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Commodity Money with Frequent Search

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Abstract

A prominent feature of the Kiyotaki and Wright (1989) model of commodity money is the multiplicity of dynamic equilibria. We show that the frequency of search is strongly related to the extent of multiplicity. To isolate the role of frequency of search in generating multiplicity, we (i) vary the frequency of search without changing the frequency of finding a trading partner and (ii) focus on symmetric dynamic equilibria, a class for which we can sharply characterize several features of the set of equilibria. For any finite frequency of search this class retains much of the multiplicity. For each frequency we characterize the full set of equilibrium payoffs, strategies played, and dynamic paths of the state variables. Indexed by any of these features, the set of equilibria converges uniformly to a unique equilibrium in the continuous search limit. We conclude that when search is frequent, the seemingly exotic dynamics are irrelevant.

KEYWORDS: Commodity Money, Search, Multiple Equilibria, Sunspots.

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1 Introduction

In their seminal contribution, Kiyotaki and Wright (1989) introduced a search theoretic model of commodity money in which goods can function as a medium of exchange, sparking a voluminous literature exploring the microfoundations of money. Characterizing the set of dynamic equilibria is an essential part of many applications. These models feature a large multiplicity of dynamic equilibria that includes cycles, sunspots, and other non-Markovian equilibria. While multiplicity is in some sense a natural feature of monetary models, it can also make analyzing dynamics a challenging task.

This paper demonstrates the tight connection between the extent of multiplicity and the frequency of search. We adapt the model parameters so that one can vary the frequency of search without altering other features of the economy such as the frequency of actually meeting trading partners in a given unit of time. Agents search each period, but are less likely to find a trading partner if the time period is short. By varying the period length, we can study economies with different search frequencies.

We focus on the set of symmetric equilibria in an economy with symmetric parameters and initial conditions. Within this class of equilibria, the role of the frequency of search is especially stark. With any strictly positive interval between search opportunities, the set of symmetric equilibria retains much of the multiplicity. In contrast, in the limiting case in which agents search continuously there is a unique dynamic equilibrium.

The fact that there are multiple equilibria with *any* strictly positive interval between search opportunities but a unique equilibrium with continuous search can give the impression that there is a qualitative difference between the two. If the relevant interval between search opportunities is indeed positive, one might think that a model with continuous search may be ignoring potentially interesting and relevant dynamics.[1] We show that despite the qualitative difference, the unique equilibrium of the continuous search limit is a good approximation to *every* equilibrium when search is frequent.

We characterize the set of payoffs consistent with a symmetric equilibrium for all points in the state space. The size of this set varies directly with the probability of meeting a trading partner within a single period, decreasing monotonically as search becomes more frequent. While we cannot characterize the strategies played within any particular period, we show that the average strategy played within any interval of time converges uniformly to the strategies of the continuous search limit. Lastly, we show that the set of equilibrium paths of the economy converges uniformly to the equilibrium path of the continuous search limit. These results imply that if search is frequent, all dynamic equilibria are well approximated by the unique equilibrium of the continuous search limit.

The connection between multiplicity and frequency of search is a subtle one. To get at this relationship, we first describe the forces driving strategic decisions. Commodity money arises because there is no double coincidence of wants. An individual may be willing to trade for a commodity hoping to later meet a trading partner who desires that commodity.

Individuals' strategies of whether to accept commodity money are linked temporally. If those who produce the good that agent j wants will not be accepting commodity money (the good j produce) in the future, then there is a stronger in-

[1] Carlstrom and Fuerst (2001) write "...as monetary theorists we must be careful in writing the basics of our models... continuous-time analysis simply sweep this fundamental issue under the rug... Since these indeterminacy issues arise for any discrete but arbitrarily small time period this resolution of the timing is artificial." Carlstrom and Fuerst (2001) were discussing how the timing assumptions of purchases requiring cash (whether real balances before or after purchases enter the utility function) affect determinacy with various interest rate rules. Fudenberg and Levine (2009) show games of imperfect monitoring are another context in which the distinction between discrete and continuous time is important.

centive for j to trade for commodity money (the good the others want) now. In this sense, trading for commodity money in one period and others not accepting it in future periods (and vice versa) are strategic complements. This provides the key to understanding the source of multiplicity.

The strategic complementarity weakens as search becomes more frequent. When search is infrequent, agents are more likely to trade within each period, so the strategy played within a single period has more influence on payoffs in subsequent periods. As a consequence, less frequent search corresponds to stronger complementarities, and hence a larger set of equilibria.

Continuous search is special in that the complementarities disappear completely. Stated differently, for there to be multiple equilibria there must be negative serial correlation in trading strategies. Just as there cannot be negative serial correlation in continuous time, there cannot be negative serial correlation when agents search continuously.

Our contribution is threefold. First, we point to little understood features of a model that has served as a blueprint for the extensive literature exploring the microfoundations of money. Many recent models share properties with the Kiyotaki and Wright (1989) model, and multiplicity comes from the same sources. While multiplicity is an important feature of these models, we paint a more full picture of the forces driving it. Second, for this class of models, we provide reassurance that when agents have frequent opportunities to search, little is lost when writing the model with agents searching continuously, as the dynamic equilibria of the discrete versions are well approximated by the continuous search model. This is useful because characterizing dynamics under continuous search can be considerably easier than in the discrete counterparts, especially in applications such as the evolution of an economy following the introduction of fiat money or changes in the quantity of money. Third, we make a methodological contribution by providing an approach to studying several features of the set of dynamic equilibria using easily accessible tools.

Our results are related to the findings of Abreu, Milgrom, and Pearce (1991), who study a continuously repeated prisoner's dilemma with imperfect monitoring. In that model, lengthening the period over which actions are held fixed increases the possibilities for cooperation, expanding the set of feasible equilibrium payoffs. Similarly, in a model of commodity money with infrequent search, the fact that strategies are held fixed for an entire period is a key feature that drives the large multiplicity of equilibria. While the game studied by Abreu, Milgrom, and Pearce (1991) is purely a game of coordination, the Kiyotaki and Wright (1989) model is an anonymous sequential game (see Jovanovic and Rosenthal (1988)) in which strategic concerns play no role. There are strategic complementarities that are inherently intertemporal, driven by real changes in the distribution of holdings.

The relationship between period length and determinacy has also arisen in the real business cycle literature. Several departures from the standard growth model lead to indeterminacy (see Benhabib and Farmer (1999)). In some of these models, the continuous time limit features a unique equilibrium. While superficially similar, these and the Kiyotaki and Wright (1989) models have disparate sources of multiplicity, and consequently the relationship between period length and multiplicity differ. In versions of the RBC model with external effects that generate increasing returns, there is typically a critical period length period length below which multiplicity disappears.[2] Determinacy for a short enough time period (i.e., large enough discount factor) is also a property of growth models that use an overlapping generations framework.[3] In contrast, in the Kiyotaki and Wright (1989) model, there is a continuum of dynamic equilibria for *any* positive period length.

Section 2 lays out the economic environment while Section 3 describes symmetric

[2] See Hintermaier (2005).

[3] See Boldrin and Montrucchio (1986).

equilibria. Section 4 gives examples of the many types of equilibria that can arise and shows that these exist for any finite frequency of search. Section 5 contains our main results: we characterize the set of perfect foresight equilibria and show how this varies with the frequency of search. Section 6 extends these results to include sunspot equilibria and we conclude in Section 7.

2 Model

There are three types of goods, labeled 1, 2, and 3, and a unit mass of infinitely lived individuals that specialize in consumption and production. There are three types of individuals, with equal proportions, indexed by the type of good they produce and like to consume: an individual of type i derives utility only from consuming good i, and produces only good $i+1$ (mod 3). Goods are indivisible and storable, but individuals can only store one good (and hence one type of good) at a time. Storage is costless.[4]

Time is discrete with h being the length of time elapsed between periods. Each period, individuals search for trading partners. Search is successful with probability $\alpha(h)$.[5] When an individual finds a potential trading partner, the two may exchange goods. If a type i individual is able to acquire and consume good i, she derives instantaneous utility $u > 0$. Immediately after consumption she produces a new unit of good $i+1$.

When two individuals meet, it will never be the case that each desires the good produced by the other. Even though there is never a double coincidence of wants, an

[4] Kiyotaki and Wright (1989) allow for storage costs to vary by good and by type. We will eventually specialize to a symmetric economic environment in which the storage cost is the same for all goods. In that environment the level of the storage cost is not relevant for any economic decisions or outcomes, so we avoid the extra notation by setting the storage cost to 0.

[5] If $\alpha(h) \approx \alpha_0 h$ then the frequency of meeting a potential trading partner in a given unit of time is roughly independent of h. We leave the functional form unspecified for greater generality.

individual may accept a good that she does not want to consume in order to exchange it later for the good she desires. In this way, the intermediate good acts as a medium of exchange.

Individuals discount future flows with the discount factor $\beta(h) < 1$. We assume that $\beta(h)$ is strictly decreasing and that $\lim_{h\downarrow 0} \beta(h) = 1$ (e.g. $\beta(h) = e^{-rh}$). In addition, we assume that $\alpha(h)$ is strictly increasing, and that $\lim_{h\downarrow 0} \frac{\alpha(h)}{h} = \alpha_0$ so that the continuous time limit is well defined.

Let $I = \{1, 2, 3\}$ be both the set of goods and the set of types, and $\mathbf{T} = \{nh\}_{n=0}^{\infty}$ denote the set of times when individuals can search.

2.1 Strategies and Equilibrium

A strategy for individual i is a function $\tau^i : I^2 \to \{0, 1\}$, where $\tau^i(j, k) = 1$ if i wants to trade good j for good k and $\tau^i(j, k) = 0$ otherwise. Following Kiyotaki and Wright (1989) and Kehoe, Kiyotaki, and Wright (1993) we make the following two assumptions. First we assume that $\tau^i(j, k) = 1$ if and only if $\tau^i(k, j) = 0$, so that if an individual trades j for k she will not trade k for j. This ensures that an agent's preferences between the good they produce and commodity money are independent of the good they are currently holding. Second, we assume that in a given period, agents of the same type choose the same (potentially mixed) trading strategy. This ensures that the agent's willingness to exchange good j for k is independent of the type, and holdings of the potential trading partner.[6]

Since $u > 0$, individuals will always want to trade for their desired good, so that $\tau^i(j, i) = 1$ for all j. Given this, a strategy can be summarized by $\tau^i(i + 1, i + 2)$,

[6] These assumptions are not without loss of generality. If the agents strictly prefer either the good they produce or commodity money, then these assumptions would in fact be equilibrium outcomes; when agents are indifferent, there is no reason for these assumptions to hold. One could easily relax the assumptions, but at the cost of more cumbersome notation. Since we will eventually focus on symmetric equilibria, additional generality at this point adds nothing of substance to our analysis.

an agent's willingness to trade the good she produces for commodity money. Let s^i be the probability that type i wants to trade for commodity, i.e., plays the strategy $\tau^i(i+1, i+2) = 1$.

There may be equilibria in which individuals coordinate their actions using the realizations of a sunspot variable. Let $\{x_t\}_{t \in \mathbf{T}}$ be an exogenous sequence of random variables that are independent across time and uniformly distributed in the $[0,1]$ interval. Let $\bar{s}_t^i$ be the strategy played by type i at time t. A history at the beginning of the period at time t, denoted by z^t, can be written as

$$z^t = \left\{ \left\{ \bar{s}_0^i, ..., \bar{s}_{t-h}^i \right\}_{i \in I}; x_0, ..., x_t \right\}$$

with $z^t \in \mathbf{Z}^t = [0,1]^{3\frac{t}{h} + \left(\frac{t}{h}+1\right)}$.

A strategy for an individual is a sequence of functions $s_t^i : \mathbf{Z}^t \to [0,1]$, giving the trading strategy for each possible history.

Type i will never store good i; upon acquiring it, she immediately consumes it and produces a new unit of good $i+1$. Let $p_t^i : \mathbf{Z}^{t-h} \to [0,1]$ denote the fraction of individuals of type i storing good $i+1$ at the beginning of the period at time t following the history z^{t-h}. The distribution of inventories at a point in time can be completely summarized by the vector $P_t(z^t) \equiv \{p_t^i(z^t)\}_{i \in I}$. Given the trading strategies used at time t, $\{s_t^i(z^t)\}_{i \in I}$, we can derive an equation describing the evolution of inventories from t to $t+h$.

Following the accounting convention of Kiyotaki and Wright (1989), let $V_t^{i,j} : \mathbf{Z}^t \to \mathbb{R}$ denote the present discounted value for type i storing good j at the *end* of the period at time t for a given history. Given i's strategies, $\{s_t^i\}_{t \in \mathbf{T}}$, the strategies of others, $\{\bar{s}_t^i\}_{i \in I, t \in \mathbf{T}}$, and an initial condition, P_0, $V_t^{i,j}$ is well defined.[7] We can write

[7] As this is an anonymous sequential game, reputation and other strategic concerns play no role.

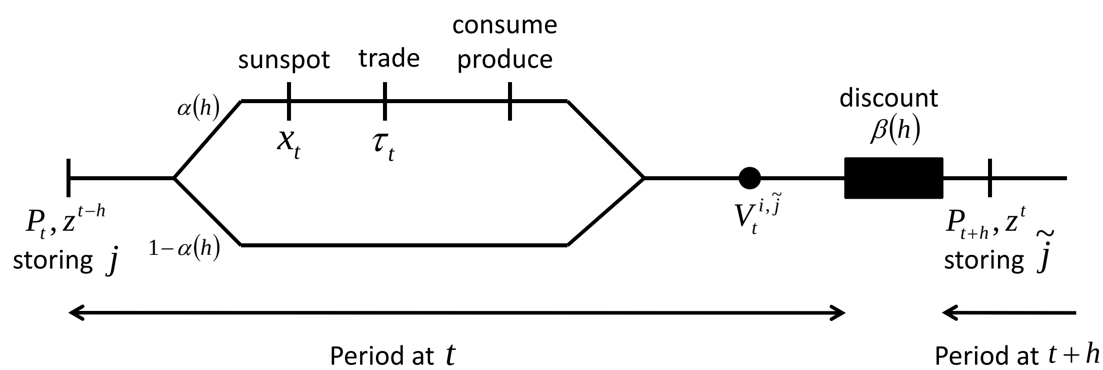
$V_t^{i,j}(z^t)$ as:

$$V_t^{i,j}(z^t) = \max_{\left\{s^i_{t+nh}\right\}_{n=1}^{\infty}} \mathbb{E}\left\{ \sum_{n=1}^{\infty} \beta(h)^n u \mathbb{I}^u_{t+nh} \middle| z^t \right\} \tag{1}$$

The expectation operator accounts for the uncertainty of meeting trading partners and realizations of the sunspot variable x_t, and $\mathbb{I}^u_t$ is an indicator of whether the individual consumes her good at time t.

Figure 1 describes the timing of the environment and the accounting of the model.

Figure 1
Timeline



At the beginning of period t, P_t describes the distribution of inventories, z^{t-h} the history of sunspots and strategies played, and individual i is holding good j. The individual meets a trading partner with probability $\alpha(h)$ and the sunspot variable x_t is realized. If she meets a trading partner, each chooses a trading strategy s_t and trade may take place. If there is a trade, the individuals may consume and produce new goods. $V_t^{i,\tilde{j}}$ denotes the present discounted value at this point, where $\tilde{j}$ is the good that individual i is storing at the end of period t. Discounting occurs in between periods.

The probability of trade and the expected payoff from a meeting depend on the types of individuals that meet and the goods each is storing. Table I shows the strategies of both individuals and the potential payoffs for all possible relevant meetings.

We can use the probabilities of trade in Table I to produce an equation describing the evolution of inventories p^i_t as a function of the strategies chosen s^i_t. Assuming the

Table I

Strategies and Payoffs from Encounters

i holding $i+1$

Trading Partner	Strategy of i	Strategy of j	Probability of Trade	Conditional Payoff	Expected Payoff
$j=i+1$ holding $i+2$	s_t^i	1	s_t^i	$V_t^{i,i+2}-V_t^{i,i+1}$	$s_t^i\left(V_t^{i,i+2}-V_t^{i,i+1}\right)$
$j=i+1$ holding i	1	1	1	u	u
$j=i+2$ holding i	1	s_t^{i+2}	s_t^{i+2}	u	$s_t^{i+2}u$
$j=i+2$ holding $i+1$	0	0	0	0	0

i holding $i+2$

Trading Partner	Strategy of i	Strategy of j	Probability of Trade	Conditional Payoff	Expected Payoff
$j=i+1$ holding $i+2$	0	0	0	0	0
$j=i+1$ holding i	1	$1-s_t^{i+1}$	$1-s_t^{i+1}$	$u+V_t^{i,i+1}-V_t^{i,i+2}$	$\left(1-s_t^{i+1}\right)\left(u+V_t^{i,i+1}-V_t^{i,i+2}\right)$
$j=i+2$ holding i	1	1	1	$u+V_t^{i,i+1}-V_t^{i,i+2}$	$u+V_t^{i,i+1}-V_t^{i,i+2}$
$j=i+2$ holding $i+1$	$1-s_t^i$	1	$1-s_t^i$	$V_t^{i,i+1}-V_t^{i,i+2}$	$(1-s_t^i)\left(V_t^{i,i+1}-V_t^{i,i+2}\right)$

This table describes the strategies and payoffs from the perspective of an individual of type i. The top panel describes these when the individual is storing good $i+1$ while the bottom panel describes these when storing $i+2$. Column 1 lists the trading partner. Columns 2 and 3 give the strategies of the individual and the trading partner respectively. Column 4 gives the probability of trade, the product of columns 2 and 3. Column 5 gives the payoff to the individual if the trade happens. Column 6 gives the expected payoff to the individual from the encounter, the product of columns 4 and 5.

history z^t can follow from z^{t-h}, we have:

$$\begin{aligned} p_{t+h}^i\left(z^t\right) \quad = \quad & p_t^i\left(z^{t-h}\right)\left[1-\tfrac{\alpha(h)}{3}p_t^{i+1}\left(z^{t-h}\right)s_t^i\left(z^t\right)\right] \\ & +\left(1-p_t^i\left(z^{t-h}\right)\right)\tfrac{\alpha(h)}{3}\left[\left(1-p_t^{i+1}\left(z^{t-h}\right)\right)\left(1-s_t^{i+1}\left(z^t\right)\right)+p_t^{i+2}\left(z^{t-h}\right)\right] \\ & +\left(1-p_t^i\left(z^{t-h}\right)\right)\tfrac{\alpha(h)}{3}\left[\left(1-p_t^{i+2}\left(z^{t-h}\right)\right)\left(1-s_t^i\left(z^t\right)\right)\right] \end{aligned} \tag{2}$$

The first term is the probability that an individual of type i was storing good $i+1$ at t and is still storing good $i+1$ at $t+h$. The second and third terms sum to the probability that she was storing good $i+2$ at t and is now storing $i+1$ at $t+h$.

We can also rewrite the sequential problem in equation (1) with a recursive representation. For an individual of type i, there are two relevant cases, one for each good that would be stored. The value of storing good $i+1$ is,

$$\begin{aligned} V_t^{i,i+1}\left(z^t\right) &= \beta(h)\mathbb{E}\{V_{t+h}^{i,i+1} + \tfrac{\alpha(h)}{3}[p_{t+h}^{i+1}s_{t+h}^{i}(V_{t+h}^{i,i+2} - V_{t+h}^{i,i+1}) \\ &\quad +(1-p_{t+h}^{i+1})u + p_{t+h}^{i+2}s_{t+h}^{i+2}u]|z^t\} \end{aligned} \tag{3}$$

while the value of holding $i+2$ is

$$\begin{aligned} V_t^{i,i+2}\left(z^t\right) &= \beta(h)\mathbb{E}\left\{V_{t+h}^{i,i+2} + \tfrac{\alpha(h)}{3}\left[\left(1-p_{t+h}^{i+1}\right)\left(1-s_{t+h}^{i+1}\right)\left(u+V_{t+h}^{i,i+1}-V_{t+h}^{i,i+2}\right)\right.\right. \\ &\quad \left.\left. +p_{t+h}^{i+2}\left(u+V_{t+h}^{i,i+1}-V_{t+h}^{i,i+2}\right)+\left(1-p_{t+h}^{i+2}\right)\left(1-s_{t+h}^{i}\right)\left(V_{t+h}^{i,i+1}-V_{t+h}^{i,i+2}\right)\right]|z^t\right\} \end{aligned} \tag{4}$$

Here we suppressed the arguments of p_{t+h}^i, s_{t+h}^i, and $V_{t+h}^{i,j}$, and the expectation is taken only over realizations of the sunspot variable x_{t+h}.

If holding good $i+1$ is more valuable than holding $i+2$, the strategy $s^i = 0$ is optimal (and $s^i = 1$ for the opposite case). If holding either good is equally valuable then any strategy can be optimal. Define $\Delta_t^i : \mathbf{Z}^t \to \mathbb{R}$ so that $\Delta_t^i\left(z^t\right) \equiv V_t^{i,i+1}\left(z^t\right) - V_t^{i,i+2}\left(z^t\right)$. Δ^i denotes the difference in value between storing $i+1$ and storing $i+2$, or equivalently the gain from exchanging $i+2$ for $i+1$. An optimal trading strategy s_t^i therefore satisfies

$$s_t^i\left(z^t\right) \in \begin{cases} \{0\} & \text{if } \Delta_t^i\left(z^t\right) > 0 \\ [0,1] & \text{if } \Delta_t^i\left(z^t\right) = 0 \\ \{1\} & \text{if } \Delta_t^i\left(z^t\right) < 0 \end{cases} \tag{5}$$

We now define an equilibrium.

Definition 1 *For an initial condition, P_0, an equilibrium is a sequence of inventories*

p_t^i, trading strategies s_t^i, and value functions $V_t^{i,i+1}, V_t^{i,i+2}$ denoted by

$$\left\{p_t^i, s_t^i, V_{t-h}^{i,i+1}, V_{t-h}^{i,i+2}\right\}_{t \in \mathbf{T}, i \in I}$$

such that (i) equations (2), (3), (4), and (5) are satisfied, and (ii) the transversality conditions $\lim_{t\to\infty} \mathbb{E}_{-h}\left[\beta(h)^{t/h} V_t^{i,j}\right] = 0$ holds for all i, j.

3 Symmetric Equilibria

We focus on symmetric equilibria in a symmetric environment: given a symmetric initial condition $P_0 = \{p_0, p_0, p_0\}$, we study equilibria in which trading strategies are symmetric ($s_t^i = s_t$ for all i) and as a consequence inventories remain symmetric ($p_t^i = p_t$ for all i). In this case, the evolution of inventories in equation (2) reduces to

$$\begin{aligned} p_{t+h}\left(z^t\right) &= p_t\left(z^{t-h}\right) - \tfrac{\alpha(h)}{3} p_t^2\left(z^{t-h}\right) s_t\left(z^t\right) \\ &\quad + \tfrac{\alpha(h)}{3}\left(1 - p_t\left(z^{t-h}\right)\right)\left[2\left(1 - p_t\left(z^{t-h}\right)\right)\left(1 - s_t\left(z^t\right)\right) + p_t\left(z^{t-h}\right)\right] \end{aligned} \tag{6}$$

and the evolution of Δ_t is

$$\Delta_t\left(z^t\right) = \beta(h)\mathbb{E}\left\{ \begin{array}{c} \Delta_{t+h} + \frac{\alpha(h)}{3} u[s_{t+h} - p_{t+h}] \\ -\frac{\alpha(h)}{3}[p_{t+h}s_{t+h} + 2(1-p_{t+h})(1-s_{t+h}) + p_{t+h}]\Delta_{t+h} \end{array} \middle| \, z^t \right\} \tag{7}$$

The following lemma will assist in the characterization of equilibria. Of particular use, we show that if $\{\Delta_t\}_{t\in T}$ corresponds to value functions that satisfy the sequence problem, then it must have a uniform bound.

Lemma 1 *Given an initial condition p_0, a sequence $\{p_t, s_t, \Delta_{t-h}\}_{t\in\mathbf{T}}$ represents a symmetric equilibrium if and only if (i) equations (5), (6), and (7) are satisfied and, (ii) there exists $B > 0$ such that for every t, $\Pr\left\{\left|\Delta_{t-h}\left(z^{t-h}\right)\right| \le B\right\} = 1$.*

Proof. See Appendix A. ■

Lemma 1 implies that we can look for equilibria in the space $\{p_t, s_t, \Delta_{t-h}\}_{t\in\mathbf{T}}$.

3.1 Symmetric Steady State Equilibrium

In this section we show existence and uniqueness of symmetric steady state equilibria. Kehoe, Kiyotaki, and Wright (1993) shows that with asymmetric storage costs there are a finite number of steady state equilibria. We show that with symmetric costs there is a unique symmetric steady state.

In any steady state equilibrium $\left\{p_t\left(z^{t-h}\right), s_t\left(z^t\right), \Delta_{t-h}\left(z^{t-h}\right)\right\} = \{p_{ss}, s_{ss}, \Delta_{ss}\}$ for all $t \in \mathbf{T}$ and $z^t \in \mathbf{Z}^t$. In this case equation (7) can be rearranged to get

$$\Delta_{ss} = \frac{u\left[s_{ss} - p_{ss}\right]}{\frac{1-\beta(h)}{\beta(h)}\left[\frac{\alpha(h)}{3}\right]^{-1} + p_{ss} + 2(1-p_{ss})(1-s_{ss}) + p_{ss}}$$

Since the denominator is positive, the value of Δ_{ss} and hence the optimal trading strategy s_{ss} depend on the sign of $s_{ss} - p_{ss}$. Consider first the possibility that $\Delta_{ss} < 0$: this would imply $s_{ss} = 1$ and hence $\Delta_{ss} \geq 0$, a contradiction. Consider next $\Delta_{ss} > 0$: this would imply $s_{ss} = 0$ and hence $\Delta_{ss} \leq 0$, also a contradiction. The only remaining possibility is $\Delta_{ss} = 0$ which holds if and only if $s_{ss} = p_{ss}$ which would be consistent with the optimal choice of the trading strategy given in equation (5). Using the evolution of inventories, equation (6), together with $p_t = s_t = p_{ss}$ for all $t \in \mathbf{T}$ provides $p_{ss} = s_{ss} = \frac{2}{3}$.

3.2 The Zero Equilibrium

We next consider a special dynamic equilibrium and label it the the Zero Equilibrium. As we will show below, an equilibrium of this type will be the unique surviving equilibrium as the interval between search opportunities, h, goes to zero. The strategies of the Zero Equilibrium will also be helpful in characterizing the set of equilibria for any fixed h.

For any h, there exists a unique equilibrium for which $\Delta_t(z^t) = 0$ for all t, z_t. This equilibrium is Markovian, and the strategy played is always $s_t = p_t$. This condition implies that the probability of being able to obtain the desired good within the period is independent of the good the agent is currently holding.[8] It is easy to see that equation (5) and equation (7) are both satisfied. For any initial condition p_0, one can find the sequence of inventories by iterating equation (6). Such a sequence $\{p_t, s_t, \Delta_{t-h}\}_{t\in\mathbf{T}}$ satisfies the conditions of Lemma 1 and is therefore an equilibrium.

We construct the Zero Equilibrium by finding a sequence of strategies and inventories so that individuals are indifferent between accepting and rejecting commodity money every period. When choosing a current strategy, individuals are concerned with three quantities: the fraction of people holding the commodity money; the strategies chosen by others; and the future relative value of holding commodity money. In the Zero Equilibrium, the future relative value is zero by construction. If a larger fraction of people are holding the good they produce ($p \uparrow$), trading for commodity becomes more advantageous. If others are more willing to accept commodity money ($s \uparrow$), there is less of a need to accept commodity money; others will accept the produced good in exchange for the desired good. In the Zero Equilibrium these two forces balance perfectly: when fewer people hold commodity money, more are willing to accept it.

The same logic explains why there is a unique symmetric steady state, as this is the Zero Equilibrium for a particular initial condition.

[8]If individual 1 is holding good 2 her probability of trading for good 1 with a type 2 is $1-p^2$ and with type 3 is p^3s^3, so that the probability of trading for the desired good is $1-p^2+p^3s^3$. If the individual 1 is holding good 3 her probability of trading for good 1 with a type 2 is $(1-p^2)(1-s^2)$ and with type 3 is p^3, so that the probability of trading for the desired good is $(1-p^2)(1-s^2)+p^3$. In a symmetric equilibrium these values are equated when $s = p$.

4 Indeterminacy

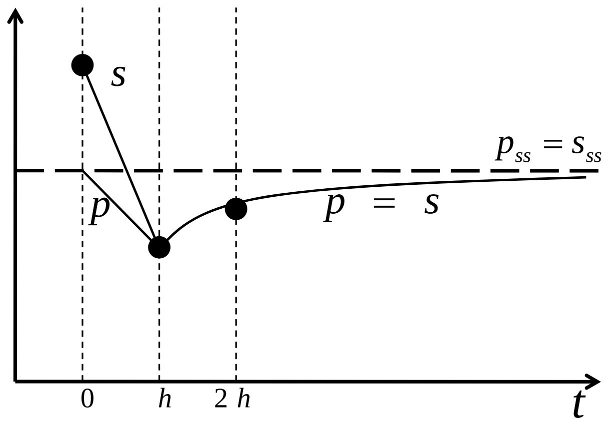
In this section we provide examples of several types of equilibria. We show that for a fixed frequency of search, there is a large multiplicity of dynamic equilibria. Several examples are taken from Kehoe, Kiyotaki, and Wright (1993), who worked with a model with asymmetric parameters and allow for asymmetric strategies. The purpose of this section is to demonstrate that many of these types equilibria are still present in an environment with symmetric parameters, even with the restriction of symmetric strategies.

First we provide examples of deterministic equilibria that eventually converge to the steady state. Figure 2 shows the evolution of the trading strategy s and inventories p as a function of the elapsed time t for a particular equilibrium. The economy starts with $p_0 = p_{ss}$. We are interested in rationalizing an equilibrium with $s_0 > s_{ss}$. Since some traders will be acquiring commodity money, the fraction of individuals holding their own produced good falls, $p_h < p_{ss}$. From period h onward, the traders play the strategies of the Zero Equilibrium. Each period, s and p move together, balancing the incentives to accept and reject commodity money. At the end of the initial period, these incentives are also balanced, regardless of the high value of the initial trading strategy, s_0.[9] Therefore this is, indeed, an equilibrium. In fact, any s_0 can be rationalized by playing $s_h = p_h$ and then following the strategies of the Zero Equilibrium.

This argument can be formalized and generalized. Notably, the argument is independent of the initial value of p_0 and, given a particular p_0, the initial strategy chosen, s_0. Given the initial condition p_0, choose any s_0. This gives p_h. From p_h

[9] It is true that both the trading strategy during the initial period and the lower fraction of individuals holding their own good in future periods make commodity money less useful during the initial period. But the only implication of this is that coming into the initial period, traders would have preferred to have been holding their own produced good, $\Delta_{-h} > 0$.

Figure 2
Rationalizing Deterministic Equilibrium Paths



An example of a deterministic equilibrium that converges to the steady state.

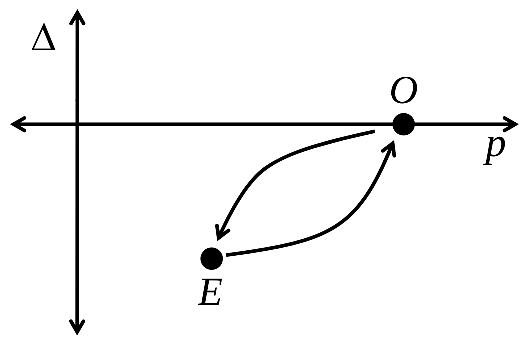
there exists strategies consistent with equilibrium such that $\Delta_t = 0$ for all $t \in \mathbf{T}$ (the strategies of the Zero Equilibrium). Since $\Delta_0 = 0$ the choice of s_0 is optimal, a fact that is independent of the value of Δ_{-h}. Because the choice of s_0 was arbitrary, each different s_0 corresponds to a different dynamic equilibrium. There is therefore a continuum of such deterministic dynamic equilibria.[10]

We can also construct cyclical equilibria. We provide an example for the following parametrization: $\alpha(h) = 0.1$, $\beta(h) = 0.98$, and $u = 1$. The economy cycles between two triples $\{p_{nh}, s_{nh}, \Delta_{(n-1)h}\}$. When n is odd the economy lies at $\{0.6737, 1, 0\}$, and lies at $\{0.6659, 0.3174, -0.0114\}$ when n is even. In contrast to the Zero Equilibrium, cyclical equilibria are rationalized by the balance between current and future incentives. In odd periods, more traders are willing to accept commodity money and more are holding their own good. Both of these make it easy to get the desired good using only the produced good, reducing the relative value of commodity money. In

[10]The idea for this type of equilibrium originates with a construction by Aiyagari and Wallace (1992) with fiat money.

even periods, it becomes harder to get the desired good using the produced good, increasing the relative value of commodity money. These cycles persist as the period length shrinks, a claim that we formalize in Appendix B.

Figure 3
Example of a cyclical equilibrium



The equilibrium is characterized by $\{p_t, s_t, \Delta_{t-h}\}$. In this case: $E = \{0.6659, 0.3174, -0.0114\}$ and $O = \{0.6737, 1, 0\}$. Parametrization: $\alpha(h) = 0.1$, $\beta(h) = 0.98$ and $u = 1$.

Finally, we can also construct non-Markovian equilibria, combining the two previous examples. For the first $2N - 1$ periods, individuals play the strategies associated with the cyclical equilibrium described above. From period $2N$ on, all individuals play the strategies associated with the Zero Equilibrium, so that $\Delta_t = 0$. In fact, we can construct an equilibrium in which every odd period the realization of the sunspot $x_{(2n+1)h}$ determines whether the individuals continue to play the cyclical strategies or the economy reverts to the Zero Equilibrium.

5 Perfect Foresight Equilibria

In this section we discuss perfect foresight equilibria. While individuals still face uncertainty in terms of meeting trading partners, p_t, s_t and Δ_t are no longer functions of the sunspot variables $\{x_t\}$ and follow deterministic paths. We can therefore drop the expectation operator in equation (7).

5.1 Continuous Search

The dynamics of the limiting model in which agents search continuously are simple and easy to describe. There is a unique equilibrium, in which agents choose $s_t = p_t$ for all $t > 0$.

For the continuous time model to be well defined, recall that we assume the following limits exist: Let $r = \lim_{h \downarrow 0} \frac{1}{h}\left(\frac{1}{\beta(h)} - 1\right)$ be the instantaneous discount rate and $\alpha_0 = \lim_{h \downarrow 0} \frac{\alpha(h)}{h}$ be the instantaneous meeting rate.

As $h \to 0$, we approach the continuous search limit. Using equation (6) and equation (7) we can show that $\dot{p}_t$ and $\dot{\Delta}_t$ exist and satisfy:

$$\dot{p}_t = \frac{\alpha_0}{3}\left[-p_t^2 s_t + (1-p_t)\left(2(1-p_t)(1-s_t) + p_t\right)\right] \tag{8}$$

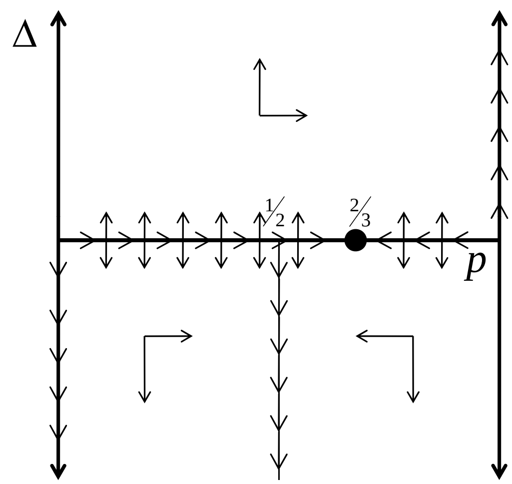
and

$$r\Delta_t = \dot{\Delta}_t + \frac{\alpha_0}{3} u\left[s_t - p_t\right] - \frac{\alpha_0}{3}\left[p_t s_t + 2(1-p_t)(1-s_t) + p_t\right]\Delta_t \tag{9}$$

It is straightforward to show the only symmetric equilibrium is the Zero Equilibrium, i.e., $\Delta_t = 0$ for all $t \geq 0$ and the optimal strategy must be $s_t = p_t$. One can extend the definition of symmetric equilibrium and Lemma 1 to the continuous case and show that for any equilibrium, $\{\Delta_t\}_{t \geq 0}$ must have a uniform bound. First, note that $\Delta_t > 0$ implies $\dot{\Delta}_t > r\Delta_t$ and similarly $\Delta_t < 0$ implies $\dot{\Delta}_t < r\Delta_t$. Together, these imply that if there is a t at which $\Delta_t \neq 0$ then $|\Delta|$ will grow exponentially and without bound, violating Lemma 1. Lastly, observe that if $\Delta_t = 0$, it must be that $s_t = p_t$ for almost every t. These dynamics are summarized by the phase diagram in Figure 4.

Note also that the paths of p_t and Δ_t are continuous as the time derivatives of these objects are uniformly bounded. This is an important difference between discrete and continuous time as it restricts the acceptable strategies that are consistent with equilibrium.

Figure 4
Phase diagram for the model in its continuous time formulation



The unique equilibrium strategy sets $s_t = p_t$ such that $\Delta_t = 0$ for all t. The equilibrium converges to the unique steady state with $p_{ss} = \frac{2}{3}$.

5.2 Properties of the Set of Perfect Foresight Equilibria

In this section we will characterize the set of state-payoff combinations that are consistent with a symmetric equilibrium for a fixed interval between search opportunities h. In order to do this, it is helpful to discuss the timing of the model. A strategy at a given point in time, s_t, affects both the fraction of individuals storing each type of good and the relationship between current and future present discounted values. Inspection of equation (6) and equation (7) reveals that s_{t+h} is relevant for the relationship between Δ_t and Δ_{t+h} on the one hand, and p_{t+h} and p_{t+2h} on the other. In other words, s_{t+h} determines the relationship between (p_{t+h}, Δ_t) and (p_{t+2h}, Δ_{t+h}).

We now characterize the set of points that are consistent with a symmetric equilibrium.

Proposition 1 *Let $\{p_t, s_t, \Delta_{t-h}\}_{t\in\mathbf{T}}$ be a sequence that satisfies equations (5), (6), and (7). This an equilibrium for an economy with initial condition p_0* ***if and only if***

$$\Delta_t \in \left[\underline{\Delta}(p_{t+h}), \overline{\Delta}(p_{t+h})\right]$$

where $\underline{\Delta}(p) \equiv -\beta(h)\gamma(h)p$ and $\overline{\Delta}(p) \equiv \beta(h)\gamma(h)(1-p)$ with $\gamma(h) \equiv \frac{\alpha(h)}{3}u$.

Proof. See Appendix E. ■

Figure 5
Phase diagram for the model in its discrete time formulation for step size h

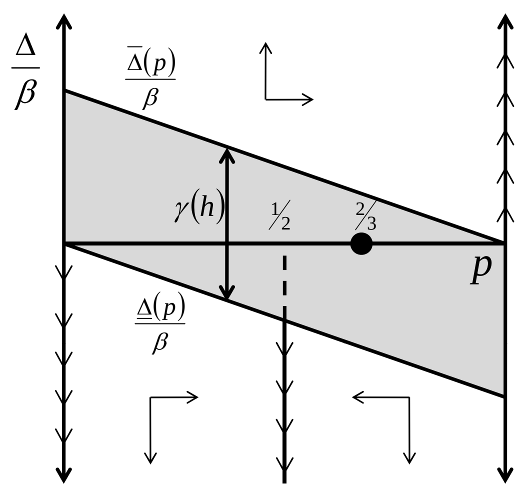


Figure 5, a partial phase diagram for a given interval between search opportunities, h, gives a graphical representation of the main ideas in the proof of Proposition 1. Note that on the vertical axis we plot $\frac{\Delta}{\beta(h)}$. This corresponds to the value at the beginning of the next time period, so that both p_{t+h} and $\frac{\Delta_t}{\beta(h)}$ refer to values at the beginning of period $t+h$.

The shaded area represents $\Gamma(h) \equiv \left\{(p, \frac{\Delta}{\beta(h)}) \text{ such that } \Delta \in \left[\underline{\Delta}(p), \overline{\Delta}(p)\right]\right\}$, the set of possible state-payoff combinations of $(p_{t+h}, \frac{\Delta_t}{\beta(h)})$ that are consistent with a symmetric equilibrium. One notable feature is that any point $(p, \frac{\Delta}{\beta(h)})$ in the shaded area is consistent with an equilibrium in which (i) $p_{t+h} = p$ and $\Delta_t = \Delta$ and (ii) $\Delta_{t+h} = 0$. In other words, the economy can go from that point to $\Delta = 0$ in one period.

Put differently, if $\Delta_{t-h} > \bar{\Delta}(p_t)$ then the value today of holding the produced good (relative to commodity money) is so high that no matter what happens this period, the individual will still prefer to hold the produced good going into next period. Therefore refusing commodity money ($s_t = 0$) is a dominant strategy. Similarly, $\Delta_{t-h} < \underline{\Delta}(p_t)$ guarantees that accepting commodity money ($s_t = 1$) is a dominant strategy.

When more traders are holding their produced good, a given individual holding her produced good has fewer potential trading partners. As a consequence, a larger portion of the relative value of holding the produced good must be expected to arrive in future periods, hence Δ_t is more likely to be positive. Therefore the threshold for Δ_{t-h} at which one can guarantee that Δ_t is positive must be decreasing in p. For analogous reasons, $\underline{\Delta}$ is also decreasing in p.[11]

If $\Delta_{t-h} > \overline{\Delta}(p_t)$ then two things happen. First, the relative advantage of not holding commodity money increases ($\Delta_t \geq \beta(h)^{-1}\Delta_{t-h}$): since others are refusing to accept commodity money this period, any advantage of holding the produced good instead of commodity money could not have come from expected utility flow within period t; the strategy can only be rationalized by expected gains in future periods. For this to happen, the future relative value of holding the produced good must

[11]From the opposite perspective, when more traders are holding their produced good, refusing to accept commodity money now makes it difficult to obtain the desired good in the future. It therefore becomes even more difficult rationalize this refusal, and hence it is more difficult to sustain such an equilibrium. As a consequence, $\overline{\Delta}(p)$ is decreasing in p.

increase by at least the discount rate. Second, the fraction of individuals holding commodity money falls ($p_{t+h} > p_t$). This means that next period there will be even fewer potential trading partners for those without commodity money, making it even harder to get utility flows next period. As a consequence, we can guarantee $\Delta_t > \overline{\Delta}(p_{t+h})$.[12] The same reasoning holds in each future period, so that $\{\Delta_t\}$ grows exponentially and eventually violates the uniform bound implied by Lemma 1. Such a path is not consistent with equilibrium because this unboundedly large future value never arrives.

When $\Delta_{t-h} < \underline{\Delta}(p_t)$, the analysis is similar, with one slight complication. Here the relative value of commodity money is so high that no matter what happens accepting commodity money ($s_t = 1$) is the dominant strategy. By an identical argument one can show that $\Delta_t \leq \beta(h)^{-1}\Delta_{t-h}$: since others will be accepting commodity money, the value of already having commodity money is low this period, and this must be made up in future periods. The change in the fraction of people holding commodity money is trickier. Among those holding commodity money, some will be able to trade their commodity money for their desired good, so there is a natural force increasing the fraction not holding commodity money (p). If every trader is accepting commodity money, then the fraction holding their produced good would fall when $p > \frac{1}{2}$ and rise when $p < \frac{1}{2}$. If p is falling, then by the same reasoning as above, we can guarantee that $\Delta_t < \underline{\Delta}(p_{t+h})$. When p is rising, commodity money is more likely to deliver the desired good next period, so one might think it is possible that even with the increased relative value of commodity money ($|\Delta_t|$) that the relative value the following period (Δ_{t+h}) need not be negative. However we can show algebraically that the increase in magnitude of Δ is large enough to dominate the rise in p, and hence we can guarantee

[12] This can be seen graphically. If $s = 0$ is played, then the changes in Δ and p are both positive, which means that the next point in the sequence is also above $\overline{\Delta}(p)$ (this can be seen from the slope of $\overline{\Delta}(p)$).

that $\Delta_t < \underline{\Delta}(p_{t+h})$ in this case as well. In either case, $\{|\Delta_t|\}$ grows exponentially and eventually violates the uniform bound.

5.3 Frequency of Search and the Set of Perfect Foresight Equilibria

The height of set of points consistent with symmetric equilibrium $\Gamma(h)$ is given by $\gamma(h) = \frac{\alpha(h)}{3}u$. As h decreases, the area of this set shrinks in proportion to $\alpha(h)$. In the limit, $\alpha(h)$, and hence $\gamma(h)$, approaches zero. In this case Figure 5 coincides exactly with the phase diagram of the continuous search model depicted in Figure 4. The only surviving equilibrium is the Zero Equilibrium.

The set of possible equilibrium payoffs is increasing with $\alpha(h)$ as individuals are more likely to find a trading partner each period. This means that a larger fraction of the payoff comes from expected utility from a single search, and less from future flows. As a consequence the set of future payoffs consistent with an initial value of Δ_{t-h} is larger, and in particular it is less likely that the sign of Δ_t, and hence the strategy s_t, is pinned down. The set of points in the state-payoff space $\Gamma(h)$ are precisely the combinations for which the sign of Δ_h is not pinned down.

We can also derive some properties of the sequence of inventories and trading strategies that are consistent with equilibrium. We show that for any equilibrium, the sequence of inventories is "close" to that of the Zero Equilibrium. More formally, the set of sequences of inventories that are consistent with equilibrium converges uniformly to the sequence of inventories of the Zero Equilibrium.

Proposition 2 *For any $h > 0$ and p_0, let $\{p_t^0\}_{t\in\mathbf{T}}$ denote the sequence of inventories of the Zero Equilibrium. For any equilibrium, for all $t \in \mathbf{T}$*

$$\left|p_t - p_t^0\right| \leq \pi(h) \tag{10}$$

where $\lim_{h \to 0} \pi(h) = 0$.

Proof. See Appendix D.1. ■

This proposition follows from the fact that both Δ_t and Δ_{t+Nh} must be within bounds that shrink as search becomes more frequent and h falls. Given Δ_t, this puts a restriction on the strategies that can be played between periods t and $t + Nh$. As the bounds on Δ shrink, the evolution of p implied by those strategies within those N periods is increasingly constricted and converges to that of the Zero Equilibrium.

To get further insight into this restriction on strategies, we can also show that the strategies played will also be "close" to those of the Zero Equilibrium. The next proposition shows that as search becomes more frequent, the local average of the trading strategies converges to the strategies of the Zero Equilibrium.

Proposition 3 *For* $\varepsilon > 0$*, let* N *be the largest integer such that* $\varepsilon \geq (2N+1)h$*. Then in any equilibrium,*

$$\left| \left(\frac{1}{2N+1} \sum_{n=-N}^{N} s_{t+nh} \right) - p_t \right| \leq \sigma(h, \varepsilon) \tag{11}$$

holds for all $t \in \mathbf{T}$*, with the property that* $\lim_{\varepsilon \to 0} \left(\lim_{h \to 0} \sigma(h, \varepsilon) \right) = 0$

Proof. See Appendix D.2. ■

6 All Equilibria

The previous section discussed deterministic, perfect-foresight equilibria. Sunspot equilibria can occur if particular strategies that are chosen depend on random variables that have no intrinsic effect on the economy; individuals may use the realizations of the random variable to coordinate their strategies.

Remarkably the set of state-payoff combinations that are consistent with sunspot equilibria coincides exactly with those of perfect foresight equilibria.

Proposition 4 *Let* $\left\{p_t\left(z^{t-h}\right), s_t\left(z^t\right), \Delta_{t-h}\left(z^{t-h}\right)\right\}_{z^t\in\mathbf{Z}^t, t\in\mathbf{T}}$ *be a sequence that satisfies equations (5), (6), and (7). This an equilibrium for an economy with initial condition* p_0 ***if and only if***

$$\Pr\left\{\Delta_t\left(z^t\right)\in\left[\underline{\Delta}\left(p_{t+h}\left(z^t\right)\right),\overline{\Delta}\left(p_{t+h}\left(z^t\right)\right)\right]\right\}=1$$

where $\underline{\Delta}(p)=-\beta(h)\gamma(h)p$ *and* $\overline{\Delta}(p)=\beta(h)\gamma(h)(1-p)$ *with* $\gamma(h)=\frac{\alpha(h)}{3}u$

Proof. See Appendix E. ■

The idea behind the proof is similar to that of the perfect foresight case. We show that if Δ_{t-h} is above $\overline{\Delta}$ then we can guarantee that there is a positive probability that agents play the strategy $s_t=0$. With this, we can show if Δ is above $\overline{\Delta}$ with positive probability, then there must be a positive probability that the sequence of Δ's eventually violate the uniform bound given by Lemma 1. For any perfect foresight equilibrium, a special case, these positive probabilities are equal to 1.

We can also extend Proposition 2 and Proposition 3 to the set of all equilibria by adding expectations operators to the left hand sides of equation (10) and equation (11).[13]

[13] One might think it would be possible to give a uniform bound on $|p_t-p_t^0|$ for almost every z^t. However, one can find sunspot equilibria in which there is an arbitrarily small probability of an arbitrarily long sequence of any trading strategies, as long as the Δ at the end of the sequence is within the bounds at the end of the sequence. Because there were no restrictions on the sequence of trading strategies, there are no restrictions on p at the end of the sequence.

7 Conclusion

The literature following Kiyotaki and Wright (1989) has studied economies with explicit search frictions to learn how these frictions affect economic behavior. We study how a feature of the environment, the frequency of search, interacts with these search frictions to shape potential equilibrium outcomes. In particular, we shed light on economic forces in the model that can generate multiplicity, and show how these change as search becomes more frequent.

We show that as search frequency increases, the set of dynamic equilibria shrinks uniformly in three dimensions: (i) equilibrium payoffs associated a given state of the economy; (ii) the average strategy played over any given unit of time; and (iii) dynamic paths of the state variables.

To do this, we focus on symmetric equilibria in a symmetric environment. This restriction allows us to cleanly characterize several dimensions of the set of dynamic equilibria. While symmetric equilibria are often the object of interest[14] a natural question is how our results generalize to asymmetric equilibria or environments. In these more general environments characterizing the set of equilibria is considerably more difficult technically. While there may not be a unique dynamic equilibrium, we conjecture that the set dynamic equilibria shrinks uniformly as agents search more frequently (indexed by the characteristics we describe above). We further conjecture that when search is frequent every equilibrium can be approximated by an element of the set of continuous search equilibria. In other words, when search is frequent, much of the multiplicity will not matter.

[14] See Kiyotaki and Wright (1993) for an early example.

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Appendix

A Proof of Lemma 1

We first show that for any equilibrium that satisfies the conditions of Definition 1, $\{\Delta_t(z^t)\}_{z^t \in \mathbf{Z}^t, t \in \mathbf{T}}$ has a uniform bound with probability 1. It is straightforward to show that value functions $V_t^{i,j}(z^t)$ can be bounded above and below by bounds that are independent of z^t and t. For the upper bound, we can assume that the individual is able to consume at every chance meeting. For the lower bound we can assume that the individual never consumes. The value function $V_t^{i,j}(z^t)$ can therefore be bounded by $0 \leq V_t^{i,j}(z^t) \leq \frac{\alpha(h)u}{1-\beta(h)}$. It follows that $\Delta_t(z^t)$ is bounded above and below by bounds that are independent of z^t and t. The other conditions of Definition 1 are trivially satisfied.

Second, we show that if a sequence $\{p_t, s_t, \Delta_{t-h}\}_{t\in\mathbf{T}}$ (i) satisfies equation (5), equation (6), and equation (7) and (ii) $\{\Delta_t(z^t)\}_{z^t\in\mathbf{Z}^t, t\in\mathbf{T}}$ is uniformly bounded, then we can construct a sequence of $\{p_t, s_t, V_{t-h}^{i,i+1}, V_{t-h}^{i,i+2}\}_{t\in\mathbf{T}}$ that is a symmetric equilibrium. We need to show that one can construct a sequence of value functions that satisfy the symmetric versions of equation (3), equation (4), and transversality.

For a given sequence, define $M_t^1(z^t)$ and $M_t^2(z^t)$ to be

$$M_t^1(z^t) = \beta(h)\frac{\alpha(h)}{3}[p_t(z^{t-h})(-\Delta_t(z^t)) + (1-p_t(z^{t-h}))u + p_t(z^{t-h})s_t(z^t)u]$$

and

$$\begin{aligned} M_t^2(z^t) &= \beta(h)\tfrac{\alpha(h)}{3}\left[\left(1-p_t(z^{t-h})\right)(1-s_t(z^t))\left(u+\Delta_t(z^t)\right)\right. \\ &\quad \left. + p_t(z^{t-h})\left(u+\Delta_t(z^t)\right) + \left(1-p_t(z^{t-h})\right)(1-s_t(z^t))\left(\Delta_t(z^t)\right)\right] \end{aligned}$$

Iterating equation (3) and equation (4), and taking the limit as $N\to\infty$ gives

$$\begin{aligned} V_{-h}^{i,i+1}(z^{-h}) &= \mathbb{E}_{-h}\left[\textstyle\sum_{n=0}^{\infty}\beta(h)^n M_{nh}^1\right] + \lim_{N\to\infty}\mathbb{E}_{-h}\left[\beta(h)^{N+1}V_{Nh}^{i,i+1}\right] \\ V_{-h}^{i,i+2}(z^{-h}) &= \mathbb{E}_{-h}\left[\textstyle\sum_{n=0}^{\infty}\beta(h)^n M_{nh}^2\right] + \lim_{N\to\infty}\mathbb{E}_{-h}\left[\beta(h)^{N+1}V_{Nh}^{i,i+2}\right] \end{aligned}$$

where $\mathbb{E}_t(\cdot) = \mathbb{E}(\cdot \mid z^t)$. The fact that $\{\Delta_t(z^t)\}$ is uniformly bounded implies that the terms $\mathbb{E}_{-h}\left[\sum_{n=0}^{\infty}\beta(h)^n M_{nh}^1(z^{nh})\right]$ and $\mathbb{E}_{-h}\left[\sum_{n=0}^{\infty}\beta(h)^n M_{nh}^2(z^{nh})\right]$ are finite. If we set $V_0^{i,i+1}(z^0) = -\mathbb{E}_{-h}\left[\sum_{n=0}^{\infty}\beta(h)^n M_{nh}^1(z^{nh})\right]$, then transversality must be satisfied. Since equation (3) and equation (4) are satisfied by construction, this is an equilibrium.

B Two Period Cycles

In this section we show that in a neighborhood around $h = 0$, we can always construct a two period cycle: in even periods the economy is at $\{\Delta_e, p_e, s_e\}$, and in odd periods at $\{\Delta_o, p_o, s_o\}$. We can evaluate equation (5), equation (6), and equation (7) at the values of odd and even periods to obtain a system of four unknowns and two restrictions. We will look for cycles in which $s_o = 1$ and $\Delta_e = 0$, so the four equations become,

$$\begin{aligned}
0 &= \Delta_o + \tfrac{\alpha(h)}{3}u[1-p_o] - \tfrac{\alpha(h)}{3}2p_o\Delta_o \\
\Delta_o &= \beta(h)\tfrac{\alpha(h)}{3}u[s_e - p_e] \\
p_o &= p_e - \tfrac{\alpha(h)}{3}p_e^2 s_e + \tfrac{\alpha(h)}{3}(1-p_e)[2(1-p_e)(1-s_e)+p_e] \\
p_e &= p_o - \tfrac{\alpha(h)}{3}p_o^2 + \tfrac{\alpha(h)}{3}(1-p_o)p_o
\end{aligned}$$

These four equations will then determine the values of the unknowns, s_e, Δ_o, p_o, and p_e. To verify that such cycle exists, we must show that (i) $\Delta_o \leq 0$ (so that $s_o = 1$ is optimal), and (ii) that $s_e \in [0,1]$ (so that the cycle is feasible).

The first equation implies that $\Delta_o \leq 0$, and with this the second equation implies that $s_e \leq p_e \leq 1$. We must now verify that $s_e \geq 0$. We can eliminate Δ_0 and reduce the system to the following three equations

$$\begin{aligned}
s_e &= S(p_o) \equiv p_o\left[1 - \tfrac{\alpha(h)}{3}p_o + \tfrac{\alpha(h)}{3}(1-p_o)\right] - \frac{1-p_o}{\left[1-\frac{\alpha(h)}{3}2p_o\right]\beta(h)} \\
p_e &= P(p_o) \equiv p_o - \tfrac{\alpha(h)}{3}p_o^2 + \tfrac{\alpha(h)}{3}(1-p_o)p_o \\
0 &= g(p_o) \equiv P(p_o) - \tfrac{\alpha(h)}{3}P(p_o)^2 S(p_o) - p_o \\
&\qquad + \tfrac{\alpha(h)}{3}\left(1-P(p_o)\right)\left[2\left(1-P(p_o)\right)\left(1-S(p_o)\right) + P(p_o)\right]
\end{aligned}$$

The third equation defines candidate values of p_o consistent with the specified cycle, while the functions S and P give the corresponding values of s_e and p_e respectively. We can show that s_e is strictly increasing in p_o,

$$S'(p_o) = 1 - \alpha(h)p_o + \frac{\alpha(h)}{3}(1-p_o) + \frac{1-2\frac{\alpha(h)}{3}}{\left[1-\frac{\alpha(h)}{3}2p_o\right]^2\beta(h)} > 0$$

Let $\bar{p}_o$ solve $S(\bar{p}_o) = 0$. The inventory level $\bar{p}_o$ is useful as it provides a lower bound for the required level of p_o for a cycle to exist. There is a unique $\bar{p}_o \in \left(\frac{1}{2}, 1\right)$, since $S(1) = 1 - \frac{\alpha(h)}{3} \in (0,1)$, $S(\frac{1}{2}) = \frac{1}{2}\left[1 - \frac{1}{\left[1-\frac{\alpha(h)}{3}\right]\beta(h)}\right] < 0$, and $S'(p) > 0$.

Because S is strictly increasing, if we can find a $p_o \in [\bar{p}_o, 1)$ that satisfies $g(p_o) = 0$ then we can guarantee that $s_e \in [0,1]$.

We show first that $g(1) < 0$ (using $P(1) = S(1)$):

$$\begin{aligned}
\left(\tfrac{\alpha(h)}{3}\right)^{-1} g(1) &= -\tfrac{\alpha(h)}{3} - P(1)^3 + 2[1-P(1)]^3 - P(1)^2 \\
&= -3P(1)^3 + 5P(1)^2 - 6P(1) - \tfrac{\alpha(h)}{3} + 2 \\
&= \tfrac{\alpha(h)}{9}\left(\alpha(h)^2 - 4\alpha(h) + 12\right) - 2
\end{aligned}$$

which is negative for any $\alpha(h) \in (0, 1]$. We next show that in a neighborhood around $h = 0$, $g(\bar{p}_o) > 0$. By the mean value theorem, this will guarantee that there is a solution $p_o \in [\bar{p}_o, 1)$, and consequently the existence of a two point cycle.

We can use the definition of P along with the fact that $p_o \geq \bar{p}_o > \frac{1}{2}$ to get

$$P(p_o) - p_o = \frac{\alpha(h)}{3} p_o(1 - 2p_o)$$

and therefore $P(p_o) < p_o$. Now we turn to evaluate $g(\bar{p}_o)$,

$$g(\bar{p}_o) = P(\bar{p}_o) - \bar{p}_o + \frac{\alpha(h)}{3}[1 - P(\bar{p}_o)][2 - P(\bar{p}_o)]$$

This can be reduced to

$$\left(\frac{\alpha(h)}{3}\right)^{-1} g(\bar{p}_o) = [1 - P(\bar{p}_o)]\,[2 - P(\bar{p}_o)] - \bar{p}_o\,(2\bar{p}_o - 1)$$

We can construct a lower bound for this object. Noting that $\bar{p}_o > P(\bar{p}_o)$,

$$\left(\frac{\alpha(h)}{3}\right)^{-1} g(\bar{p}_o) > -\left(\bar{p}_o^2 + 2\bar{p}_o - 2\right)$$

If $\bar{p}_o \in \left(\frac{1}{2}, \sqrt(3) - 1\right)$ then we can guarantee that $g(\bar{p}_0) > 0$, and that there is a cycle. Now, from the definition of S, we can see that $lim_{h \to 0} \bar{p}_o = \frac{1}{2}$, so along with the continuity of S, this implies that in a neighborhood around $h = 0$ this condition will be satisfied.

C Proof of Proposition 1

We develop the proof as a sequence of claims. Let $\{p_t, s_t, \Delta_{t-h}\}_{t \in \mathbf{T}}$ be a sequence that satisfies equation (5), equation (6), and equation (7).

Claim 1 *If $\Delta_t > \overline{\Delta}(p_{t+h})$ then $\Delta_{t+h} > \underline{\Delta}(p_{t+2h})$ and $\Delta_{t+h} \geq \Delta_t / \beta(h)$. Similarly, if $\Delta_t < \underline{\Delta}(p_{t+h})$ then $\Delta_{t+h} < \underline{\Delta}(p_{t+2h})$ and $\Delta_{t+h} \leq \Delta_t / \beta(h)$.*

Proof. Rearranging the perfect foresight version of equation (7) gives

$$\Delta_{t+h} = \frac{\Delta_t - \beta(h) \frac{\alpha(h)}{3} u(s_{t+h} - p_{t+h})}{\beta(h) \Omega_{t+h}}$$

where $\Omega_t = 1 - \frac{\alpha(h)}{3}[p_t s_t + 2(1-p_t)(1-s_t) + p_t] \in (0,1]$. $\Delta_t > \overline{\Delta}(p_{t+h})$ guarantees that $\Delta_{t+h} > 0$ and hence $s_{t+h} = 0$. Similarly, $\Delta_t < \underline{\Delta}(p_{t+h})$ guarantees that $\Delta_{t+h} < 0$ and hence $s_{t+h} = 1$.

Another rearrangement of equation (7) gives

$$\beta(h)\Delta_{t+h} - \Delta_t = -\beta(h)\frac{\alpha(h)}{3}u(s_{t+h} - p_{t+h}) + \beta(h)(1 - \Omega_{t+h})\Delta_{t+h}$$

If $\Delta_{t+h} > 0$, then $s_{t+h} = 0$ and hence $\beta(h)\Delta_{t+h} \geq \Delta_t$. Similarly, if $\Delta_{t+h} < 0$, then $s_{t+h} = 1$ and hence $\beta(h)\Delta_{t+h} \leq \Delta_t$.

We can also rearrange equation (6) to be

$$p_{t+2h} - p_{t+h} = \frac{\alpha(h)}{3}\left\{p_{t+h}(1 - 2p_{t+h})s_{t+h} + (2 - p_{t+h})(1 - p_{t+h})(1 - s_{t+h})\right\}$$

If $s_{t+h} = 0$ then $p_{t+2h} \geq p_{t+h}$. If $s_{t+h} = 1$ then the sign of $p_{t+2h} - p_{t+h}$ depends on whether $p_{t+h} \gtrless 1/2$.

Consider first the case of $\Delta_t > \overline{\Delta}(p_{t+h})$. We have shown that $\Delta_{t+h} \geq \Delta_t$ and that $p_{t+2h} \geq p_{t+h}$. These, along with the fact that $\overline{\Delta}$ is decreasing in p imply that $\Delta_{t+h} > \overline{\Delta}(p_{t+2h})$.

Now consider $\Delta_t < \underline{\Delta}(p_{t+h})$. We have shown that $\Delta_{t+h} \leq \Delta_t$. If in addition $p_{t+h} \geq 1/2$, then $p_{t+2h} \leq p_{t+h}$. These along with the fact that $\overline{\Delta}$ is decreasing in p imply that $\Delta_{t+h} < \underline{\Delta}(p_{t+2h})$.

If, however, $p_{t+h} < 1/2$ then we cannot rely on this argument because $p_{t+2h} > p_{t+h}$. Instead we check algebraically that $\Delta_{t+h} < \underline{\Delta}(p_{t+2h})$. We can write

$$\begin{aligned}
\frac{\Delta_{t+h} - \Delta_t}{p_{t+2h} - p_{t+h}} &= \frac{(1 - \beta(h))\Delta_{t+h} - \beta(h)\left[\frac{\alpha(h)}{3}u(1 - p_{t+h}) - \frac{\alpha(h)}{3}2p_{t+h}\Delta_{t+h}\right]}{\frac{\alpha(h)}{3}p_{t+h}(1 - 2p_{t+h})} \\
&< -\frac{\beta(h)u(1 - p_{t+h})}{p_{t+h}(1 - 2p_{t+h})} \\
&< -\beta(h)u\frac{\alpha(h)}{3} \\
&= -\beta(h)\gamma(h)
\end{aligned}$$

where the last inequality follows as $p_{t+h} < \frac{1}{2}$ and $\frac{\alpha(h)}{3} < 1$.

Starting with $\Delta_t < -\beta(h)\gamma(h)p_{t+h}$, we have that

$$\begin{aligned}
\Delta_{t+h} &< -\beta(h)\gamma(h)p_{t+2h} + \beta(h)\gamma(h)p_{t+h} + \Delta_t \\
&< -\beta(h)\gamma(h)p_{t+2h} \\
&< \underline{\Delta}(p_{t+2h})
\end{aligned}$$

which completes the proof ■

Claim 2 *Let $\{p_t, s_t, \Delta_{t-h}\}_{t\in\mathbf{T}}$ be a sequence that satisfies equations (5), (6), and (7). This an equilibrium for an economy with initial condition p_0 if and only if $\Delta_t \in \left[\underline{\Delta}(p_{t+h}), \overline{\Delta}(p_{t+h})\right]$.*

Proof. If $\Delta_t \notin \left[\underline{\Delta}(p_{t+h}), \overline{\Delta}(p_{t+h})\right]$ for some t, then the previous claim implies that $\Delta_{t+Nh} \notin \left[\underline{\Delta}(p_{t+(N+1)h}), \overline{\Delta}(p_{t+(N+1)h})\right]$ for all $N > 0$. Therefore $|\Delta_{t+Nh}| \geq \beta(h)^{-N}|\Delta_t|$. This would violate the uniform bound on $\{\Delta_t\}$, so the sequence cannot be an equilibrium.

If, however, $\Delta_t \in \left[\underline{\Delta}(p_{t+h}), \overline{\Delta}(p_{t+h})\right]$ for all $t \in \mathbf{T}$ then the sequence $\{\Delta_t\}$ has a uniform bound. By Lemma 1, the sequence is consistent with equilibrium. ■

D Proofs of Proposition 2 and Proposition 3

We first prove a preliminary result that will help us prove Proposition 2 and Proposition 3. Let $\{p_t, s_t, \Delta_{t-h}\}_{t\in\mathbf{T}}$ be an sequence consistent with equilibrium.

Lemma 2 *For any $N > 0$, the following inequality holds:*

$$\sum_{n=1}^{N} \omega_{t,n,N}\left(s_{t+nh} - p_{t+nh}\right) \leq 2\frac{1-\beta(h)\left(1-\alpha(h)\right)}{1-\left[\beta(h)\left(1-\alpha(h)\right)\right]^N}$$

where

$$\omega_{t,n,N} = \frac{\prod_{j=1}^{n-1}\rho_{t+jh}}{\sum_{\tilde{n}=1}^{N}\left(\prod_{j=1}^{\tilde{n}-1}\rho_{t+jh}\right)}$$

and $\rho_t = \beta(h)\left(1 - \frac{\alpha(h)}{3}\left[p_t\left(1+s_t\right) + 2\left(1-p_t\right)\left(1-s_t\right)\right]\right)$.

Proof. Under perfect foresight equation (7) can be written as

$$\Delta_t = \beta(h)\frac{\alpha(h)}{3}u\left(s_{t+h} - p_{t+h}\right) + \rho_{t+h}\Delta_{t+h}$$

We can iterate this equation to get

$$\Delta_t = \beta(h)\frac{\alpha(h)}{3}u\sum_{n=1}^{N}\left(\prod_{j=1}^{n-1}\rho_{t+jh}\right)\left(s_{t+nh} - p_{t+nh}\right) + \left(\prod_{n=1}^{N}\rho_{t+nh}\right)\Delta_{t+Nh}$$

where $\prod_{j=1}^{0}$ is defined to be 1.

Reordering terms, dividing by $\sum_{n=1}^{N}\left(\prod_{j=1}^{n-1}\rho_{t+jh}\right)$, and using the definition of $\omega_{t,n,N}$ provides

$$\beta(h)\frac{\alpha(h)}{3}u\sum_{n=1}^{N}\omega_{t,n,N}\left(s_{t+nh}-p_{t+nh}\right)=\frac{\Delta_t-\left(\prod_{n=1}^{N}\rho_{t+nh}\right)\Delta_{t+Nh}}{\sum_{n=1}^{N}\left(\prod_{j=1}^{n-1}\rho_{t+jh}\right)}$$

Since $\rho_t\in\left(\beta(h)\left(1-\frac{2}{3}\alpha(h)\right),\beta(h)\right]$, we can bound the right hand side of this equation. The denominator is greater than $\sum_{n=1}^{N}\left[\beta(h)(1-\frac{2}{3}\alpha(h))\right]^{n-1}$, while the magnitude of the numerator is less than $2\beta(h)\gamma(h)$. These give the following bound:

$$\sum_{n=1}^{N}\omega_{t,n,N}\left(s_{t+nh}-p_{t+nh}\right)\leq 2\frac{1-\beta(h)\left(1-\frac{2}{3}\alpha(h)\right)}{1-\left[\beta(h)\left(1-\frac{2}{3}\alpha(h)\right)\right]^{N}} \tag{12}$$

We can also use the bounds on ρ to bound each individual ω

$$\omega_{t,n,N}\in\left(\frac{\left[1-\frac{2}{3}\alpha(h)\right]^{N}}{N},\frac{1}{N\left[1-\frac{2}{3}\alpha(h)\right]^{N}}\right) \tag{13}$$

which completes the proof. ■

D.1 Proof of Proposition 2

In any equilibrium, the sequence of inventories follows the equation

$$p_{t+h}=p_t+\frac{\alpha(h)}{3}\left[-p_t^2 s_t+2(1-p_t)^2(1-s_t)+p_t(1-p_t)\right]$$

Similarly, the sequence of inventories for the Zero Equilibrium must also follow the law of motion. Combining these equations give

$$p_{t+h}-p_{t+h}^0=\Phi_t(s_t-p_t)-\lambda_t(p_t-p_t^0)$$

where Φ_t and λ_t are defined and bounded as follows

$$\begin{aligned}\Phi_t &= 1-\tfrac{\alpha(h)}{3}\left\{-\left[3(p_t+p_t^0)-4\right]p_t^0-3+(p_t+p_t^0)+p_t^2+2(1-p_t)^2\right\}\\ &\in \left[1-\tfrac{5}{3}\alpha(h),1-\tfrac{2}{3}\alpha(h)\right]\end{aligned} \tag{14}$$

and

$$\lambda_t=\frac{\alpha(h)}{3}\left[p_t^2+2(1-p_t)^2\right]\in\left[\frac{2}{9}\alpha(h),\frac{2}{3}\alpha(h)\right] \tag{15}$$

We can iterate this equation over N periods to get

$$p_{t+Nh}-p^0_{t+Nh} = \sum_{n=0}^{N-1}\left(\prod_{j=n+1}^{N-1}\Phi_{t+jh}\right)\lambda_{t+nh}(s_{t+nh}-p_{t+nh})+\left(\prod_{n=0}^{N-1}\Phi_{t+nh}\right)(p_t-p^0_t) \quad (16)$$

where again the product $\prod_{n=N}^{N-1}$ is defined to be one.

We now provide a bound on the divergence of inventories from those of the Zero Equilibrium among the first N periods. Since $p_0 = p^0_0$ and $|s_t - p_t| \leq 1$ we can use equation (16) and the upper bounds on Φ and λ given by equation (14) and equation (15) to get:

$$\left|p_{Nh} - p^0_{Nh}\right| \leq \frac{2}{3}\alpha(h)\sum_{n=0}^{N-1}\left(1-\frac{2}{3}\alpha(h)\right)^n = 1-\left[1-\frac{2}{3}\alpha(h)\right]^N$$

Define $\tilde{\pi}_0(h,N) \equiv 1-\left[1-\frac{2}{3}\alpha(h)\right]^N$ to be this bound.

We next provide a bound on the subsequent divergence of inventories from those of the Zero Equilibrium. We can write equation (16) as

$$\begin{aligned} p_{t+Nh} - p^0_{t+Nh} \;=\; & \left(1-\prod_{n=0}^{N-1}\Phi_{t+nh}\right)\chi_{t,N}\sum_{n=0}^{N-1}\frac{\lambda_{t+nh}}{\lambda_t}\phi_{t,n,N}(s_{t+nh}-p_{t+nh}) \\ & + \left(\prod_{n=0}^{N-1}\Phi_{t+nh}\right)(p_t-p^0_t) \end{aligned} \quad (17)$$

where χ and ϕ are defined by

$$\chi_{t,N} = \lambda_t\frac{\sum_{\tilde{n}=0}^{N-1}\left(\prod_{j=\tilde{n}+1}^{N-1}\Phi_{t+jh}\right)}{1-\prod_{n=0}^{N-1}\Phi_{t+nh}}$$

$$\phi_{t,n,N} = \frac{\prod_{j=n+1}^{N-1}\Phi_{t+jh}}{\sum_{\tilde{n}=0}^{N-1}\left(\prod_{j=\tilde{n}+1}^{N-1}\Phi_{t+jh}\right)}$$

We will show that the term $\chi_{t,N}\sum_{n=0}^{N-1}\frac{\lambda_{t+nh}}{\lambda_t}\phi_{t,n,N}(s_{t+nh}-p_{t+nh})$ can be bounded by a function $\tilde{\pi}_1(h,N)$. This is useful because equation (17) would then imply that if $|p_t - p^0_t| \leq \varepsilon$ for some $\varepsilon \geq \pi_1(N,h)$, then we also have $\left|p_{t+Nh}-p^0_{t+Nh}\right| \leq \varepsilon$.

To do this, we first show that $|\chi_{t,N}| \leq 1$. Since χ is increasing in each Φ_t, we can use the upper bounds on λ and Φ to get

$$|\chi_{t,N}| \leq \left|\frac{2}{3}\alpha(h)\frac{\sum_{n=0}^{N-1}\left(1-\frac{2}{3}\alpha(h)\right)^n}{1-\left(1-\frac{2}{3}\alpha(h)\right)^N}\right| = 1$$

Next we can bound $\sum_{n=0}^{N-1} \frac{\lambda_{t+nh}}{\lambda_t}\phi_{t,n,N}(s_{t+nh} - p_{t+nh})$ by decomposing it into three parts using

$$\frac{\lambda_{t+nh}}{\lambda_t}\phi_{t,n,N} = \left(\frac{\lambda_{t+nh} - \lambda_t}{\lambda_t}\phi_{t+nh}\right) + (\phi_{t,n,N} - \omega_{t,n,N}) + (\omega_{t,n,N})$$

Using $|s_t - p_t| \leq 1$ and $\phi_{t,n,N} > 0$ gives

$$\begin{aligned}\left|\sum_{n=0}^{N-1} \frac{\lambda_{t+nh}}{\lambda_t}\phi_{t,n,N}(s_{t+nh} - p_{t+nh})\right| \;\; &\leq \;\; \sum_{n=0}^{N-1}\left|\frac{\lambda_{t+nh} - \lambda_t}{\lambda_t}\right|\phi_{t,n,N} + \sum_{n=0}^{N-1}|\phi_{t,n,N} - \omega_{t,n,N})| \\ &\quad + \left|\sum_{n=0}^{N-1}\omega_{t,n,N}(s_{t+nh} - p_{t+nh})\right|\end{aligned}$$

We will bound each of these three terms separately. First, note that equation (6) implies

$$|p_{t+h} - p_t| = \frac{\alpha(h)}{3}\left|(1-2p_t)\,p_t s_t + (2-p_t)\,(1-p_t)\,(1-s_t)\right| \leq \frac{2}{3}\alpha(h)$$

and hence $|p_{t+nh} - p_t| \leq n\left(\frac{2}{3}\alpha(h)\right)$. We can also use the definition of λ to write

$$\left|\frac{\lambda_{t+nh} - \lambda_t}{\lambda_t}\right| = \left|\frac{\frac{\alpha(h)}{3}\,(p_{t+nh} - p_t)\,(3(p_{t+nh} + p_t) - 4)}{\lambda_t}\right| \leq \frac{\frac{\alpha(h)}{3}\left(n\frac{2}{3}\alpha\right)4}{\frac{2}{9}\alpha(h)} \leq 4N\alpha(h)$$

Since $\sum_{n=0}^{N-1}\phi_{t,n,N} = 1$, we have

$$\left|\sum_{n=0}^{N-1}\left|\frac{\lambda_{t+nh} - \lambda_t}{\lambda_t}\right|\phi_{t,n,N}\right| \leq 4N\alpha(h)$$

We can use the bound on Φ given by equation (14) to get upper and lower bounds for ϕ:

$$\phi_{t,n,N} \in \left(\frac{1}{N}\left(\frac{1-\frac{5}{3}\alpha(h)}{1-\frac{2}{3}\alpha(h)}\right)^N, \frac{1}{N}\left(\frac{1-\frac{2}{3}\alpha(h)}{1-\frac{5}{3}\alpha(h)}\right)^N\right)$$

This, in combination with the bounds on ω from equation (13) imply that

$$\begin{aligned} |\phi_{t,n,N} - \omega_{t,n,N}| &\leq \tfrac{1}{N} \max_{\iota \in \{-1,1\}} \left| \left(1 - \tfrac{2}{3}\alpha(h)\right)^{\iota N} - \left(\tfrac{1-\frac{2}{3}\alpha(h)}{1-\frac{5}{3}\alpha(h)}\right)^{\iota N} \right| \\ &= \left(1 - \tfrac{2}{3}\alpha(h)\right)^{N} \left[\left(1 - \tfrac{5}{3}\alpha(h)\right)^{-N} - 1\right] \end{aligned}$$

Lastly, the third term can be bounded using Lemma 2. In total, these give the result that

$$\chi_{t,N} \sum_{n=0}^{N-1} \frac{\lambda_{t+nh}}{\lambda_t} \phi_{t,n,N}(s_{t+nh} - p_{t+nh}) \leq \tilde{\pi}_1(h,N)$$

with

$$\tilde{\pi}_1(h,N) \equiv 4N\alpha(h) + \left(1 - \tfrac{2}{3}\alpha(h)\right)^{N} \left(\left(1 - \tfrac{5}{3}\alpha(h)\right)^{-N} - 1\right) + 2\frac{1-\beta(h)\left(1-\frac{2}{3}\alpha(h)\right)}{1-\left[\beta(h)\left(1-\frac{2}{3}\alpha(h)\right)\right]^N}$$

At this point we have shown that for any N, inventories in the first N periods are within $\tilde{\pi}_0(h,N)$ of those of the Zero Equilibrium. We have also shown that if inventories in the first N periods are within ε of those of Zero Equilibrium for any quantity $\varepsilon \geq \tilde{\pi}_1(h,N)$, then inventories in all subsequent periods are as well. We can combine these two statements to arrive at a uniform bound for the entire sequence. Define $\tilde{\pi}(h,N) = \max\{\tilde{\pi}_0(h,N), \tilde{\pi}_1(h,N)\}$. We therefore have that for any $N > 0$ and any $t \in \mathbf{T}$, inventories are within $\tilde{\pi}(N,h)$ of those of the Zero Equilibrium:

$$\left|p_t - p_t^0\right| \leq \tilde{\pi}(h,N)$$

Let $\pi(h) = \min_N \tilde{\pi}(h,N)$. This will be a bound for $|p_t - p_t^0|$.

Lastly, we can show that $\lim_{h\to 0} \pi(h) = 0$. Let $\nu(h) = h^{-1/2}$. From the definitions of $\tilde{\pi}_0$ and $\tilde{\pi}_1$ it is straightforward to show that $\lim_{h\to 0} \tilde{\pi}_0(h,\nu(h)) = \lim_{h\to 0} \tilde{\pi}_1(h,\nu(h)) = 0$. Since $\pi(h) \leq \tilde{\pi}(h,\nu(h))$, these imply that $\lim_{h\to 0} \pi(h) = 0$.

D.2 Proof of Proposition 3

In a similar way, we can show that, at least locally, the average trading strategy played coincides with that of the Zero Equilibrium.

For $\varepsilon > 0$, let N be the largest integer such that $\varepsilon \geq (2N+1)h$. We can form a bound on the local average trading strategy:

$$
\begin{aligned}
\left|\left(\tfrac{1}{2N+1}\sum_{n=-N}^{N} s_{t+nh}\right)-p_t\right| \quad &\leq \quad \left|\sum_{n=-N}^{N}\left(\tfrac{1}{2N+1}-\omega_{t-N,n+N,2N+1}\right)(s_{t+nh}-p_{t+nh})\right| \\
&\quad + \left|\sum_{n=-N}^{N}\omega_{t-N,n+N,2N+1}(s_{t+nh}-p_{t+nh})\right| \\
&\quad + \left|\tfrac{1}{2N+1}\sum_{n=-N}^{N}(p_{t+nh}-p_t)\right|
\end{aligned}
$$

The first sum can be bounded using the bound on ω given by equation (13)

$$
\begin{aligned}
\left|\sum_{n=-N}^{N}\left(\tfrac{1}{2N+1}-\omega_{t-N,n+N,2N+1}\right)(s_{t+nh}-p_{t+nh})\right| \quad &\leq \quad \sum_{n=-N}^{N}\left|\tfrac{1}{2N+1}-\omega_{t-N,n+N,2N+1}\right| \\
&\leq \quad \left(1-\tfrac{2}{3}\alpha(h)\right)^{-(2N+1)}-1
\end{aligned}
$$

The second summation can be bounded using equation (12). The third term can be bounded using the fact that $|p_{t+h}-p_t| \leq \alpha(h)$, which can be seen from equation (6). This implies that

$$
\left|\frac{1}{2N+1}\sum_{n=-N}^{N}(p_{t+nh}-p_t)\right| \leq N\alpha(h)
$$

We can combine these to form a single bound for a fixed ε:

$$
\sigma(\varepsilon,h)=\left(1-\frac{2}{3}\alpha(h)\right)^{-\varepsilon/h}-1+2\frac{1-\beta(h)\left(1-\frac{2}{3}\alpha(h)\right)}{1-\left[\beta(h)\left(1-\frac{2}{3}\alpha(h)\right)\right]^{\varepsilon/h}}+\frac{\varepsilon}{2}\alpha(h)
$$

For a fixed ε, each of these three bounds goes to a finite number as $h \to 0$:

$$
\lim_{h\to 0}\sigma(\varepsilon,h)=e^{\frac{2}{3}\alpha_0\varepsilon}-1+\frac{\varepsilon}{2}\alpha_0
$$

It follows that $\lim_{\varepsilon\to 0}\left(\lim_{h\to 0}\sigma(\varepsilon,h)\right)=0$.

E Proof of Proposition 4

We develop the proof as a sequence of claims. For ease of exposition we drop the argument z from p_t, s_t, and Δ_t.

Let $\{p_t, s_t, \Delta_{t-h}\}_{z^t\in\mathbf{Z}^t, t\in\mathbf{T}}$ be a sequence that satisfies equation (5), equation (6), and equation (7). Also, Let $G_{t,n}$ be the event that $\Delta_{t-jh} \notin [\underline{\Delta}(p_{t-(j-1)h}), \overline{\Delta}(p_{t-(j-1)h})]$ for all $j \in (0,...,n)$. We can make the following claims about the sequence:

Claim 3 *If* $\Pr(G_{t,n})>0$ *then* $\Pr\left(G_{t+h,n+1} \text{ and } |\Delta_{t+h}| \geq \frac{|\Delta_t|}{\beta(h)}\right)>0$

Proof. The following definitions will assist in the exposition of the proof. As before, let $\Omega_t = \frac{\alpha(h)}{3}[p_t s_t + 2(1-p_t)(1-s_t) + p_t]$. Note that $\Omega_t \in [0,1]$. Also let $X_{t+h} = -\Delta_t + \beta(h)\Delta_{t+h} + \beta(h)\frac{\alpha(h)}{3}(s_{t+h} - p_{t+h})u - \beta(h)\Omega_{t+h}\Delta_{t+h}$. equation (7) can be rewritten as $0 = \mathbb{E}_t[X_{t+h}]$, where $\mathbb{E}_t(\cdot) = \mathbb{E}(\cdot \mid z^t)$. This implies both that $\Pr(X_{t+h} \geq 0|z^t) > 0$ and also that $\Pr(X_{t+h} \leq 0|z^t) > 0$ for all z^t. We therefore have that if $\Pr(G_{t,n}) > 0$ then either $\Pr\big(G_{t,n} \text{ and } \Delta_t > \overline{\Delta}(p_{t+h}) \text{ and } X_{t+h} \geq 0\big) > 0$ or $\Pr(G_{t,n} \text{ and } \Delta_t < \underline{\Delta}(p_{t+h}) \text{ and } X_{t+h} \leq 0) > 0$. We will show that in either case

$$\Pr\left(G_{t+h,n+1} \text{ and } |\Delta_{t+h}| \geq \frac{|\Delta_t|}{\beta(h)}\right) > 0$$

.

First, consider the event in which $\Delta_t > \overline{\Delta}(p_{t+h})$. If $\Delta_{t+h} \leq 0$, then it must be that $X_{t+h} < 0$, because $\Delta_t > \overline{\Delta}(p_{t+h}) \geq \beta(h)\frac{\alpha(h)}{3}(s_{t+h} - p_{t+h})u$. Consequently, if $X_{t+h} \geq 0$, then $\Delta_{t+h} > 0$ and therefore $s_{t+h} = 0$. The combination of $X_{t+h} \geq 0$ and $s_{t+h} = 0$ imply that $\Delta_{t+h} \geq \frac{\Delta_t}{\beta(h)}$ and $p_{t+2h} > p_{t+h}$. Since $\overline{\Delta}(p)$ is decreasing in p, these also imply that $\Delta_{t+h} > \overline{\Delta}(p_{t+2h})$. We therefore have that in the event that $\Delta_t > \overline{\Delta}(p_{t+h})$ and $X_{t+h} \geq 0$, then $\Delta_{t+h} \notin [\underline{\Delta}(p_{t+2h}), \overline{\Delta}(p_{t+2h})]$ and $|\Delta_{t+h}| \geq \frac{|\Delta_t|}{\beta(h)}$.

Now we turn to the event in which $\Delta_t < \underline{\Delta}(p_{t+h})$, $X_{t+h} \leq 0$, and $\Delta_{t+h} < 0$. We will show that in this case $\Delta_{t+h} < \underline{\Delta}(p_{t+2h})$. This is more difficult because the change in p is not a monotonic function of p. If $\Delta_t < \underline{\Delta}(p_{t+h})$ then in a similar manner as above we can show that $\Delta_{t+h} < 0$ and $s_{t+h} = 1$. This means that we can write

$$\Delta_t \geq \beta(h)\left[\Delta_{t+h} + \frac{\alpha(h)}{3}u(1 - p_{t+h}) - \Omega_{t+h}\Delta_{t+h}\right]$$

and

$$p_{t+h} = p_t + \frac{\alpha(h)}{3}p_t(1 - 2p_t)$$

We take two cases separately. For each we will show that if Δ_t is below the bound, than Δ_{t+h} is below the bound as well. (i) If $p_{t+h} \geq \frac{1}{2}$, then we can show this in a similar manner as above. Since $p_{t+2h} \leq p_{t+h}$ and $\Delta_{t+h} < \Delta_t < 0$, the fact that $\underline{\Delta}(p)$

is decreasing in p implies that $\Delta_{t+h} < \underline{\Delta}(p_{t+2h})$. (ii) If $p_{t+h} < 1/2$ then we can write

$$\begin{aligned}
\frac{\Delta_{t+h} - \Delta_t}{p_{t+2h} - p_{t+h}} &\leq \frac{(1-\beta(h))\Delta_{t+h} - \beta(h)\left[\frac{\alpha(h)}{3}u(1-p_{t+h}) - \Omega_{t+h}\Delta_{t+h}\right]}{\frac{\alpha(h)}{3}p_{t+h}(1-2p_{t+h})} \\
&< -\frac{\beta(h)u(1-p_{t+h})}{p_{t+h}(1-2p_{t+h})} \\
&< -\beta(h)u\frac{\alpha(h)}{3} \\
&= -\beta(h)\gamma(h)
\end{aligned}$$

where the last inequality follows because $p_{t+h} < \frac{1}{2}$ and $\frac{\alpha(h)}{3} < 1$. We start with $\Delta_t < -\beta(h)\gamma(h)p_{t+h}$. We then have that

$$\begin{aligned}
\Delta_{t+h} &< -\beta(h)\gamma(h)p_{t+2h} + \beta(h)\gamma(h)p_{t+h} + \Delta_t \\
&< -\beta(h)\gamma(h)p_{t+2h} \\
&< \underline{\Delta}(p_{t+2h})
\end{aligned}$$

For both cases we also know that $X_{t+h} \leq 0$. This, in combination with $s_{t+h} = 1$, implies that $\Delta_{t+h} \leq \frac{\Delta_t}{\beta(h)}$.

If $\Delta_{t+h} \geq 0$ then we know that $X_{t+h} > 0$ because $\Delta_t < \underline{\Delta}(p_{t+h}) \leq \beta(h)\frac{\alpha(h)}{3}(s_{t+h} - p_{t+h})u$. This implies that if $X_{t+h} \leq 0$, then $\Delta_{t+h} < 0$. We have therefore shown that if $\Delta_t < \underline{\Delta}(p_{t+h})$ and $X_{t+h} \leq 0$, then $\Delta_{t+h} \notin [\underline{\Delta}(p_{t+2h}), \overline{\Delta}(p_{t+2h})]$ and $|\Delta_{t+h}| \geq \frac{|\Delta_t|}{\beta(h)}$. ∎

Claim 3 shows that if $\Delta_t \notin \left[\underline{\Delta}(p_{t+h}), \overline{\Delta}(p_{t+h})\right]$, then the following Δ is outside the bounds with positive probability and the magnitude grows exponentially.

Claim 4 *If* $\Pr\left(\Delta_t \notin [\underline{\Delta}(p_{t+h}), \overline{\Delta}(p_{t+h})]\right) > 0$ *then a sequence of inventories, strategies, and value functions* $\left\{p_t\left(z^{t-h}\right), s_t\left(z^t\right), \Delta_{t-h}\left(z^{t-h}\right)\right\}_{z^t \in \mathbf{Z}^t, t \in \mathbf{T}}$ *is not consistent with equilibrium.*

Proof. Let B be the uniform bound implied by Lemma 1. Assume that there exists a t_0 such that $\Pr\left(\Delta_{t_0} \notin [\underline{\Delta}(p_{t_0+h}), \overline{\Delta}(p_{t_0+h})]\right) > 0$. This implies that there exists $\epsilon > 0$ such that $\Pr\left(\Delta_{t_0} \notin [\underline{\Delta}(p_{t_0+h}), \overline{\Delta}(p_{t_0+h})], |\Delta_{t_0}| > \epsilon\right) > 0$. Iterating Claim 3 gives the result

$$\Pr\left(\Delta_{t_0+nh} \notin [\underline{\Delta}(p_{t_0+(n+1)h}), \overline{\Delta}(p_{t_0+(n+1)h})], |\Delta_{t_0+nh}| \geq \frac{|\Delta_{t_0}|}{\beta(h)^n} \geq \frac{\epsilon}{\beta(h)}\right) > 0$$

Since there exists an $N > 0$ such that $\frac{\epsilon}{\beta(h)^N} > B$, we have $\Pr\left(|\Delta_{t+Nh}| > B\right) > 0$. ∎