1-\epsilon)$. There is a unit mass of firms, index by $k \in [0,1]$, and each of them produces $n$ goods, index by $i = 1, ..., n$. There is a preference shock $A_{k,i}(t)$ associated with good $i$ produced by firm $k$ at time $t$, which acts as a multiplicative shifter of the demand of each good $i$. Let $c_{k,i}(t)$ be the consumption of the product $i$ produced by firm $k$ at time $t$. The composite Dixit-Stiglitz consumption good $c$ is

$$c(t) = \left[\int_0^1 \left(\sum_{i=1}^n A_{k,i}(t)^{\frac{1}{\eta}}\, c_{k,i}(t)^{\frac{\eta-1}{\eta}}\right) dk\right]^{\frac{\eta}{\eta-1}} \tag{33}$$

For firm $k$ to produce $y_{k,i}(t)$ of the $i$ good at time $t$ requires $\ell_{k,i}(t) = y_{k,i}(t) Z_{k,i}(t)$ units of labor, so that $W(t)Z_{k,i}(t)$ is the marginal cost of production. We assume that $A_{k,i}(t) = Z_{k,i}(t)^{\eta-1}$ so the (log) of marginal cost and the demand shock are perfectly correlated. We assume that $Z_{k,i}(t) = \exp\left(\sigma \mathcal{W}_{k,i}(t)\right)$ where $\mathcal{W}_{k,i}$ are standard BM's, independent across all $i, k$.

The budget constraint of the representative agent is

$$M(0) + \int_0^\infty Q(t)\left[\bar{\Pi}(t) + \tau(t) + (1+\tau_\ell)W(t)\ell(t) - R(t)M(t) - \int_0^1 \sum_{i=1} P_{k,i}(t) c_{k,i}(t) dk\right] dt = 0$$

where $R(t)$ is the nominal interest rates, $Q(t) = \exp\left(-\int_0^t R(s)ds\right)$ the price of a nominal bond, $W(t)$ the nominal wage, $\tau(t)$ the lump sum nominal transfers, $\tau_\ell$ a constant labor subsidy rate, and $\bar{\Pi}(t)$ the aggregate (net) nominal profits of firms.

The first order conditions for the household problem are (with respect to $\ell, m, c, c_{k,i}$):

$$\begin{aligned}
0 &= e^{-rt}\alpha - \lambda_0(1+\tau_\ell)Q(t)W(t) \\
0 &= e^{-rt}\frac{1}{M(t)} - \lambda_0 Q(t)R(t) \\
0 &= e^{-rt}c(t)^{-\epsilon} - \lambda_0 Q(t)P(t) \\
0 &= e^{-rt}c(t)^{-\epsilon}c(t)^{1/\eta}c_{k,i}(t)^{-1/\eta}A_{k,i}(t)^{1/\eta} - \lambda_0 Q(t)P_{k,i}(t)
\end{aligned}$$

where $\lambda_0$ is the Lagrange multiplier of the agent budget constraint. If the money supply follows $M(t) = M(0)\exp(\mu t)$, then in an equilibrium

$$\lambda_0 = \frac{1}{(\mu + r)M(0)} \quad \text{and for all } t: \; R(t) = r + \mu \ , \ W(t) = \frac{\alpha}{1+\tau_\ell}(r+\mu)M(t) \tag{34}$$

Moreover the foc for $\ell$ and the one for $c$ give the output equation

$$c(t)^{-\epsilon} = \frac{\alpha}{1+\tau_\ell}\frac{P(t)}{W(t)} \tag{35}$$

From the household's f.o.c. of $c_{k,i}(t)$ and $\ell(t)$ we can derive the demand for product $i$ of firm $k$, given by:

$$c_{k,i}(t) = c(t)^{1-\epsilon\eta} A_{k,i}(t) \left(\frac{\alpha}{1+\tau_\ell}\frac{P_{k,i}(t)}{W(t)}\right)^{-\eta} \tag{36}$$

In the impulse response analysis of Section 5 we assume $\mu = 0$, $\tau_\ell = 0$, and that the initial value of $M(0)$ is such that $M(0)/P(0)$, computed using the invariant distribution of prices charged by firms, is different from its steady state value.

The nominal profit of a firm $k$ from selling product $i$ at price $P_{k,i}$, given the demand shock is $A_{k,i}$, marginal cost is $Z_{k,i}$, nominal wages are $W$ and aggregate consumption $c$, is (we omit the time index):

$$c^{1-\epsilon\eta} A_{k,i} \left(\frac{\alpha}{1+\tau_\ell}\frac{P_{k,i}}{W}\right)^{-\eta} [P_{k,i} - W Z_{k,i}]$$

or, collecting $W Z_{k,i}$ and using that $A_{k,i}\, Z_{k,i}^{1-\eta} = 1$, gives

$$W c^{1-\epsilon\eta} \left(\frac{\alpha}{1+\tau_\ell}\frac{P_{k,i}}{W Z_{k,i}}\right)^{-\eta} \left[\frac{P_{k,i}}{W Z_{k,i}} - 1\right]$$

so that the nominal profits of firm $k$ from selling product $i$ with a price gap $p_{k,i}$ is

$$W(t)\, c(t)^{1-\epsilon\eta}\, \Pi(p_{k,i}(t)) \quad \text{where} \quad \Pi(p_{k,i}) \equiv \left(\frac{\alpha}{1+\tau_\ell}\frac{\eta}{\eta-1}\right)^{-\eta} e^{-\eta\, p_{k,i}} \left[e^{p_{k,i}}\frac{\eta}{\eta-1} - 1\right] \tag{37}$$

where we rewrite the actual markup in terms of the price gap $p_{k,i}$, defined in equation (15), i.e. $\frac{P_{k,i}}{W Z_{k,i}} = e^{p_{k,i}}\frac{\eta}{\eta-1}$. This shows that the price gap $p_{k,i}$ is sufficient to summarize the value of profits for product $i$. Note also that, by simple algebra, $\Pi(p_{k,i})/\Pi(0) = \; e^{-\eta\, p_{k,i}}\, [1 + \eta\, e^{p_{k,i}} - \eta]$, which we use below.

Next we show that the ideal price index $P(t)$, i.e. the price of one unit of the composite good, can be fully characterized in terms of the price gaps. Using the definition of total expenditure (omitting time index) $P\, c = \int_0^1 \sum_{i=1}^n (P_{k,i}\; c_{k,i})\; dk$, replacing $c_{k,i}$ from equation (36), and using the first order condition with respect to $c$ to substitute for the $c^{-\epsilon}$ term, gives

$$P = W \left(\int_0^1 \sum_{i=1}^n \left(\frac{P_{k,i}}{W Z_{k,i}}\right)^{1-\eta} dk\right)^{\frac{1}{1-\eta}} \tag{38}$$

which is the usual expression for the ideal price index, and can be written in terms of the price gaps using $\frac{P_{k,i}}{WZ_{k,i}} = e^{p_{k,i}} \frac{\eta}{\eta-1}$.

## B.1 The firm problem

We assume that if firm $k$ adjusts any of its $n$ nominal prices at time $t$ it must pay a fixed cost equal to $\psi_\ell$ units of labor. We express these units of labor as a fraction $\psi$ of the steady state frictionless profits from selling one of the $n$ products, i.e. the dollar amount that has to be paid in the event of a price adjustment at $t$ is $\psi_\ell \, W(t) = \psi \; W(t) \bar{c}^{1-\eta\epsilon} \, \Pi(0)$. To simplify notation, we omit the firm index $k$ in what follows, and denote by $p$ the vector of price gaps and by $p_i$ its $i-th$ component.

The time 0 problem of a firm selling $n$ products that starts with a price gap vector $p$ is to choose $\{\boldsymbol{\tau}, \boldsymbol{\Delta p}\} \equiv \{\tau_j, \Delta p_i(\tau_j)\}_{j=1}^{\infty}$ to minimize the negative of the expected discounted (nominal) profits net of the menu cost. The signs are chosen so that the value function is comparable to the loss function in equation (1):

$$-\mathbb{E}\left[\int_0^\infty e^{-rt} \left(\sum_{i=1}^{n} W(t)\, c(t)^{1-\epsilon\eta}\, \Pi\left(p_i(t)\right)\right) dt - \sum_{j=1}^{\infty} e^{-r\tau_j} W(t) \;\psi_\ell \,\middle|\; p(0) = p\right]$$

Letting $\hat{\Pi}(p_i) \equiv \Pi(p_i)/\Pi(0)$, using that equilibrium wages are constant $W(t)/\bar{W} = e^{\delta}$, and the parameterization of fixed cost in terms of steady state profits: $\psi_\ell = \psi \, \bar{c}^{1-\eta\epsilon} \, \Pi(0)$ gives (where bars denote steady state values):

$$\mathcal{V}(\boldsymbol{\tau}, \boldsymbol{\Delta p}, \mathbf{c}; p) \equiv -\bar{W} e^{\delta} \; \bar{c}^{1-\epsilon\eta} \Pi(0)\, \mathbb{E}\left[\int_0^\infty e^{-rt} \sum_{i=1}^{n} \mathcal{S}\left(c(t), p_i(t)\right) \, dt - \sum_{j=1}^{\infty} e^{-r\tau_j} \psi \;\middle|\; p(0) = p\right] \tag{39}$$

subject to equation (2), $\Delta p_i(\tau_j) \equiv \lim_{t \downarrow \tau_j} p_i(t) - \lim_{t \uparrow \tau_j} p_i(t)$ for all $i \leq n$ and $j \geq 0$, where $\boldsymbol{c} = (c(t))_{t \geq 0}$ and where the function $\mathcal{S} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ gives the normalized per-product profits as a function of aggregate consumption $c$ and the price gap of the product $g$ as follows:

$$\mathcal{S}(c, g) \equiv \left(\frac{c}{\bar{c}}\right)^{1-\eta\epsilon} \; \hat{\Pi}\left(g\right) \quad = \quad \left(\frac{c}{\bar{c}}\right)^{1-\eta\epsilon} \; e^{-\eta \, g} \left[1 + \eta \, e^{g} - \eta\right] \; .$$

Expanding $\mathcal{S}(c, g)$ around $c = \bar{c}, g = 0$ and using that:

$$\left.\frac{\partial \mathcal{S}(c, g)}{\partial g}\right|_{g=0} = \left.\frac{\partial^2 \mathcal{S}(c, g)}{\partial p \, \partial c}\right|_{g=0} = 0 \; , \; \left.\frac{\partial^2 \mathcal{S}(c, g)}{\partial g \, \partial g}\right|_{g=0, c=\bar{c}} = \eta(1-\eta)$$

into equation (39), we obtain:

$$
\begin{aligned}
& \mathcal{V}(\boldsymbol{\tau}, \boldsymbol{\Delta p}, \mathbf{c}; p) = \bar{W}\, \Pi(0)\, \bar{c}^{1-\eta\epsilon}\, e^{\delta} \ \Bigg\{ \mathbf{V}(\boldsymbol{\tau}, \boldsymbol{\Delta p}; p) - \frac{1}{r} \\
- & \ (1-\epsilon\eta) \int_0^\infty e^{-rt} \left( \frac{c(t)-\bar{c}}{\bar{c}} + \frac{1}{2}\eta\epsilon \left( \frac{c(t)-\bar{c}}{\bar{c}} \right)^2 - \frac{1}{6}\eta\epsilon(1+\eta\epsilon) \left( \frac{c(t)-\bar{c}}{\bar{c}} \right)^3 \right) dt \\
- & \ \mathbb{E} \left[ \int_0^\infty e^{-rt} \frac{(2\eta-1)\eta(\eta-1)}{6} \left( \sum_{i=1}^n p_i(t)^3 \right) \, dt \, \middle| \, p(0) = p \right] \\
+ & \ \mathbb{E} \left[ \int_0^\infty e^{-rt} (1-\epsilon\eta) \frac{\eta(\eta-1)}{2} \left( \frac{c(t)-\bar{c}}{\bar{c}} \sum_{i=1}^n p_i(t)^2 \right) \, dt \, \middle| \, p(0) = p \right] \\
+ & \ \mathbb{E} \left[ \int_0^\infty e^{-rt} o(||(p(t), c(t)-\bar{c})||^3) \, dt \, \middle| \, p(0) = p \right] \Bigg\}
\end{aligned}
$$

where $\mathbf{V}(\boldsymbol{\tau}, \boldsymbol{\Delta p}; p)$ is given by equation (1) with $B = (1/2)\eta(\eta-1)$. We can then write:

$$
\mathcal{V}(\boldsymbol{\tau}, \boldsymbol{\Delta p}, \mathbf{c}; p) \ = \ \Upsilon e^{\delta}\, \mathbf{V}(\boldsymbol{\tau}, \boldsymbol{\Delta p}; p) + \mathbb{E} \left[ \int_0^\infty e^{-rt} o\left( ||(p(t), c(t)-\bar{c})||^2 \right) \, dt \, \middle| \, p(0) = p \right] + \iota(\delta, \mathbf{c})
$$

where the constant $\Upsilon = \bar{W}\, \Pi(0)\, \bar{c}^{1-\eta\epsilon}$ is the per product maximum (frictionless) nominal profits in steady state, and where the function $\iota$ does not depend of $(\boldsymbol{\tau}, \boldsymbol{\Delta p})$.

# C Numerical accuracy of the approximations

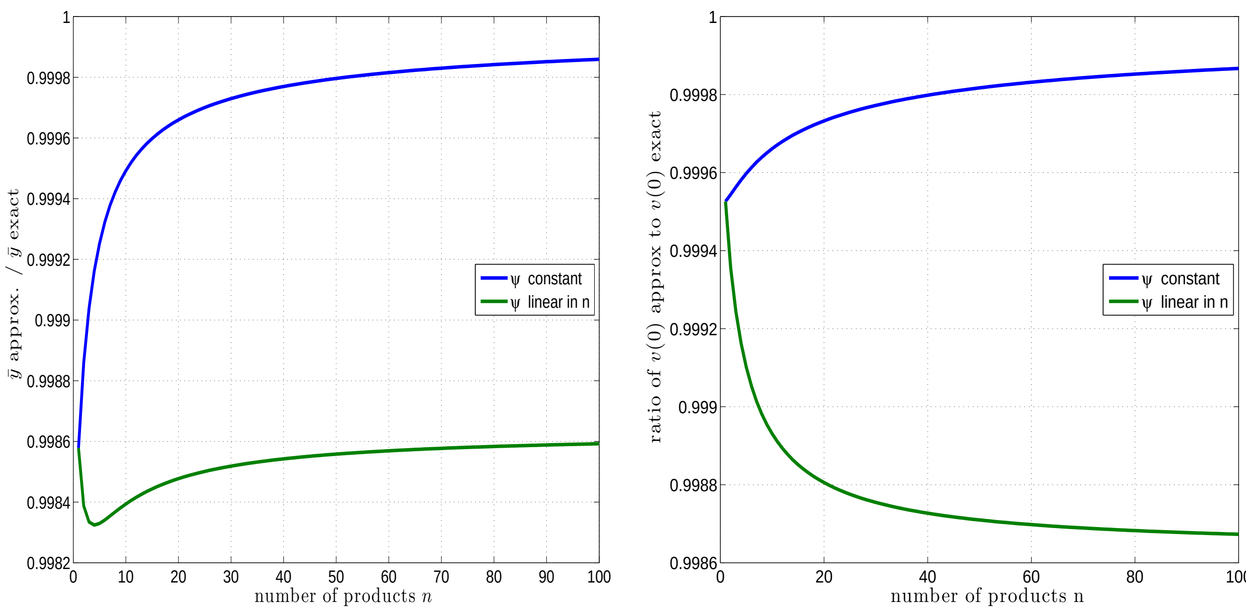
The technical challenges in the analytical study of price setting problems with menu cost have led researchers to consider simple environments. For instance a quadratic profit function, or a quadratic approximation to it, has been used in the seminal work on price setting problems with menu cost by Barro (1972), Dixit (1991), Tsiddon (1993), section 5 of Sheshinski and Weiss (1992), Caplin and Leahy (1997), and chapter 12 of Stokey (2008), among others. Moreover the idiosyncratic shocks considered are stylized, e.g. random walks with constant volatility, as used in Barro (1972), Tsiddon (1993), Gertler and Leahy (2008) and Danziger (1999) among others. Facing the same challenges, we adopt similar assumptions for *each* of the goods, which allows the complete characterization. Our analytical solution rests on carefully chosen approximations, with simplifications that preclude the analysis of the case of asymmetric demands for goods, although section Section 6 considers extensions such as correlation on the shocks across products, the presence of inflation, and different elasticities of substitution of products within and between firms.

This appendix documents the precision of our analytical results in comparison to the exact numerical solution of a model that uses no approximations. In particular, recall that our solution used a second order approximation of the profit function, no drift in the price gaps, and that the impulse response functions were computed using the steady state decision rules i.e. ignoring the general equilibrium feedback effect which, as stated in Proposition 7, were shown to be second order. This section explores the robustness of our approximations compared to a model that features an asymmetric profit function, the presence of drift, and takes into account the general equilibrium feedback on decision rules following the aggregate shock.

The section has two parts. In the first one we show that the steady state decision rule for the quadratic problem $\bar{y}$ of equation (11) discussed in Section 3 gives a very good approximation to the exact solution of the quadratic problem for the parameters used (this should not come as a surprise given point (iii) of Proposition 3). The second part is substantially more involved. It discusses the accuracy of our impulse response analysis in comparison to the impulse response generated by a model that uses a non-quadratic objective function, features drift in the price gap, and takes account of the general equilibrium feedback effect on decision rules. To this end we solve numerically two models that can be computed: one for the case of $n = 1$ and one for the case of $n = \infty$. We compare the results with the ones produced by our approximations of Section 5. We show that the approximate results are very close to the exact results. The reason, explained in Proposition 7, is that the general equilibrium feedback effect on the decision rules is second order.

## C.1 On the accuracy of the approximate decision rules

Figure 9: Ratio of $\bar{y}$ and of $v(0)$ for the approximation relative to the "exact" solution




Note: parameter values are $B = 20$ , $\sigma = 0.25$ , $\psi = 0.03$ and $r = 0.03$.

Next we present some evidence on the numerical accuracy of the approximation for the decision rule $\bar{y}$. Figure C.1 compares the value of $\bar{y}$ obtained from the quadratic approximation to $v$ described above, with what we call the "exact" solution, which is the numerical solution using up to 30 terms for $\beta_i$ in its the expansion. The approximation are closer for smaller values of $\sigma$ and $\psi$, which we regard as more realistic.

## C.2 On the accuracy of the impulse responses

First we describe the case of $n = 1$. We solve for the optimal firms policy of a firm in steady state. This is done using the non-quadratic objective function from the implied CES preferences described in the general equilibrium setup of Appendix B. The optimal policy is of the $sS$ nature, but given the lack of symmetry on the objective function, the thresholds are not symmetric (i.e. the distance between the optimal return point and the lowest adjustment threshold -which gives the size of the price increases- is not equal to the distance between the highest thresholds and the optimal return point -which gives the size of price decreases). Another difference with the model in the main body of the paper is that we reported results assuming the price gap had no drift, due to zero inflation an no drift in the real marginal costs. In this section we assume that marginal cost has a negative drift, due to productivity growth $Z_t$, equal to 2%. Similar results are obtained by assuming a small inflation rate. Because of these differences the optimal return point (i.e. the optimal price upon resetting) does not need to be equal the zero, i.e. the frictionless optimal.[19] A positive drift in the price gap will give the firm a motive to set a positive price gap to hedge against the anticipated depreciation of the sale price. Another motive for the non zero price gap is that the profit function associated to the CES demand is asymmetric, so that price below the optimum are more costly (in terms of foregone revenues) than prices above the optimum. Both forces will give the firm a motive for setting a positive price gap upon resetting.

We solve numerically a discrete time model, with a very small time period (half a day), where the shock to the (log) of the firm marginal cost follows a discrete time analog to the Brownian motion (used in the main model) with drift equal to the trend growth of productivity, so that the price gap will have a small drift. The parameterization of the non-linear model is chosen to be the same as the one of the quadratic model (with the noted exception of the small drift in the price gap).

Solving for the impulse response involves the following steps:

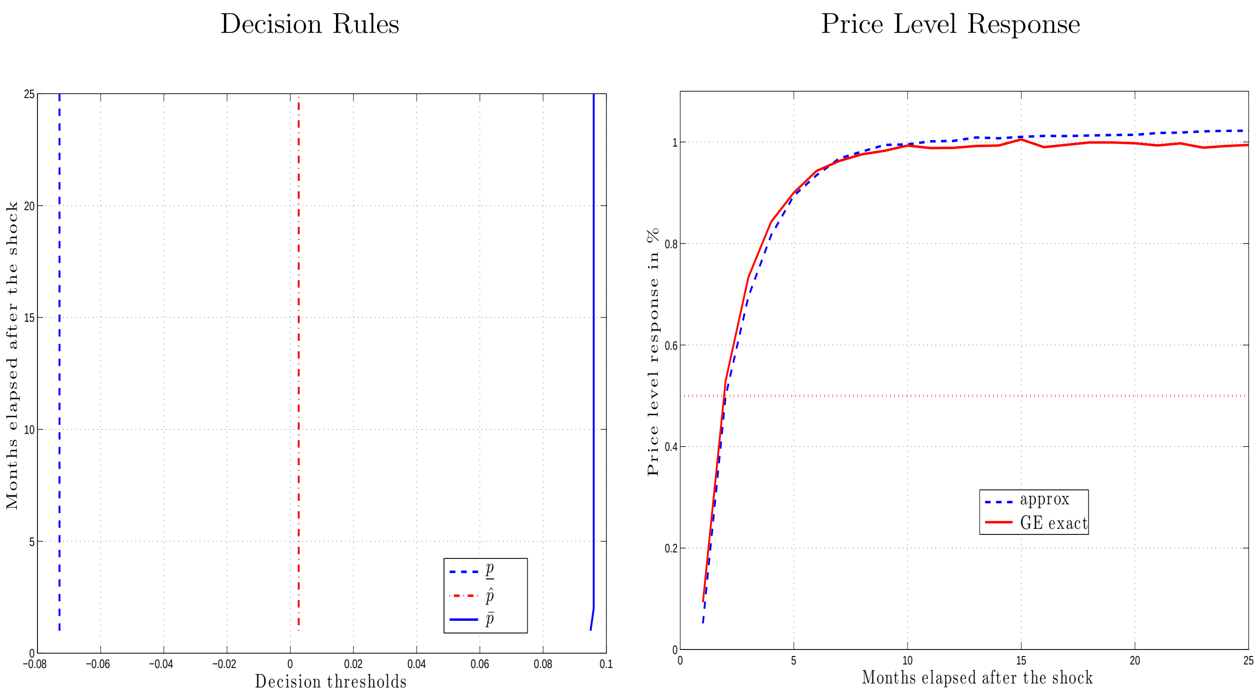
- We compute the steady state (invariant) distribution of the price gaps. Since the thresholds are not symmetric the distribution is not necessarily symmetric either.

- We draw a large number of firms ($N = 500,000$) with price gaps distributed according to the invariant distribution. In the cases of $n = 1$ and $n = \infty$ such invariant distribution can be derived analytically solving the ODE of the associated Kolmogorov forward equation.

- We shock the nominal value of each firm's price gap, by the same proportion at time $t = 0$. This uses the fact that, as in Golosov and Lucas (2007), the equilibrium path of nominal wages, and nominal interest rates, can be solved independently of the aggregate output and the distribution of prices of the final good (see Appendix B).

[19] Of course these differences vanish as the adjustment cost $\psi$ goes to zero.

Figure 10: Approximate vs exact solution after a 1% shock to money supply for $n = 1$




Note: the parameter values are: $\eta = 6.8$ (so that $B = 20$), $\sigma = 0.10$, $\psi = 0.035$, $\rho = 0.02$, the productivity drift is 2%; in the approximate model these produce $N_a = 1$, $Std(\Delta p) = 0.10$. Very similar values are produced by the exact model. The simulation to compute the impulse response function uses a cross section of 500,000 firms.

- (1) We simulate the shocks for each of the $N$ firms until $T$ years, keeping track of the price gap of each of the $j$ firms in each period. We use the decision rules obtained for a given assumed path of future aggregate consumption $\{c_t\}$
- (2) For each time period between $t = 0$ and $t = T$ we use the cross section of the $N$ firms's price gap to compute the ideal price index and the associated aggregate consumption. At the end of this procedure we have a path for the aggregate consumption and one for the price level: $\{c'_t, P_t\}_{t=0}^{T}$.
- If the assumed aggregate consumption path $\{c_t\}$ equals the new path $\{c'_t\}$, up to numerical tolerance, we stop the algorithm. If it does not, we let $\{c_t\} = \{c'_t\}$, return to (1) and iterate again until convergence.

The left panel of Figure 10 plots the decision rules produced by this procedure for the model with $n = 1$ for a shock $\delta = 1\%$, as considered in the main body of the paper. The threshold levels for the price gaps $\underline{p}$, $\bar{p}$ , delimiting the inaction range, and the optimal return point $\hat{p}$. The vertical axis measures the time elapsed since the shock occurred. These lines are virtually vertical, indicating that the optimal decision rules are virtually overlapping with the steady state ones. The only visible effect appear for $\bar{p}$ in the periods immediately following the shocks. Much larger shocks are needed, in the order $\delta = 10\%$, to see more actions (still rather small) in the decision thresholds. The reason was given in Proposition 7 where it was shown that the general equilibrium feedback effect on the decision rules is second order.

The right panel of Figure 10 plots the "exact" impulse response function that takes into account general equilibrium effect, drift and the non-quadratic profit function as well as the impulse response produced by our model for the analogue parametrization, which was presented in Figure 4 of Section 5. The two curves appear almost on top of each other, and their half life is virtually identical. The figures shows that our model provides a very good approximation for a 1% monetary shock (which is not a small shock historically).

### C.2.1 The decision rule along a transition for the $n = 1$ model

Using equation (39) we write the profit function relative to the steady state frictionless profit. We do this for the $n = 1$ case. Let $T$ be the time when consumption reverts to the steady state. For each $t \in (0, T)$ there is a triplet, two inaction bands and an optimal return point, that satisfy value matching and smooth pasting for the following value functions

$$\bar{v}(c_t, p_t) = \mathcal{S}(c_t, p_t)\Delta + \frac{1}{1 + \Delta r}\mathbb{E} \;\; v(c_{t+\Delta}, p_{t+\Delta})$$

and

$$\hat{v}(c_t, p_t) = \max_{\hat{p}} \left( \mathcal{S}(c_t, p_t)\Delta - \psi + \frac{1}{1 + \Delta r}\mathbb{E} \;\; v(c_{t+\Delta}, p_{t+\Delta}) \middle| \, p_t = \hat{p} \right)$$

where

$$\mathcal{S}(c, p) = \left(\frac{c}{\bar{c}}\right)^{1-\eta\epsilon} \; e^{-\eta p} \left[1 + \eta\, e^{p} - \eta\right] \; .$$

### C.2.2 On the exact solution of the $n = \infty$ case

Consider a value function defined in the augmented state: $\tilde{V}(p_1, ..., p_n; \tau)/n$ where $(p_1, ...., p_n)$ is the vector of price gaps and $\tau$ is the time since last adjustment. The period return for this Bellman equation is $\sum_{i=1}^{n} c(t)^a \ \Pi(p_i(t))/n$ where $a = 1 - \eta\epsilon$ and $\Pi(p_i)$ is the function in equation (37) deflated by nominal wages. As $n \to \infty$ the law of large numbers gives that we can write the period return as

$$\sum_{i=1}^{n} c(\tau)^a \ \Pi(p_i(\tau))/n \to c(t)^a \ \mathbb{E}[\Pi(p(\tau)) \mid p(0)] = c(\tau)^a \quad F(\tau, p(0))$$

where $F(\tau, p(0))$ is a function that gives the expected value of $\Pi(p(\tau))$ after $\tau$ periods since resetting each price gap at $p(0)$. We can write the steady state Bellman equation as:

$$\hat{V} = \max_{p,T} \int_0^T e^{-rt} \ c^a \ F(t,p) \ dt + e^{-rT}[\hat{V} - \psi_\ell]$$

where, abusing notation, we use $\hat{V}$ to denote the value of the value function right after an adjustment of prices, i.e.:

$$\hat{V} = \max_{p_1,...,p_n} \ \tilde{V}(p_1, ...., p_n, 0)/n$$

and $\psi_\ell = \Pi(0)c^a \ \psi$ as assumed in Section B.1. Recall, from equation (37), that

$$\Pi(p) \equiv \left(\frac{\alpha}{1+\tau_\ell}\frac{\eta}{\eta-1}\right)^{-\eta} e^{-\eta p}\left[e^p \frac{\eta}{\eta-1} - 1\right]$$

Recall that the price gap $p$ is given by equation (15) so it has the following diffusion

$$\mathrm{d}p = (\gamma - \mu)\mathrm{d}t + \sigma \mathrm{d}B$$

Next define the function $f(\tau, p)$ as the ratio between the expected profits $\tau$ periods after resetting a price gap $p$ and the frictionless profit term $\Pi(0)$:

$$\begin{aligned} f(\tau, p) \ &\equiv \ \mathbb{E}\left[\frac{\Pi(p_\tau)}{\Pi(0)}\Big| p_0 = p\right] = \frac{F(\tau,p)}{F(0,0)} \\ &= \ \eta e^{(1-\eta)p} \, e^{\left((\eta-1)(\mu-\gamma)+\frac{\sigma^2}{2}(\eta-1)^2\right)\tau} - (\eta-1)e^{-\eta p} e^{\left(\eta(\mu-\gamma)+\frac{\sigma^2}{2}\eta^2\right)\tau} \ . \end{aligned} \tag{40}$$

Then the firm's value function, scaled by the frictionless profits $\Pi(0)c^a$, solves the Bellman equation

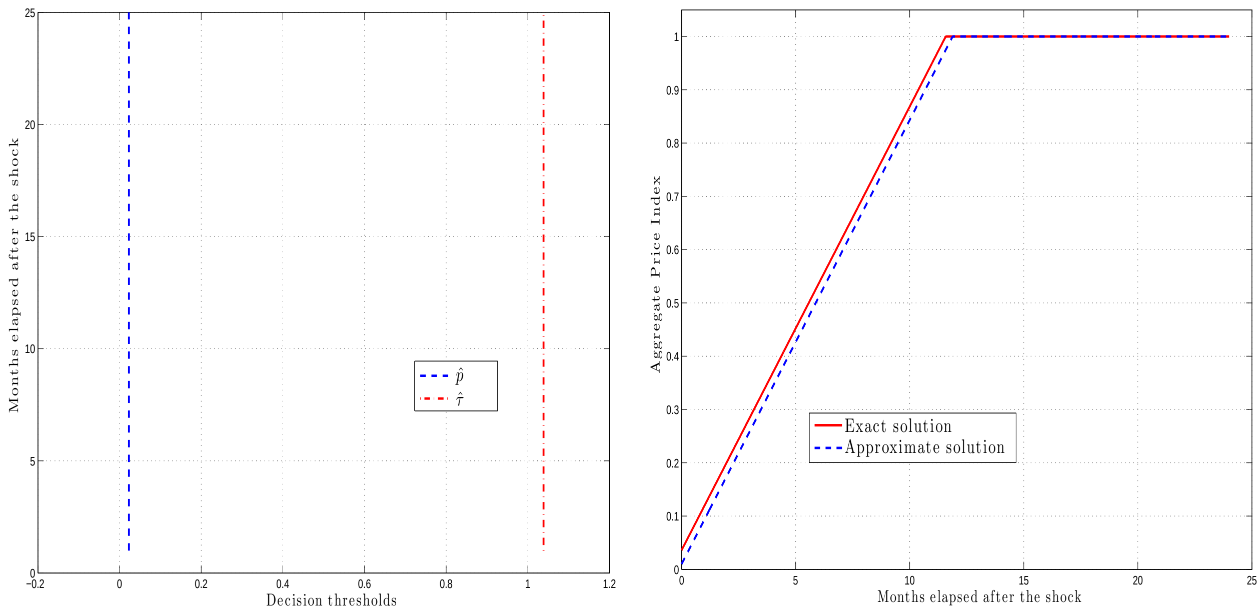
$$\hat{v} = \max_{p,T} \int_0^T e^{-rt} \ f(t,p) \ dt + e^{-rT}(\hat{v} - \psi) \tag{41}$$

To match the model moments to the observables note the following. To keep the number of adjustments finite let $\psi = n \ \psi_1$ so that as $n$ increases the cost per good stays constant at $\psi_1$. Thus as $n \to \infty$ we have that $N_a = \sqrt{\frac{B\sigma^2}{2\psi_1}}$ and $Std(\Delta p) = \sqrt{\sigma^2/N_a}$. Under the invariant the distribution of the $y/\bar{y}$ is uniform in $(0, 1)$ as in Section 5.2. After the shock hits the distribution is shifted.

Figure 11: Approximate vs exact solution after a 1% shock to money supply for $n = \infty$

Decision Rules

Price Level Response




Note: the parameter values are: $\eta = 6.8$ (so that $B = 20$), $\sigma = 0.10$, $\psi = 0.035$, $\rho = 0.02$, the productivity drift is 2%; in the approximate model these produce $N_a = 1$, $Std(\Delta p) = 0.10$. Very similar values are produced by the exact model. The simulation to compute the impulse response function uses a cross section of 500,000 firms.

### C.2.3 The decision rule along a transition for the $n = \infty$ model

Let $T$ be the time when consumption reverts to the steady state. For each $t \in (0, T)$ there is a value function $v_t$, an optimal return point $\hat{p}_t$ and a time until the next review $\tau_t$, that solve the following Bellman equation (scaled by the steady state frictionless profits $\Pi(0)c^a$),

$$\hat{v}_t = \max_{p_t, \tau_t} \int_0^{\tau_t} e^{-rs} \left( \frac{c_{t+s}}{\bar{c}} \right)^{1-\eta\epsilon} f(s, p_t)\, ds + e^{-r\tau_t} \left( \hat{v}_{t+\tau_t} - \psi \right) \tag{42}$$

and use $\hat{v}_T = \hat{v}$, i.e. the steady state value function. For a given guess of the aggregate consumption profile $c_t$ the value functions can be solved backward.

We first determine which firms will adjust prices immediately as the shock arrives. Let $\hat{p}$ be the price gap chosen by firms in the steady state and $\hat{T}$ be the time until the next adjustment in the steady state. After a monetary shock all firms find their price gaps reduced by $\delta$, so their value function corresponds to one in which the last price gap upon was reset at $\hat{p} - \delta$. This determines a new planned date for adjusting prices: $\tau_0$ which by the first order condition with respect to $\tau_t$ in equation (42) solves

$$f(\tau_0, \hat{p} - \delta) - r(\hat{v}_{\tau_0} - \psi) = 0 \quad . \tag{43}$$

After computing the value functions $v_t$ one can thus determine the new times until adjustment $\tau_0$. All firms who adjusted prices $t$ periods ago with $t \in (\tau_0, \hat{T})$ will immediately adjust prices. Thus the fraction of firms that will jump on impact after the monetary shock is given by $\frac{\hat{T}-\tau_0}{\hat{T}}$. All other firms will adjust when the age of their price will reach $\tau_0$, and that that point use the decision rules $\{\tau_t, p_t\}$ prescribed by equation (42). Numerically, as occurred for the $n = 1$ case, the left panel of Figure 11 shows that for a shock $\delta = 1\%$, as those considered in the main text, these rules are virtually identical to the ones of the steady state. The main difference compared to the approximate rule derived in the main text of the paper concerns the size of the impact effect which the model slightly under-estimates which is due to the fact that the rule in the paper uses $\hat{T}$ as the optimal adjustment date whereas the exact model prescribes $\tau_0$. The right panel of Figure 11 shows that the difference between the approximate and the exact impulse response is tiny.[20]

# D Asymptotic hazard rates

Table 3: Normalized limit hazard rates for various values of $n$

| | number of products $n$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 6 | 8 | 10 | 20 | 50 | 100 |
| first zero of $J_{\frac{n}{2}-1}(\cdot) : q_{n,1}$ | 1.6 | 2.4 | 3.1 | 3.8 | 5.1 | 6.4 | 7.6 | 13 | 30 | 56 |
| $\mathcal{T}(0) \cdot \lim_{t\to\infty} h_n(t)$ | 1.2 | 1.4 | 1.6 | 1.8 | 2.2 | 2.5 | 2.9 | 4.5 | 8.8 | 16 |

$\mathcal{T}(0)$ is the expected duration and $\mathcal{T}(0) \cdot \lim_{t\to\infty} h_n(t) = q_{n,1}^2 \,/\, (2n)$ is the normalized limit hazard rate.

For completeness, Table 3 computes the first zero -denoted by $q_{n,1}$- for the relevant Bessel functions and the (normalized) asymptotic hazard rate for several value of $n$.

[20]Likewise, Figure 7 in Golosov and Lucas (2007) compares an impulse response that includes the general equilibrium feedback effect with one computed ignoring this effect, i.e. keeping the firms decision rules constant. The authors conclude that "Evidently, the approximation works very well for the effects of a one-time shock, even a large one." Likewise, small general equilibrium feedback effects are found in Alvarez, Lippi, and Paciello (2012) and in Appendix C where we solve numerically a model with the general equilibrium structure of Golosov and Lucas using idiosyncratic shocks and adjustment cost corresponding to the ones of this paper for $n = 1$.

# E Details on the Impulse response to Monetary Shocks

## E.1 Numerical Evaluation of Impact Effects

The expression in equation (23) is readily evaluated by either numerical integration, or using that $S(z)$ is proportional to the hypergeometric function ${}_2F_1(\cdot)$. Likewise, equation (24) is easy to evaluate, since it has the following closed form solution:

$$S(z) = \frac{{}_2F_1\left(\frac{1}{2}, \frac{3-n}{2}, \frac{3}{2}, \frac{z^2}{n}\right)}{Beta\left(\frac{n-1}{2}, \frac{1}{2}\right)\ \sqrt{n}} \quad \text{and} \quad \int_{-\sqrt{n}}^{\nu} z\ s(z) dz = -\frac{n\left(1-\frac{\nu^2}{n}\right)^{(n-1)/2}}{(n-1)Beta\left(\frac{n-1}{2}, \frac{1}{2}\right)\ \sqrt{n}}\ .$$

## E.2 Analytical Characterization of Impact Effects

The next lemma characterize the impact effect of an aggregate shock in terms of properties of the $\Theta_n$ and $\Phi_n$ functions.

Lemma 2. Let $\Phi_n$ and $\Theta_n$ be, respectively, the fraction of firms that change prices and the average price change across the $n$ goods after a monetary shock of size $\delta$. We have:

(i) Fix $n \geq 1$ and $\bar{y} > 0$. If $\delta \geq 2\sqrt{\bar{y}/n}$ (large shocks) then $\Phi_n(\delta, \bar{y}) = 1$ and $\Theta_n(\delta, \bar{y}) = \delta$ .

(ii) $\Phi_n, \Theta_n$ are homogenous in $(\delta, \sqrt{\bar{y}})$: $\Phi_n(\delta, \bar{y}) = \Phi_n\left(\frac{\delta}{\sqrt{\bar{y}}}, 1\right)$ and $\frac{1}{\sqrt{\bar{y}}}\Theta_n\left(\delta, \bar{y}\right) = \Theta_n\left(\frac{\delta}{\sqrt{\bar{y}}}, 1\right)$.

(iii) $\Phi_n$ and $\Theta_n$ are both weakly increasing in $\delta$; the function $\Phi_n$ is decreasing in $\bar{y}$ and, for small $\delta$, $\Theta_n$ is decreasing in $\bar{y}$.

(iv) Impact effects are second order

$$\frac{\partial \Phi_n(0, \bar{y}_n)}{\partial \delta} = \frac{\partial \Theta_n(0, \bar{y}_n)}{\partial \delta} = 0\ ,$$

**Proof.** (of Lemma 2 )

Point (i): follows by noting that the "last" firm to be pushed out of the inaction region is a firm with a price gap $y = \bar{y}$ and all individual price gaps equal to $\sqrt{\bar{y}/n}$ so that $\sum_{i=1}^{n} p_i = \sqrt{n\bar{y}}$. Intuitively, this is the firm with the largest possible positive price gaps and the shock is helping the firm to reduce them. Using this assumption in equation (21) gives the smallest $\delta = 2\sqrt{\bar{y}/n}$ such that all prices adjust, so that $\Phi_n = 1$. The symmetry of the density of price changes immediately gives the result $\Theta_n = \delta$.

Point (ii): Let us start by stating the homogeneity property, and defining two related functions $\hat{\Phi}_n$ and $\hat{\Theta}_n$:

$$\hat{\Phi}_n\left(\frac{\delta}{\sqrt{\bar{y}}}\right) \equiv \Phi_n\left(\frac{\delta}{\sqrt{\bar{y}}}, 1\right) = \Phi_n(\delta, \bar{y}) \quad \text{and} \quad \hat{\Theta}_n\left(\frac{\delta}{\sqrt{\bar{y}}}\right) \equiv \Theta_n\left(\frac{\delta}{\sqrt{\bar{y}}}, 1\right) = \frac{\Theta_n(\delta, \bar{y})}{\sqrt{\bar{y}}}\ .$$

The homogeneity follows from a change of variables in equation (23) and equation (24), taking into account that $f(y)$ is homogenous of degree $-1$ in $(\bar{y}, y)$ as displayed in equation (20), and that $\nu(y, \delta)$ is homogenous of degree zero in $(\sqrt{y}, \sqrt{\bar{y}}, \delta)$, as displayed in equation (21). In contrast $S(z)$ and $s(z)$, evaluated at a given $z$, do not depend on $\delta$ nor on $\bar{y}$.

Point (iii): The monotonicity of $\Phi_n$ and $\Theta_n$ with respect to $\delta$ follows, after manipulating the derivatives, from the monotonicity of $\nu(\cdot)$ and $S(\cdot)$. To show that for small $\delta$ the function $\Theta_n$ is decreasing in $\bar{y}$, differentiate totally with respect to $\bar{y}$ the expression stating the homogeneity of this function obtaining

$$\Theta_{n,2}(\delta, \bar{y}) = \frac{1}{2}\frac{\delta}{\sqrt{\bar{y}}}\left[\frac{\Theta_n\left(\frac{\delta}{\sqrt{\bar{y}}}, 1\right)}{\delta} - \Theta_{n,1}\left(\frac{\delta}{\sqrt{\bar{y}}}, 1\right)\right] ,$$

where $\Theta_{n,i}$ denote the derivative with respect to the $i^{th}$ argument of this function. Use that, for small values of $\delta$, the function $\Theta_n(\delta/\sqrt{\bar{y}}, 1)$ is increasing and convex on $\hat{\Theta}$, so that the expression in squares brackets is negative. That this function is increasing and convex follows since its first derivative is zero at $\delta = 0$ and the function is increasing. For $\Phi_n$ it follows from differentiating with respect to $\bar{y}$ the definition of homogeneity, obtaining:

$$\Phi_{n,2}(\delta, \bar{y}) = -\frac{1}{2}\frac{\delta}{\sqrt{\bar{y}}}\Phi_{n,1}\left(\frac{\delta}{\sqrt{\bar{y}}}, 1\right) ,$$

hence this function is decreasing in $\bar{y}$ in its domain. That $\Phi$ is strictly increasing in $\delta$ follows from the monotonicity of $S$ and $\nu$.

Point (iv): The zero derivative of $\Phi_n$ and $\Theta_n$ with respect to $\delta$ at zero can be obtained by considering two related functions, $\bar{\Phi}_n$ and $\bar{\Theta}_n$ which are obtained by replacing $\nu$ with $\bar{\nu} = \delta/(2\sqrt{y})$ and, without loss of generality, by replacing the lower extreme of integration w.r.t. $y$ by the $\underline{y}(\delta)$, which solves $\nu(\underline{y}(\delta), \delta) = -\sqrt{n}$. Note that $\underline{y}(\delta)$ goes to $\bar{y}$ as $\delta \downarrow 0$, and that, by monotonicity, these functions are upper bounds for $\Phi_n$ and $\Theta_n$. The result follows by differentiating $\bar{\Phi}_n$ and $\bar{\Theta}_n$ w.r.t. $\delta$ and letting $\delta$ go to zero.

**Zero first derivatives of the $\hat{\Phi}_n$ and $\hat{\Theta}_n$ functions at $\delta = 0$**

To show that $\Phi'(0) = 0$ we use the following. i) equation (21) implies that

$$S(\nu(y, \delta)) = 0 \;\text{ for } 0 \leq y \leq \underline{y}(\delta) \equiv \left(\max\left\{\sqrt{\bar{y}} - \delta\sqrt{n}\;,\; 0\right\}\right)^2 \;, \tag{44}$$

hence the integration with respect to $y$ in equation (23) and equation (24) can be done between $\underline{y}(\delta)$ and $\bar{y}$. ii) note that $\underline{y}(\delta) \to \bar{y}$ as $\delta \downarrow 0$, iii) once we have defined the integral w.r.t $y$ in $\Phi$ in the interval $(\underline{y}, \bar{y})$, we can replace the function $\nu$ by an upper bound $\bar{\nu}(y, \delta) \equiv n\delta/(2\sqrt{y})$ and define upper bounds for $\Phi$ as

$$\bar{\Phi} \equiv \int_{\underline{y}(\delta)}^{\bar{y}} f(y)\; S\left(\bar{\nu}(y, \delta)\right) dy \leq \Phi = \int_{\underline{y}(\delta)}^{\bar{y}} f(y)\; S\left(\nu(y, \delta)\right) dy \tag{45}$$

iv) the density $f(\bar{y}) = 0$. Then differentiating the expression for $\bar{\Phi}$ w.r.t. $\delta$, evaluating the derivative at $\delta = 0$ we obtain that $\bar{\Phi}'(0, \sqrt{\bar{y}}) = 0$. v) since $\bar{\Phi} \geq \Phi \geq 0$ this establishes the desired result.

To show that $\hat{\Theta}'(0) = 0$ notice that $\Theta = \Phi \times \mathbb{E}_\delta[\Delta p | \Delta p \neq 0]$, i.e. the change in prices is equal to the fraction of firms that change prices times the expectation that of a price change, conditional on having a price change. Using that $\Phi(0, \sqrt{\bar{y}}) = \Phi'(0, \sqrt{\bar{y}}) = 0$ we obtained that $\Theta'(0, \sqrt{\bar{y}}) = 0$.

□

As a benchmark, note that in a flexible price economy all firms change prices, i.e. $\Phi_n(\delta) = 1$ and hence the average price change equals the monetary impulse, i.e. $\Theta_n(\delta) = \delta$ for all $\delta$. Part (i) of the lemma states that, for *large* shocks i.e. for $\delta \geq 2\sqrt{\bar{y}/n}$, all the firms adjust prices change on average by $\delta$, so that the economy behaves like one with no frictions.

Part (ii) illustrates a convenient homogeneity property of the $\Phi_n, \Theta_n$ functions: after normalizing the monetary shock in terms of the price gap, these functions have *only* one argument. Part (iii) states that the fraction of adjusters and the response of the aggregate price level are increasing in the size of the shock.

## E.3 Sensitivity analysis of the IRF to the size of monetary shock

We explore the sensitivity of the impulse responses of considering different values for the monetary shock, by considering the case when the monetary shock is half and twice as large as in the benchmark case of Figure 4. The results, which are aligned to the benchmark case, are shown in Figure 12 and Figure 13.

# F Correlation, drift and cross products

Our paper studies the problem of a firm who controls an $n$-dimensional vector of price gaps $p \in \mathbb{R}^n$ subject to a common menu cost $\psi$. Assuming that the individual price gaps $p_i$ had no drift and were mutually uncorrelated, and that the objective function was to minimize the square of the price gaps we showed that the $n$ dimensional state of the problem could be collapsed into a single state variable $y = \sum_{i=1}^n p_i^2$, measuring the squared norm of the price gaps. This delivered a lot of analytical tractability.

This appendix extends the model to the possibility that the price gaps $p_i$ are mutually correlated and/or that the gaps have a common drift and that the objective function has non-zero cross partial terms (all equal to a common constant). These extensions impair the symmetry of the problem so that one might fear to lose the tractability that was obtained before. Surprisingly (to us at least), we show that despite the apparent complexity of these extensions, the modified problem remains tractable: instead of the single state variable $y$ defined above, the state of the problem with either drift, correlation, and cross product, or any combination of them now includes only one additional variable measuring the sum of the coordinates of the vector, namely $z = \sum_{i=1}^n p_i$. Importantly, this not only allows to solve numerically the steady state problem of the firm for any $n \geq 1$, but also to compute the impulse response, since the effect on the aggregate price level can be obtained by keeping track of $z$ for each firm.

Figure 12: Impulse Response of CPI, smaller shock $\delta$

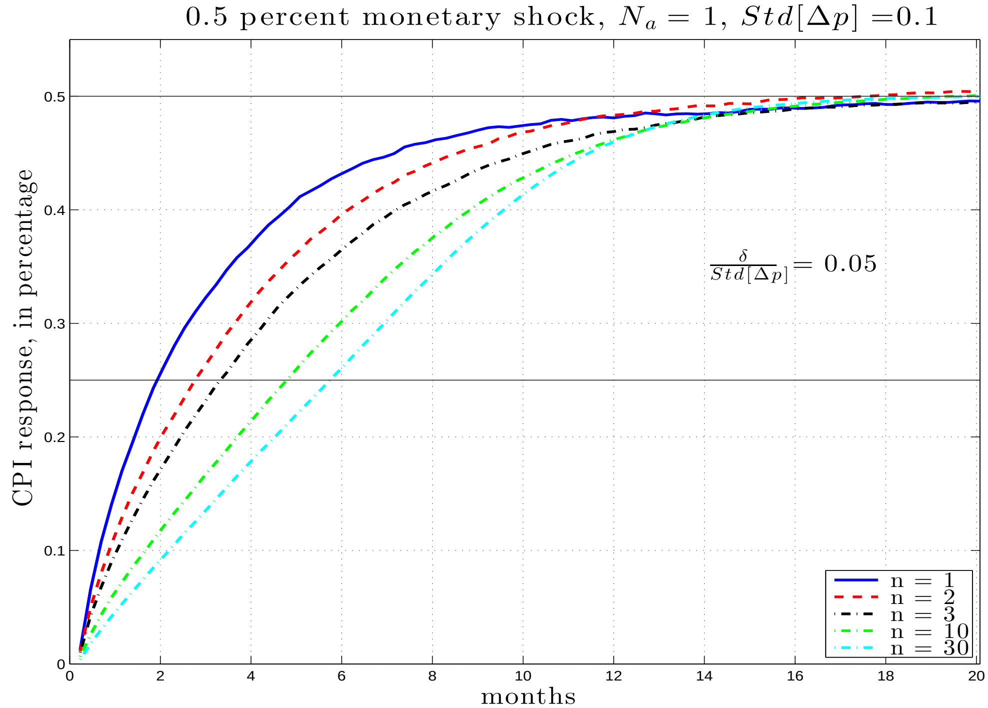
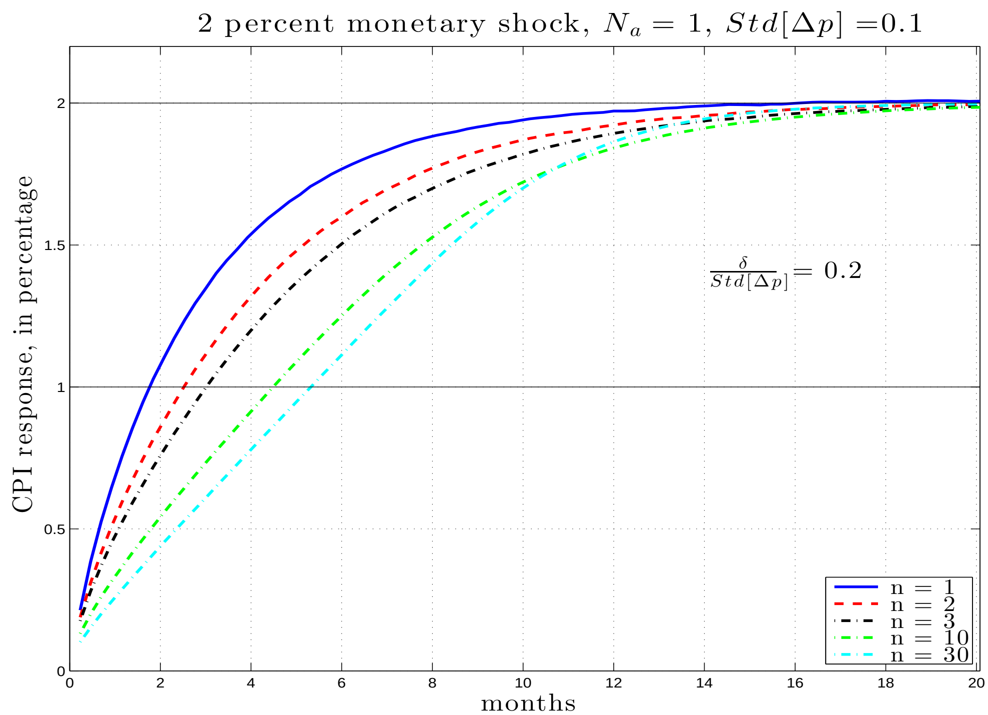



Figure 13: Impulse Response of CPI, larger shock $\delta$



For ease of exposition and because its implications are more important to judge the robustness of the benchmark case, the next section shows how to solve the firm problem when the price gaps are correlated but there is no drift. The value function and decision rules for the problem are presented in Section F.1. Section F.2 illustrates the cross section implications of an economy where firms follow these decision rules, presenting the implications for the cross-section distribution of price changes, a statistic that is central to the empirical analyses of the price setting problem. Section F.3 moves on to characterize how the aggregate economy will respond to a monetary shocks. We will show how the response of the economy to a monetary shock varies as we change (1) the number of goods $n$ sold by each firm and (2) the correlation $\rho$ between the shocks of the price gaps of the firm. Finally, Section F.4 shows how to further extend the firm problem to include a common drift in all all price gaps, e.g. inflation, and Section F.5 shows how to include non-zero cross-partial derivatives (between the price gaps of the different goods) in the instantaneous return function.

## F.1 The case of correlated price gap

We assume that the price gaps are diffusions that satisfy:

$$\mathbb{E}\left[\mathrm{d}p_i(t)\right] = 0\,\mathrm{d}t,\ \mathbb{E}\left[\mathrm{d}p_i^2(t)\right] = \hat{\sigma}^2\,\mathrm{d}t, \quad \text{and} \quad \mathbb{E}\left[\mathrm{d}p_i(t)\mathrm{d}p_j(t)\right] = \rho\ \hat{\sigma}^2\,\mathrm{d}t \tag{46}$$

for all $i = 1, ..., n$ and $j \neq i$ and for two positive constants $\hat{\sigma}^2$ and $\rho$. Then we can write that each price gap follows

$$\mathrm{d}p_i(t) = \bar{\sigma}\,\mathrm{d}\bar{\mathcal{W}}(t) \ + \sigma\mathrm{d}\mathcal{W}_i(t) \ \text{ for all } i = 1, ..., n \ . \tag{47}$$

where $\bar{\mathcal{W}}, \mathcal{W}_i(t)$ are independent standard BMs, so that $\hat{\sigma}^2 = \bar{\sigma}^2 + \sigma^2$ and the correlation parameter is $\rho = \frac{\bar{\sigma}^2}{\bar{\sigma}^2+\sigma^2}$. Define:

$$y(t) = \sum_{i=1}^{n} p_i^2(t) \text{ and } z(t) = \sum_{i=1}^{n} p_i(t) \tag{48}$$

Using Ito's Lemma:

$$\mathrm{d}y(t) = \left[n\sigma^2 + n\bar{\sigma}^2\right]\mathrm{dt} + 2\sigma\sum_{i=1}^{n} p_i(t)\mathrm{d}\mathcal{W}_i(t) + 2\bar{\sigma}\left[\sum_{i=1}^{n} p_i(t)\right]\mathrm{d}\bar{\mathcal{W}}(t)$$

and

$$\mathrm{d}z(t) = n\bar{\sigma}\mathrm{d}\bar{\mathcal{W}}(t) + \sigma\sum_{i=1}^{n}\mathrm{d}\mathcal{W}_i(t)$$

This implies that :

$$
\begin{aligned}
\mathbb{E}\left[\,\mathrm{d}y(t)\,\right]^2 &= 4\sigma^2\left(\sum_{i=1}^n p_i^2(t)\right)\mathrm{d}t + 4\bar{\sigma}^2\left(\sum_{i=1}^n p_i(t)\right)^2\mathrm{d}t \\
&= 4\sigma^2\,y(t)\,\mathrm{d}t + 4\bar{\sigma}^2\,z(t)^2\,\mathrm{d}t\ , && (49)\\
\mathbb{E}\left[\,\mathrm{d}z(t)\,\right]^2 &= \sigma^2\,n\,\mathrm{d}t + \bar{\sigma}^2\,n^2\,\mathrm{d}t \quad \text{and} && (50)\\
\mathbb{E}\left[\,\mathrm{d}y(t)\,\mathrm{d}z(t)\,\right] &= 2\sigma^2\left(\sum_{i=1}^n p_i(t)\right)\mathrm{d}t + 2n\bar{\sigma}^2\left(\sum_{i=1}^n p_i(t)\right)\mathrm{d}t \\
&= 2\left(\sigma^2 + n\bar{\sigma}^2\right)\,z(t)\,\mathrm{d}t && (51)
\end{aligned}
$$

Thus define the diffusions

$$
\mathrm{d}y(t) = n\left[\sigma^2 + \bar{\sigma}^2\right]\mathrm{d}t + 2\sigma\sqrt{y(t)}\ \mathrm{d}\mathcal{W}^a(t) + 2\bar{\sigma}z(t)\ \mathrm{d}\mathcal{W}^c(t) \tag{52}
$$

$$
\mathrm{d}z(t) = n\,\bar{\sigma}\ \mathrm{d}\mathcal{W}^c(t) + \sqrt{n}\sigma\left[\frac{z(t)}{\sqrt{n\,y(t)}}\ \mathrm{d}\mathcal{W}^a(t) + \sqrt{1-\left(\frac{z(t)}{\sqrt{n\,y(t)}}\right)^2}\ \mathrm{d}\mathcal{W}^b(t)\right] \tag{53}
$$

where $(\mathcal{W}^a, \mathcal{W}^b, \mathcal{W}^c)$ are three standard independent BM's.

Note that if $\bar{\sigma} = 0$ then $z$ does not affect $y$, and hence the state of the problem is $y$. Also note that if $\bar{\sigma} > \sigma = 0$ then the specification coincides with a one good model ($n = 1$ and no correlation), so the state can be taken to be $y$ too. In the case where $\sigma$ and $\bar{\sigma}$ are positive the state of this problem will be the pair $(y, z)$. We offer some preliminary characterization of the problem:

1. We require that:
$$
z(t)^2 \leq n\,y(t) \tag{54}
$$
If $z(0)^2 \leq n\,y(0)$ and $\{y(t), z(t)\}_{t>0}$ generated by equation (52)-equation (53) satisfy this inequality. To see why, consider the case where $z = \sqrt{y\,n}$, and use Ito's lemma to compute $\mathrm{d}(yn)$ and $\mathrm{d}(z^2)$. At this point the two process have the same drift and diffusion. A similar argument follows at $z = -\sqrt{y\,n}$ where the diffusions coefficient differ only on their sign. Thus $z^2/(y\,n)$ stays in $[0, 1]$.

2. The diffusions defined by equation (52)-equation (53) satisfy: equation (49), equation (50) and equation (51).

3. The value function has arguments $(y, z)$, denoted by $v(y, z)$. Alternatively $V(p_1, p_2, ..., p_n) = v\left(\sum_{i=1}^n p_i^2, \sum_{i=1}^n p_i\right)$.

4. The value function is symmetric in $z$ around zero, so $v(y, z) = v(y, -z)$ for all $(y, z) \in \mathbb{R}^2_+$. This follows because clearly $V(p) = V(-p)$.

5. The optimal policy is to have an inaction region $\mathcal{I} = \{(y, z) : 0 \leq \bar{y}(z)\}$ for some function $\bar{y}(z)$.

6. At the threshold we have value matching and, if the function is $C^1$ in the entire domain, smooth pasting:

$$V_i(p) = 0 \text{ and } V(p) = V(0) + \psi \text{ if } \bar{y}\Big(\sum_{i=1}^{n} p_i\Big) = \sum_{i=1}^{n} p_i^2 \text{ or} \tag{55}$$

$$v(\bar{y}(z), z) = v(0,0) + \psi \text{ and } v_1(\bar{y}(z), z)2z + nv_2(\bar{y}(z), z) = 0 \tag{56}$$

Differentiating value matching w.r.t $z$ and comparing with smooth pasting we have:

$$v_1(\bar{y}(z), z)\bar{y}'(z) + v_2(\bar{y}(z), z) = 0 \text{ all } z \implies v_1(\bar{y}(z), z)\left[\bar{y}'(z) - \frac{2z}{n}\right] = 0 \tag{57}$$

We conjecture that for all $z$ we have $v_1(\bar{y}(z), z) = 0$ and hence $v_2(\bar{y}(z), z) = 0$ too. These are required if $v$ is $C^1$ in the entire domain.

7. The threshold $\bar{y}(z)$ satisfies $\bar{y}(z) = \bar{y}(-z) > 0$ for al $z > 0$ and $\bar{y}'(z) = -\bar{y}'(-z)$ for all $z > 0$.

There are two special cases of interest for which we can solve for $\bar{y}(z)$. One is when $\bar{\sigma} = 0 < \sigma$ so the correlation is zero, which is our benchmark case for which we have an analytical solution of the problem. In this case $\bar{y}(z)$ does not depend on $z$ i.e. $\bar{y}'(z) = 0$ for all $z$. The second case corresponds to perfect correlation, i.e. when $\sigma = 0 < \bar{\sigma}$. This case corresponds to the case with only one product, since for any history where $p_i(0) = 0$ for all $i = 1, .., n$ we have $p_i(t) = p(t)$ and $y(t) = n\,p(t)^2 = (1/n)\,[n\,p(t)]^2 = z(t)^2/n$. In this case only the values of $\bar{y}(z)$ at the edges of the state space can be achieved. The two diffusions give:

$$dy(t) = n\,\bar{\sigma}^2\,dt + 2\,z(t)\,\bar{\sigma}\,d\mathcal{W}^c(t) \tag{58}$$

$$dz(t) = n\,\bar{\sigma}\,d\mathcal{W}^c(t) \tag{59}$$

where $\mathcal{W}^c$ is standard BM. If $y = z^2/n$ we can write this also as:

$$dy(t) = (n\,\bar{\sigma}^2)\,dt + 2\sqrt{y(t)\,(n\,\bar{\sigma}^2)}\,d\mathcal{W}^c(t) \tag{60}$$

which coincides with the law of motion of the case of one product with an innovation variance of $n\bar{\sigma}^2$. The common optimal value for $\bar{y}$ at these two points can be found by solving the problem with no correlation with the same $r, B$ and $\psi$ but with $(\sigma', n') = (\sqrt{n}\,\bar{\sigma}, 1)$.

### F.1.1 The case of a large number of products

In this section we analyze the limit case as $n \to \infty$ in the presence of correlation. The process for the average square gap is the sum of two processes obtained in the benchmark case of no correlation. One process corresponds to the case of $n = 1$ with instantaneous variance $\bar{\sigma}^2$, and the other one is the deterministic process corresponding to the case of $n = \infty$ with drift

$\sigma^2$. Let us define

$$\tilde{y}(t) \equiv \frac{y(t)}{n} = \frac{1}{n}\left[\sum_{i=1}^{n} \sigma^2\, W_i(t)^2 \;+\; \bar{\sigma}^2\, \bar{W}(t)^2 \;+\; 2\,\sigma\bar{\sigma} W_i(t)\, \bar{W}(t)\right]$$

with $\tilde{y}(0) = 0$. Thus

$$\begin{aligned} Var\left[\tilde{y}(t)\,|\,\tilde{y}(0)=0\right] &= \frac{1}{n^2}\left[\sum_{i=1}^{n} \sigma^4\, E[W_i(t)^4] \;+\; \bar{\sigma}^4\, E[\bar{W}(t)^4] \;+\; 2\,\sigma^2\bar{\sigma}^2 E[W_i(t)^2]E[\bar{W}(t)^2]\right] \\ &+ \frac{n(n-1)}{n^2}\bar{\sigma}^4 E[\bar{W}(t)]^4 = \frac{t^2}{n}\left[\sigma^4 + \bar{\sigma}^4 + 2\,\sigma^2\bar{\sigma}^2\right] \;+\; \frac{t^2(n-1)}{n}\bar{\sigma}^4 \end{aligned}$$

so we can write

$$\mathrm{d}\tilde{y}(t) \;=\; \left[\sigma^2 + \bar{\sigma}^2\right]\mathrm{d}t + 2\bar{\sigma}\;\tilde{z}(t)\;\mathrm{d}\mathcal{W}(t) \quad \text{and} \quad \mathrm{d}z(t) = \bar{\sigma}\;\mathrm{d}\mathcal{W}(t) \tag{61}$$

or

$$\tilde{y}(t) = \sigma^2 t + y_1(t) \text{ where } \mathrm{d}y_1 = \bar{\sigma}^2\,\mathrm{d}t + 2\bar{\sigma}\sqrt{y_1}\,\mathrm{d}\,\mathcal{W} \tag{62}$$

Thus we can consider a problem where the objective function depends on $y_1$ and time since last adjustment: $B(y_1 + \sigma^2 t)$ and the law of motion of $y_1$ is given by equation (62).

### F.1.2 Discrete time approximation

Let $\Delta > 0$ be the length of the time period. We approximate the pair of diffusions as follows:

$$y' \;=\; \mathcal{Y}(y,z,\mathrm{e}) \equiv \max\left\{y + n\left[\sigma^2 + \bar{\sigma}^2\right]\Delta + 2\sqrt{\Delta}\sigma\sqrt{y}\;\mathrm{e}^a + 2\sqrt{\Delta}\,\bar{\sigma}z\;\mathrm{e}^c\;,\;0\right\} \tag{63}$$

$$\begin{aligned} z' \;=\; &\mathcal{Z}(y',y,z,\mathrm{e}) \equiv \max\Bigg[-\sqrt{y'\,n}\;, \\ &\min\left\{z + n\sqrt{\Delta}\,\bar{\sigma}\;\mathrm{e}^c + \sqrt{n\Delta}\sigma\left[\frac{z}{\sqrt{n\,y}}\;\mathrm{e}^a + \sqrt{1-\left(\frac{z}{\sqrt{n\,y}}\right)^2}\;\mathrm{e}^b\right]\;,\;\sqrt{n\,y'}\right\}\Bigg] \end{aligned} \tag{64}$$

where $\mathrm{e} = \{\mathrm{e}^a, \mathrm{e}^b, \mathrm{e}^c\}$ is a vector of three independent random variables, with zero mean and unit variance. An example will be three binomials, each taking the values $\pm 1$ with probability $1/2$. The set of binomial shocks is denoted by $\mathrm{E} = \{\mathrm{e}^i \in \{-1, 1\}$ for $i = a, b, c\}$. We let E denote the set of innovations, and for notational purposes we use $F$ for its its CDF. The max and min operators in the previous definitions ensure that $y$ stays positive and that $z^2 \le ny$. Let the state space be $\mathbb{S} = \left\{(y,z) : y \ge 0\,,\, -\sqrt{y\,n} \le z \le \sqrt{y\,n}\right\} \subset \mathbb{R}_+ \times \mathbb{R}$.

To simplify the notation we let $\mathcal{S} : \mathbb{S} \times \mathrm{E} \to \mathbb{S}$ mapping $(y, z, \mathrm{e})$ into $(y', z')$ as follows:

$$(y', z') = \mathcal{S}(y,z,\mathrm{e}) \equiv (\,\mathcal{Y}(y,z,\mathrm{e})\;,\;\mathcal{Z}(\mathcal{Y}(y,z,\mathrm{e})\,,y,z,\mathrm{e})\,) \tag{65}$$

The discrete time Bellman equation becomes for all $(y,z) \in \mathbb{S}$:

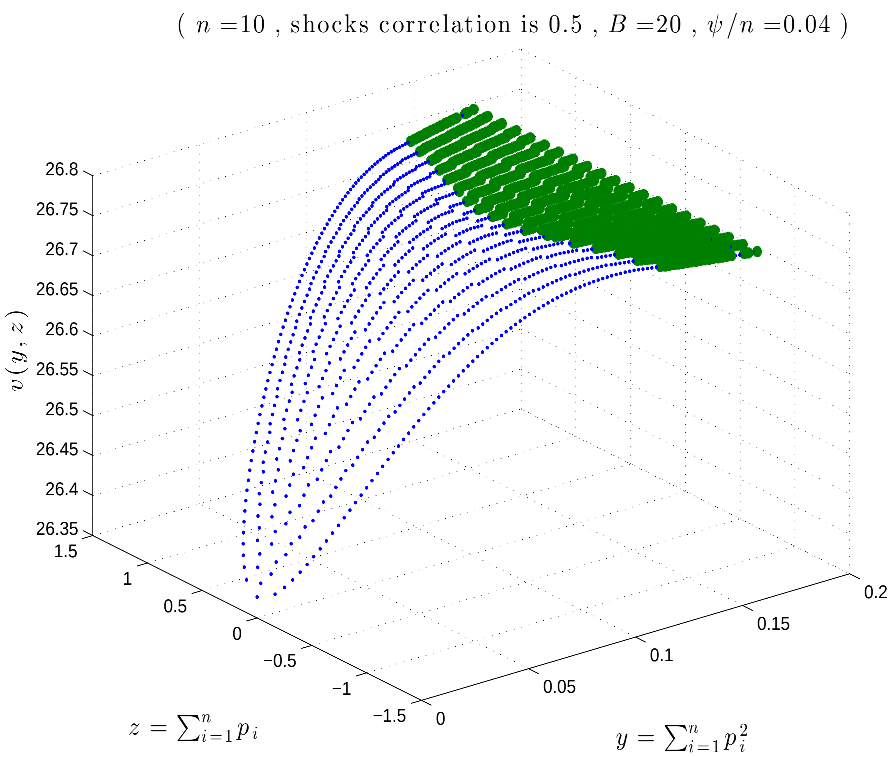
$$v(y,z) = \min \left\{ \psi + v(0,0) \ , \ \Delta B y + e^{-\Delta r} \int_{\mathrm{e} \in \mathrm{E}} v \left( \mathcal{S}\left(y, z, \mathrm{e}\right)\right) dF(\mathrm{e}) \right\} \tag{66}$$

We solve $v(y,z)$ by repeated iterations in a grid included in $\mathbb{S}$. We use the value function for the uncorrelated case (with the same volatility for each price gap, i.e. $\sqrt{\sigma^2 + \bar{\sigma}^2}$) as the initial function. To compute expected value of the value function in each iteration we need to be able to evaluate the value function outside the grid points. Let's us denote a set of $N$ grid points in $\mathbb{S}$ by $\mathbb{G}$. To do so, in each iteration we fit a polynomial in $(y,z)$ to the grid points that are in the inaction region. We use the following polynomial:

$$v(y,z) = \beta_0 + \beta_1\, y + \beta_2\, y^2 + \beta_3\, y^3 + \beta_4\, z^2 + \beta_5\, z^4 + \beta_6\, y\, z^2 + \beta_7\, y^2\, z^2 + \beta_8\, y^3\, z^2 \tag{67}$$

for $(y,z) \in \mathbb{G}$. Note that this polynomial imposes symmetry w.r.t. $z$, and includes the third order approximation for the case of no correlation. For the case with no correlation we have found that this functional form gives very accurate results. The coefficient of this polynomial are fitted to the grid points $(y_i, z_i)$ for which $v(y_i, z_i) < \max_{j\in\mathbb{G}} v(y_j, z_j)$, i.e. it is fitted to the inaction set.

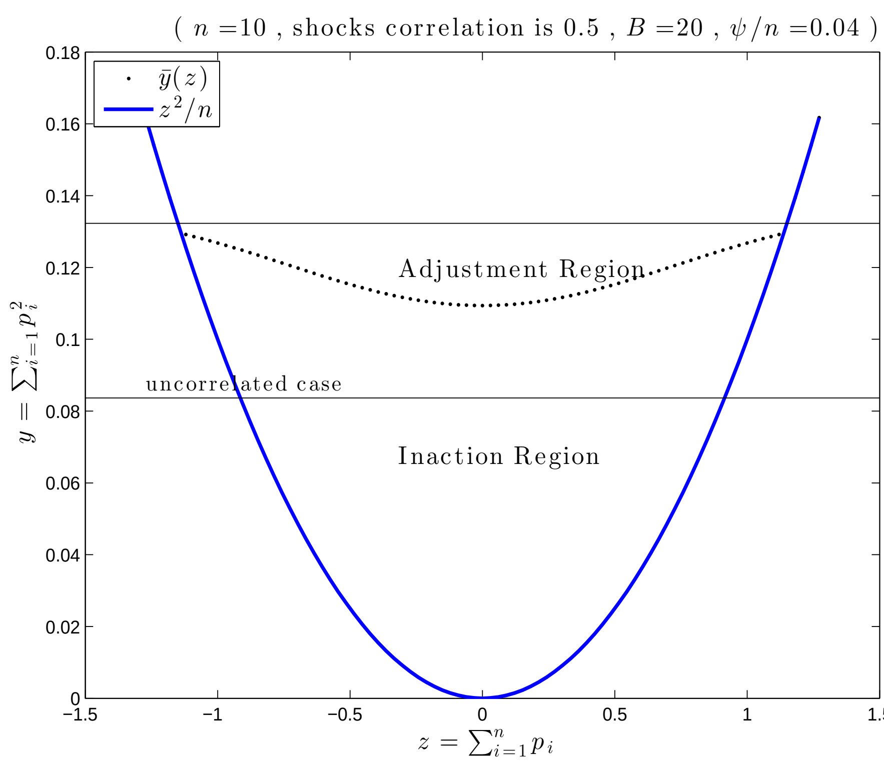
Figure 14: Value function $v(y,z)$




We display two numerical examples of the value function and policies for the following parameters. We measure time in years and let the real discount rate by 5% or $r = 0.05$, use a markup of about 15% which implies $B = 20$, and a volatility of each price gap of 10% with a pair-wise correlation of 1/2, so $\sigma = \bar{\sigma} = 0.05$. The menu cost is 4% of friction-less profits per good, so $\psi/n = 0.04$. We solve the model for daily periods, so $\Delta = 1/365$. We display results for the case of $n = 10$ products per firm. Figure 14 plots the value function as a function of $y$ and $z$.

Figure 15 plots the decision rule of the firm. This figure plots the level of the value function in all the grid points we have used to compute it. The values of the value function for which control (i.e. price adjustment) is optimal are marked with green starts. The feasible state space for the firm is given by the region in the $y, z$ inside the parabola. For each $z$ the value function has a similar shape as the one for the case of no correlation. Fixing $y$, the value function is decreasing in $|z|$. This is because higher $|z|$ implies higher conditional variance of $y$, and hence higher option value. As anticipated the function $\bar{y}(z)$ is symmetric around $z = 0$ and increasing for larger values of $|z|$. The fact that $\bar{y}$ is increasing in $|z|$ reflects the option value effect of $z$ just described. For comparison we plot an horizontal line with the value $\bar{y}$ for the case on uncorrelated price gaps but with the same innovation variance per unit of time of each of the price gaps, i.e. with with $\sigma^2 + \bar{\sigma}^2$ as well as the case with perfectly correlated price gaps. While the inaction set can be summarized in an $\mathbb{R}^2$ space, we emphasize that the state of the problem is $n$ which can be much higher, for instance it is $n = 10$ for this example.

Figure 15: Decision rules



## F.2 Cross section implications with correlated shocks

We use the decision rules described above to produce the invariant distribution of a cross section of firms using simulations. The model parameterization is close to the one used in the main body of the paper, i.e. it produces a frequency of adjustments per year that is $N_a = 1.3$ and a standard deviation $Std(\Delta p) = 0.11$. Figure 16 plots the standardized distribution of price changes $w(\Delta p)$ for different values of $n = 1, 2, 3, 50$ and different levels of correlation between the shocks: $\rho = 0$, our baseline case, as well as $\rho = 0.5$ and $\rho = 0.75$.

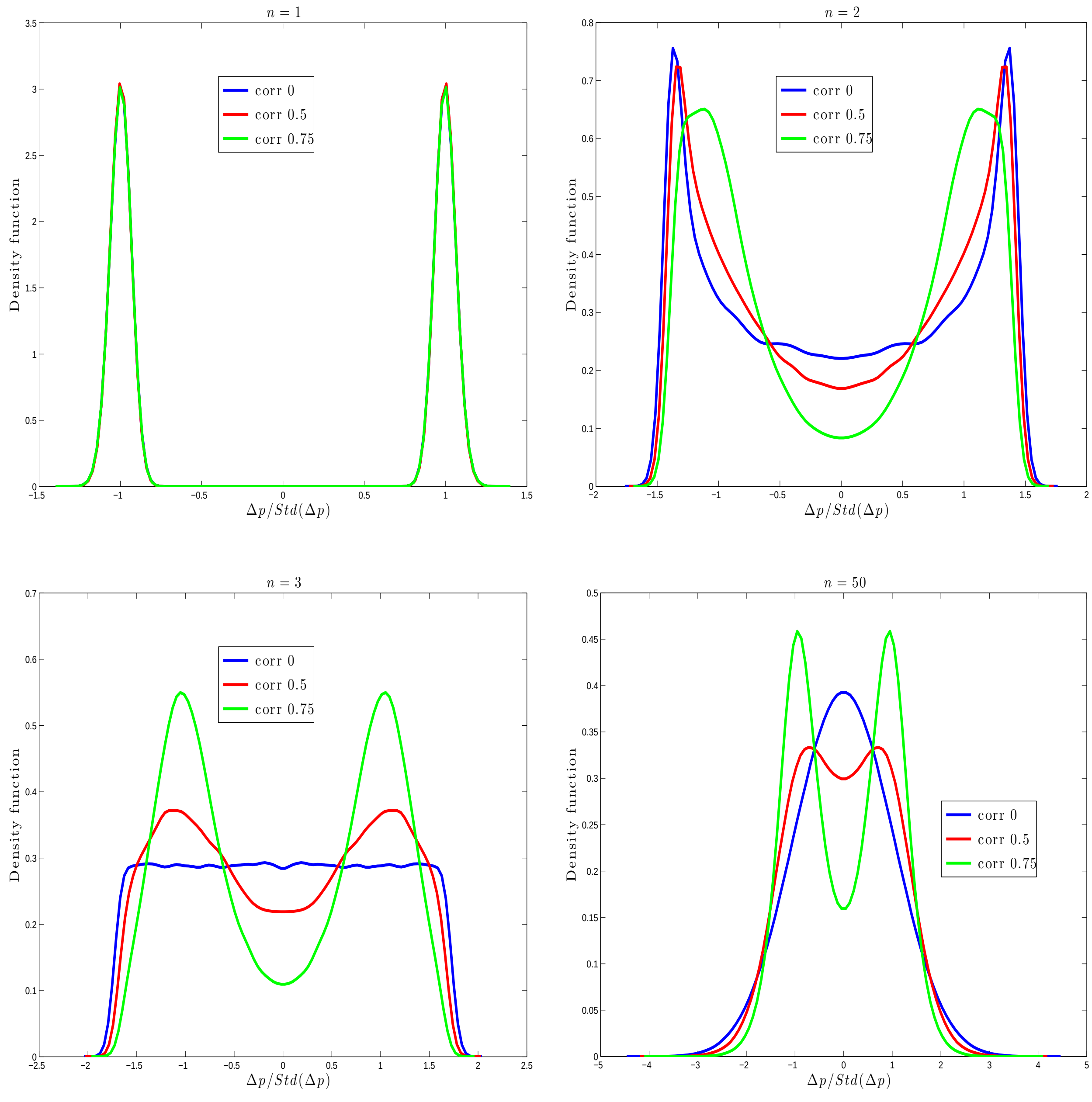
The marginal distribution of price changes is obtained as follows. First we solve for the optimal decision rules, which gives us the function $\bar{y}(\cdot)$. Then we simulate a discrete time version of the $n$-dimensional process $\{p_t\}$ described equation (47), and use the optimal policy to stop it the first time it reaches the adjustment region, upon which the $n$ price gaps are set to zero. In particular each draw of the joint $n-$dimensional distribution is obtained by starting $p_{0,i} = 0$ for all $i = 1, ..., n$, simulating $\{p_t\}$ and the associated $y_t, z_t$ defined by equation (69). Letting the first time $\tau$ that $y_\tau \geq \bar{y}(z_\tau)$, we obtain each of the $n$ price changes as $\Delta p_i = -p_{\tau,i}$. We set the length of the time period $\Delta = 1/(2 \times 365)$, i.e. half a day, and simulate 50,000 price changes of the $n$ products.[21]. We represent the outcome of the simulations by fitting a smooth kernel density to the simulated data.

The first panel contains the distribution of price changes for the case of one product, i.e. $n = 1$. In this case $\bar{y}$ is flat, and the correlation should make no difference. The distribution of price changes should be degenerate, but given that we simulate a discrete time process, albeit with a small time period, the price changes are distributed tightly, but not degenerately, around two values. This is included as a check of the procedure and to control the difference that is due to the discretization of the model. The case of $n = 2$ shows that as the correlation increases the distribution has more mass for small price changes. Not surprisingly, adding correlation to the shocks makes the $n = 2$ case to be closer to the $n = 1$ case, a feature that is important for both its empirical plausibility (i.e. the comparison with empirical distribution of price changes) and for the predicted effect of monetary shocks. The case of $n = 3$ is particularly revealing since for zero correlation the distribution is uniform, but as the correlation is positive the density decreases to have a minimum at zero and two maxima at a high values of the absolute value, as in the case of $n = 1$ . The case of $n = 50$ is also informative because with zero correlation the marginal distribution of price changes is essentially normal. Nevertheless with positive correlation the distribution of price changes remains bimodal, with a minimum of its density at zero.

Interestingly the simultaneously near normality and bimodality (or the dip on the density of the distribution on a central value of price changes) which is displayed in the figure for $n = 50$ is apparent in several data sets such as Midrigan (2009) using scanner AC Nielsen data for US (see Figure 1, bottom two panels), in Wulfsberg (2010) using Norway's CPI data (see Figure 4), and has been explicitly tested and estimated by Cavallo and Rigobon (2010) using online supermarket data for 23 countries.

[21] We simulate half as many, and then we use a symmetry to reflect it and obtain a sample twice as large, a standard importance sampling procedure

Figure 16: Distribution of price changes: $\Delta p_i$




All distributions have been standardized to have $Std(\Delta p) = 0.1$.

## F.3 Impulse responses with correlated shocks

In this section we compute the impulse response function of the price level to a once and for all shock to the money supply. We investigate the effect of correlation on the results. In particular for a shock of the same size, and for several values of $n$ we compare the IRF of prices for correlation $\rho = 0$, $\rho = 0.4$ and $\rho = 0.75$. We stress that to solve for the IRF for any $n$ we only need to keep track of a two dimensional object, which makes the procedure computationally feasible.

We obtain the IRF as follows. We start with the optimal steady state decision rules, summarizing them by the function $\bar{y}(\cdot)$.

- We simulate a discrete time version of the the process for $\{y_t^j, z_t^j\}$ for a large number of firms, say $j = 1, ..., M$. We use $M = 500,000$.

- We let $t = 0$ the first period, $t = T$ the period where the aggregate monetary shock of size $\delta$ occurs, and $t = T + T'$ the last period of the simulation.

- The first $T$ periods discrete time versions of the firms' sate are simulated so that at $t = T$ the distribution of $(y_t^j, z_t^j)$ across $j = 1, ..., M$ gives an accurate representation of the invariant distribution without aggregate shocks. During the first $T$ periods whenever a firm's state reaches $y_t^j \geq \bar{y}(z^j)$ we set $y_t^j = z_t^j = 0$, corresponding to a price adjustment on the $n$ products, and we keep simulating the process for $y_{t+1}^j, z_{t+1}^j$ according to its law of motion.

- At time $t = T$ we shock the values of $(z_t^j, y_t^j)$ of each of the $j$ firms by decreasing the price gap in each of the $n$ component by $\delta > 0$.

- The value of the state for each firm right after the shock but right before the adjustment can be characterized as a the following two dimensional shift. We denoting with $Y'$ and $Z'$ the post monetary shock (but pre-adjustment) value of the state for a firm with state $y, z$:

$$Y'(\delta, y, z) \equiv \sum_{i=1}^{n} (p_i - \delta)^2 = y - 2\,\delta\,z \,+\, n\,\delta^2 \quad \text{and} \quad Z'(\delta, y, z) \equiv \sum_{i=1}^{n} (p_i - \delta) = z \,-\, n\,\delta \,.$$

- At time $t = T$ before the adjusting decision takes place we replace $y_t^j$ by $Y'(\delta, y_t^j, z_t^j)$ and $z_t^j$ by $Z'(\delta, y_t^j, z_t^j)$ for all $j = 1, ..., J$.

- We simulate the process for state for each firm $j$ up to the first time $t = \tau_j \geq T$ in which it adjust its prices. In particular the first time $\tau_j$ were $y_{\tau_j}^j \geq \bar{y}\left(z_{\tau_j}^j\right)$. Note that at time $\tau_j$ firm $j$ sum of the price changes across the $n$ goods equals the negative of $z_{\tau_j}^j$. If at time $t = T + T'$ the firm $j$ has not adjusted its price, we force to change it.

- For each time $t = T, ...T + T'$ we compute the contribution of each firm to the change in equal weighted aggregate price level:

$$\theta_t = -\frac{1}{M}\sum_{j=1}^{M} z^j_{\tau_j} \times \mathcal{I}_{t=\tau_j} \quad \text{for} \quad t = T, T+1, ..., T+T' \ .$$

where $\mathcal{I}_{t=\tau_j}$ is the indicator function taking the value of one if the firm $j$ adjust the price at time $t$ and zero otherwise.

- The effect on the equal-weighted price level at time $t$ is:

$$\mathcal{P}(\delta, t) = \sum_{s=T}^{t} \theta_t \quad \text{for} \quad t = T, T+1, ..., T+T' \ .$$

Figure 17 displays the result of IRF of the price level with respect to a monetary shock. Each panel of this figure consider different values of the number of product ($n = 2, 3, 10$ and $n = 50$), and for each $n$ we plot 3 cases: the case of $n = 1$ (which is the same as the case with perfect correlation), correlation equal to zero ($\rho = 0$) and correlation equal to one half ($\rho = 1/2$). Motivated by the scaling and stretching results we have shown for the case of zero correlation, we normalize the parameters so that the expected number of price changes per year is 1 ($N_a = 1$) and consider a shock of 10% of the size of the steady state standard deviation of price changes (i.e. say $\delta = 0.01$ and $Std(\Delta p) = 0.1$, i.e. one percent change in money supply and 10% steady state standard deviation of price changes). Thus, each figure corresponds to an economy with the same steady state. The case of $n = 2$ shows that going from zero correlation to one half makes reduces by more than half the distance between the $n = 2$ and $n = 1$ case, i.e. it significantly increases the price flexibility at all horizons. The other cases are even more extreme, i.e. the vertical distance between the IRF with correlation $\rho = 1/2$ and $\rho = 1$ is very small compared with the distance between the IRF with $\rho = 1/2$ and the IRF for $\rho = 0$. Recall that the effect on ouput is proportional to the vertical distance between the level of the IRF and a constant at $\delta$, so a correlation of one half reduced the effect of output significantly towards the case of $n = 1$, i.e. towards the Golosov and Lucas case.

## F.4 The case with drift and correlation

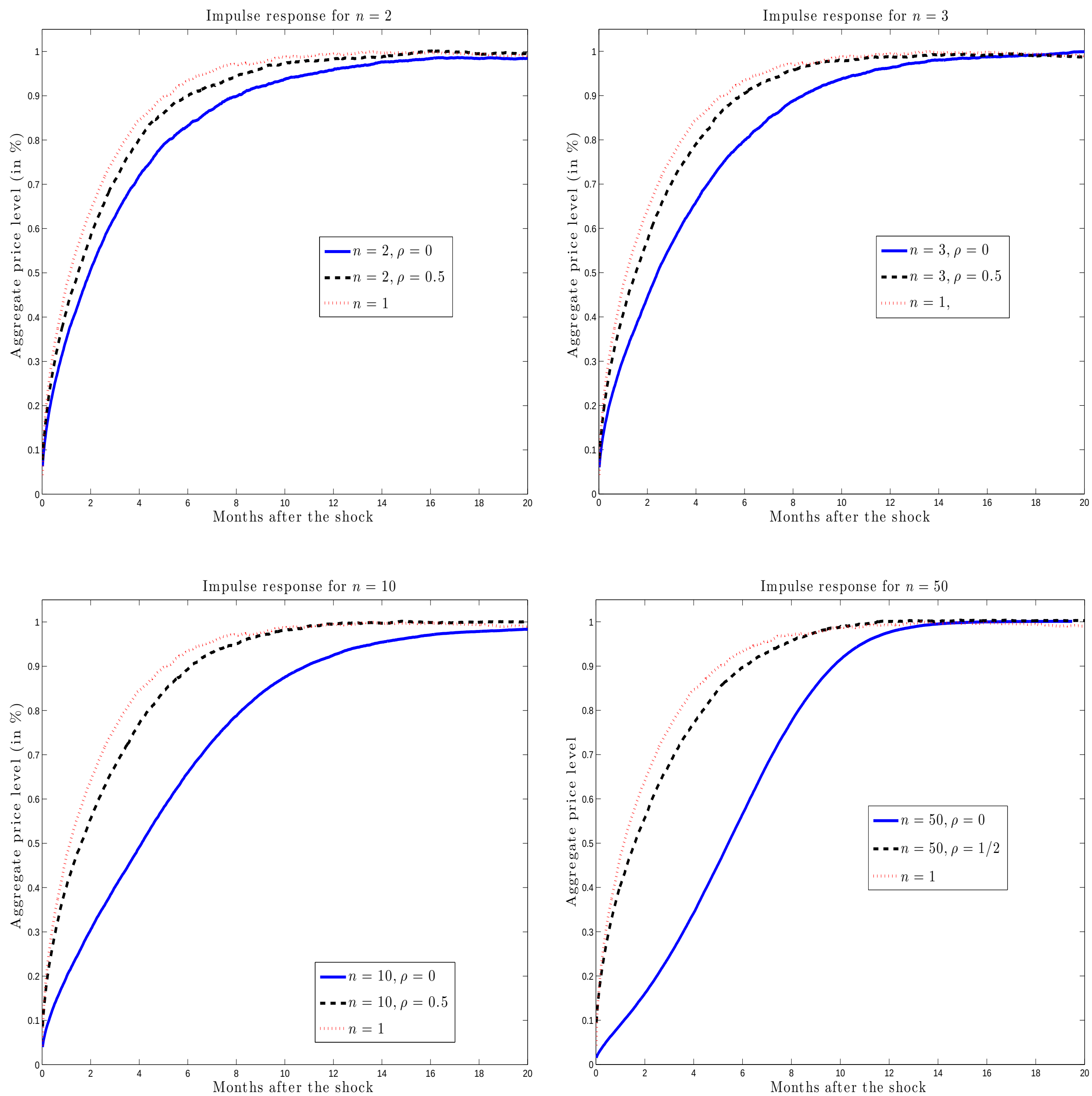
This section further extends the problem to the case of the joint presence of drift and correlation. Let each price gap follow

$$\mathrm{d}p_i(t) = -\mu\, \mathrm{d}t + \bar{\sigma}\, \mathrm{d}\bar{\mathcal{W}}(t) \ + \sigma \mathrm{d}\mathcal{W}_i(t) \ \text{ for all } i = 1, ..., n \ . \tag{68}$$

where $\bar{\mathcal{W}}, \mathcal{W}_i(t)$ are independent standard BMs. Define:

$$y(t) = \sum_{i=1}^{n} p_i^2(t) \text{ and } z(t) = \sum_{i=1}^{n} p_i(t) \tag{69}$$

Figure 17: Impulse response to a monetary shock: $\delta/Std(\Delta p) = 0.1$



Using Ito's Lemma:

$$\mathrm{d}y(t) = \left[n\sigma^2 + n\bar{\sigma}^2 - 2\mu z(t)\right] \mathrm{d}t + 2\sigma \sum_{i=1}^{n} p_i(t)\mathrm{d}\mathcal{W}_i(t) + 2\bar{\sigma} \left[\sum_{i=1}^{n} p_i(t)\right] \mathrm{d}\bar{\mathcal{W}}(t)$$

and

$$\mathrm{d}z(t) = -n\mu\, \mathrm{d}t + n\bar{\sigma}\mathrm{d}\bar{\mathcal{W}}(t) + \sigma \sum_{i=1}^{n} \mathrm{d}\mathcal{W}_i(t)$$

This implies that :

$$\begin{aligned}
\mathbb{E}\left[\,\mathrm{d}y(t)\,\right]^2 &= 4\sigma^2 \left(\sum_{i=1}^{n} p_i^2(t)\right) \mathrm{d}t + 4\bar{\sigma}^2 \left(\sum_{i=1}^{n} p_i(t)\right)^2 \mathrm{d}t \\
&= 4\sigma^2\, y(t)\, \mathrm{d}t + 4\bar{\sigma}^2\, z(t)^2\, \mathrm{d}t \;, && (70) \\
\mathbb{E}\left[\,\mathrm{d}z(t)\,\right]^2 &= \sigma^2\, n\, \mathrm{d}t + \bar{\sigma}^2\, n^2\, \mathrm{d}t \quad \text{and} && (71) \\
\mathbb{E}\left[\,\mathrm{d}y(t)\, \mathrm{d}z(t)\,\right] &= 2\sigma^2 \left(\sum_{i=1}^{n} p_i(t)\right) \mathrm{d}t + 2n\bar{\sigma}^2 \left(\sum_{i=1}^{n} p_i(t)\right) \mathrm{d}t \\
&= 2\left(\sigma^2 + n\bar{\sigma}^2\right)\, z(t)\, \mathrm{d}t && (72)
\end{aligned}$$

Thus define the diffusions

$$\begin{aligned}
\mathrm{d}y(t) &= \left[n\sigma^2 + n\bar{\sigma}^2 - 2\mu z(t)\right] \mathrm{d}t + 2\sigma\sqrt{y(t)}\; \mathrm{d}\mathcal{W}^a(t) + 2\bar{\sigma} z(t)\; \mathrm{d}\mathcal{W}^c(t) && (73) \\
\mathrm{d}z(t) &= -n\mu\, \mathrm{d}t + n\,\bar{\sigma}\; \mathrm{d}\mathcal{W}^c(t) + \\
&\quad \sqrt{n}\sigma \left[\frac{z(t)}{\sqrt{n\, y(t)}}\; \mathrm{d}\mathcal{W}^a(t) + \sqrt{1 - \left(\frac{z(t)}{\sqrt{n\, y(t)}}\right)^2}\; \mathrm{d}\mathcal{W}^b(t)\right] && (74)
\end{aligned}$$

where $(\mathcal{W}^a, \mathcal{W}^b, \mathcal{W}^c)$ are three standard independent BM's.

## F.5 Cross partials and different elasticities within and across firms

In this section we show that:

1. A quadratic approximation to a cost function that is symmetric across the $n$ price gaps but with non-zero cross partial derivative can be accommodated by adding the term $D\, z$ for a constant $D$ to the flow cost function which becomes $B\, y \,+\, D\, z^2$.

2. The approximation with a non-zero cross partial derivative can be used to consider a nested CES case, where the aggregate of products produced by a firm have elasticity of substitution $\eta$ between firms, and the products produced by a firm have elasticity $\varrho$ between them. This yields the following expressions for the cost function, $B$, and $D$:

$$B\, y \,+\, D\, z^2 \equiv \frac{1}{n}\left(\frac{\varrho(\eta-1)}{2}\, y - \frac{(\varrho-\eta)(\eta-1)}{2n}\, z^2\right) \tag{75}$$

which becomes $By = \eta(\eta-1)/(2n)\, y$ in the benchmark case.

3. The effect of the different elasticities described in Item (2) in the constant $D$ of the cross product is proportional to $1/n$, so it vanishes for moderately high $n$, as can be seen in equation (75).

4. The effect on $B$ of incorporating different elasticities is that the value of $B$ can be larger than the implied by the elasticity $\eta$ and its optimal markup in the frictionless case, or equivalently the model produces the same behaviour with larger fixed cost $\psi$. Equation (75) shows that $B$ is essentially the product of the two elasticities, $\eta$ and $\varrho$.

5. From the previous analysis, one can conclude that as $n \to \infty$ the dynamics for the model different elasticities is identical to the one with the same elasticities.

6. The effect of introducing two different elasticities is quantitatively very small in both the shape of the distribution of price changes and the IRF to monetary shocks, especially for moderately large values of $n$.

The rest of this section develops the ideas presented below in detail. Before getting into that we offer few remarks on the results listed above.

To see that introducing symmetric cross partials yields the expression in Item (1) one just develops the squares in the relevant expressions. For completeness we include the relevant algebra below.

To understand the expressions for $B$ and $D$ as a function of the elasticities in equation (75), and the effect in Item (4) where if the products sold by the same firm are better substitutes than the aggregate across firms $B$ is larger, we consider two simple examples. Assume that $\varrho$ is almost $\infty$ and hence products sold by the same firm are almost perfect substitutes. Furthermore, just o simplify, assume that there are only to products, $n = 2$. We consider two examples, where $y$ is the same but $z^2$ differs, as a way to understand the expressions on equation (75). In the first example the price gaps across the two goods are equal in absolute value and of opposite sign, so $z = 0$ and in the second example the price gaps are equal in absolute value and sign, so $z^2 > 0$. In the first case the firm only sells the good with the lower price. In the second case profits for the firm higher (cost is smaller) since the relative prices are same.

The reason why Item (6) holds is that is that, contrary to the case with correlation, there is no change in the law of motion on $y$ and $z$, just different optimal function $\bar{y}(\cdot)$. But given the previous result the expressions for $D$ and $B$ the function $\bar{y}$ is almost flat for moderate $n$.

### Cross Products in the approximation of the profit function

Consider a profit function of the firm as a function $\Pi(p)$ of the $n$ price gaps $p = (p_1, ..., p_n)$ and assume that the price gaps are interchangeable, so that profits are the same for any permutation of the price gaps such, for example $\Pi(a, b, ...) = \Pi(b, a, ...)$. Evaluating this function around the maximizing choice $p_i = 0$ for all $i$ we have

$$\bar{b} \equiv -\frac{1}{\Pi(0,0,...,0)} \frac{\partial \Pi^2(0,...,0)}{\partial p_i \partial p_i} \text{ and } \bar{d} \equiv -\frac{1}{\Pi(0,0,...,0)} \frac{\partial \Pi^2(0,...,0)}{\partial p_j \partial p_i} \text{ if } i \neq j \ ,$$

where the negative sign is included to define the cost problem. We can write:

$$\begin{aligned}\frac{\Pi(0,0,...,0)-\Pi(p_1,p_2,....,p_n)}{\Pi(0,0,...,0)} &= \frac{\bar{b}}{2}\left(\sum_{i=1}^{n} p_i^2\right)+\bar{d}\left(\sum_{1\leq i\neq j\leq 1} p_i\,p_j\right)+o\left(||p||^2\right)\\ &= \frac{\bar{b}-\bar{d}}{2}y+\frac{\bar{d}}{2}z^2+o\left(||p||^2\right)\equiv B\,y\,+\,D\,z^2\,+o\left(||p||^2\right)\end{aligned}$$

Thus we can define the second order approximation of $\Pi(\cdot)$ in terms of $y$ and $z$ as defined above. For $\partial\Pi^2/(\partial p\partial p)$ to be negative semi-definite around $p=0$ (or equivalently for the cost problem to be convex) we require: $\bar{b}-\bar{d}>0$ and $\bar{b}+(n-1)\bar{d}>0$, since $0\leq z^2\leq ny$ and $y\geq 0$. Note that if $\bar{d}=0$ we recover our benchmark case setting $\bar{b}/2=B$.

**Different elasticities between firms and within firms' products**

Now we consider the particular case where the cross product comes from a different elasticity of substitution between products produced by the firm, denoted by $\varrho$ and between the composite good produced by different firms, denoted by $\eta$. Let the period $t$ utility be:

$$\frac{c(t)^{1-\epsilon}}{1-\epsilon} \quad \text{with} \quad c(t)=\left[\int_0^1 c_k(t)^{1-1/\eta}dk\right]^{\frac{\eta}{\eta-1}} \quad \text{and} \quad c_k(t)=\left[\sum_{i=1}^{n}\left(Z_{ki}(t)\,c_{ki}(t)\right)^{1-1/\varrho}\right]^{\varrho/(\varrho-1)}.$$

Using CES structure of preference we can write the demand from the product $i$ of the firm $k$ at time $t$ as:

$$c_{ik}(t)=\left(\frac{P_{ik}(t)}{P_k(t)}\right)^{-\varrho}\;Z_{ik}(t)^{\varrho-1}\left(\frac{P_k(t)}{P(t)}\right)^{-\eta}\;c(t)$$

where $P_k(t)$ is the ideal price index of the products produced by firm $k$ and $P(t)$ is the ideal price index of all the goods produced in the economy:

$$P(t)=\left[\int_0^1 P_k(t)^{1-\eta}\,dk\right]^{\frac{1}{1-\eta}} \quad \text{and} \quad \frac{P_k(t)}{W(t)}=\left[\sum_{i=1}^{n}\left(\frac{P_{ki}(t)}{W(t)Z_{ki}(t)}\right)^{1-\varrho}\right]^{\frac{1}{1-\varrho}}$$

The time $t$ nominal profits of the firm $k$ are:

$$\begin{aligned}&\sum_{i=1}^{n}\left(P_{ik}(t)-Z_{ik}(t)W(t)\right)c_{ik}(t)\\ =\;& W(t)\left(\frac{P_k(t)}{P(t)}\right)^{-\eta}c(t)\sum_{i=1}^{n}Z_{ik}(t)^{\varrho-1}\left(\frac{P_{ik}(t)}{P_k(t)}\right)^{-\varrho}\left(\frac{P_{ik}(t)}{W(t)}-Z_{ik}(t)\right)\\ =\;& W(t)\left(\frac{W(t)}{P(t)}\right)^{-\eta}\left(\frac{P_k(t)}{W(t)}\right)^{\varrho-\eta}c(t)\sum_{i=1}^{n}\left(\frac{P_{ik}(t)}{W(t)Z_{ki}(t)}\right)^{-\varrho}\left(\frac{P_{ik}(t)}{W(t)Z_{ki}(t)}-1\right)\end{aligned}$$

Using the foc for $\ell(t)$ and $c(t)$:

$$\frac{W(t)\,(1+\tau_\ell)}{P(t)} = \alpha\; c(t)^\epsilon$$

we can write the the nominal profit of the firm $k$ at time $t$ as:

$$\begin{aligned} & \sum_{i=1}^{n} \left(P_{ik}(t) - Z_{ik}(t)W(t)\right) c_{ik}(t) \\ = \;\; & W(t) \left(\frac{\alpha}{1+\tau_\ell}\right)^{-\eta} c(t)^{1-\epsilon\eta} \left(\frac{P_k(t)}{W(t)}\right)^{\varrho-\eta} \sum_{i=1}^{n} \left(\frac{P_{ik}(t)}{W(t)Z_{ki}(t)}\right)^{-\varrho} \left(\frac{P_{ik}(t)}{W(t)Z_{ki}(t)} - 1\right) \end{aligned}$$

or omiting time indices, using $p_i$ for the price gap of the firm $k$ defined as $\exp(p_i) = \frac{\eta}{\eta-1} P_{ki}/(WZ_{ki})$ we get

$$\Pi(p_1, ..., p_n) \;\; = \;\; \left(\frac{\eta}{\eta-1}\right)^{-\eta} \frac{1}{\eta-1} \left[\sum_{i=1}^{n} e^{p_i(1-\varrho)}\right]^{\frac{\varrho-\eta}{1-\varrho}} \sum_{i=1}^{n} e^{-p_i\varrho}\, \left(\eta e^{p_i} - (\eta-1)\right)$$

where the scaled profit satisfy:

$$\sum_{i=1}^{n} \left(P_{ik}(t) - Z_{ik}(t)W(t)\right) c_{ik}(t) = W(t) \left(\frac{\alpha}{1+\tau_\ell}\right)^{-\eta} c(t)^{1-\epsilon\eta}\; \Pi\left(p_{1k}(t), ..., p_{kn}(t)\right)$$

so that $\Pi(0, ..., 0) = \left(\frac{\eta}{\eta-1}\right)^{-\eta} 1/(\eta-1)\, n^{1+\frac{\varrho-\eta}{1-\varrho}}$ We have:

$$\begin{aligned} \frac{\Pi_j(p_1, ..., p_n)}{\Pi(0, ..., 0)} \;\; &= \;\; \frac{1}{n} \left[\frac{1}{n}\sum_{i=1}^{n} e^{p_i(1-\varrho)}\right]^{\frac{\varrho-\eta}{1-\varrho}} \left\{ e^{(1-\varrho)p_j} \left[(\varrho-\eta)\eta \;\; - (\varrho-\eta)\,(\eta-1)\, \frac{\sum_{i=1}^{n} e^{-\varrho p_i}}{\sum_{i=1}^{n} e^{p_i(1-\varrho)}}\right] \right. \\ &+ \;\; \left. \left[(1-\varrho)\eta e^{p_j(1-\varrho)} + \varrho(\eta-1)e^{-\varrho p_j}\right] \right\} \end{aligned}$$

Thus

$$\begin{aligned} 0 \;\; &= \;\; \frac{\Pi_j(0, ..., 0)}{\Pi(0, ..., 0)} \;\; \text{for all } j = 1, ..., n, \\ \bar{b} \;\; &\equiv \;\; \frac{\Pi_{jj}(0, ..., 0)}{\Pi(0, ..., 0)} = -\frac{1}{n} \left\{ (\varrho-\eta)(1-\varrho) + \frac{(\varrho-\eta)(\eta-1)}{n} + (1-\varrho)^2\eta + \varrho^2(1-\eta) \right\}, \\ \bar{d} \;\; &= \;\; \frac{\Pi_{ji}(0, ..., 0)}{\Pi(0, ..., 0)} = -\frac{1}{n}\frac{(\varrho-\eta)(\eta-1)}{n} \;\; \text{for} j \neq i \end{aligned}$$

The conditions for concavity of the profit function (or convexity of the cost function) are

$$\begin{aligned} \bar{b} - \bar{d} \;\; &= \;\; -(\varrho-\eta)(1-\varrho) - (1-\varrho)^2\eta - \varrho^2(1-\eta) = \varrho(\eta-1) > 0 \;\; \text{and} \\ \bar{b} + (n-1)\bar{d} \;\; &= \;\; -(\varrho-\eta)(1-\varrho) + (\varrho-\eta)(\eta-1) - (1-\varrho)^2\eta - \varrho^2(1-\eta) = \eta(\eta-1) > 0 \end{aligned}$$

which are satisfied provided that $\eta > 1$.

**Effect of different elasticities for large $n$**

We finish this section with an asymptotic result: as $n$ get large one can ignore the presence of cross products. The form of the coefficient for the cross-products derived above means that we can write the period return as as:

$$\frac{\varrho(\eta-1)}{2}\frac{y}{n} - \frac{(\varrho-\eta)(\eta-1)}{2}\left(\frac{z}{n}\right)^2$$

As we let $n \to \infty$, by the law of large numbers, $z/n$ converges with probability one to its expected value, namely 0. In this case the objective function, and thus decision rules, converge to the same ones derived for the case with no cross-products, i.e. $\bar{y}(z)$ is flat, i.e. independent of $z$. As $n \to \infty$, the process for $y/n$ becomes the same deterministic process as in the benchmark case with one common elasticity. Thus all the analysis for the case of no cross product apply as $n \to \infty$. The only difference in this case is the interpretation of $B$.