Identification of the long-run β structure

A graduate course in the Cointegrated VAR model: Special topics in Rome

Katarina Juselius

University of Copenhagen

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Identification of common stochastic trends

The identification problem of common driving trends, $\alpha'_{\perp}\sum\varepsilon_i$, is similar to the identification the long-run β relations in the following sense: One can always choose a normalization and $(p-r-1)$ restrictions without changing the value of the likelihood function, whereas additional restrictions are overidentifying and, hence, testable. I shall discuss how to impose econometrically plausible restrictions on the underlying common trends, but aviod attaching a (wishful) structural interpretation to the estimated shocks.

The discussion here will be based on the VAR model with a linear trend restricted to the cointegration relations but, for simplicity, with no dummy variables:

$$\begin{aligned} \Delta x_t &= \Gamma_1 \Delta x_{t-1} + \alpha\beta' x_{t-1} + \mu_0 + \alpha\beta_1 t + \varepsilon_t, \\ \varepsilon_t &\sim IN(0, \Omega) \end{aligned} \tag{1}$$

The corresponding moving-average representation is given by:

$$x_t = C \sum_{i=1}^{t} \varepsilon_i + tC\mu_0 + C^*(L)(\varepsilon_t + \mu_0 + \alpha\beta_1 t) + \tilde{X}_0, \tag{2}$$

where

$$C = \beta_\perp (\alpha'_\perp \Gamma \beta_\perp)^{-1} \alpha'_\perp. \tag{3}$$

It is useful to express the C matrix as a product of two matrices (similarly to $\Pi = \alpha\beta'$)

$$C = \widetilde{\beta}_\perp \alpha'_\perp, \tag{4}$$

where $\widetilde{\beta}_\perp = \beta_\perp(\alpha'_\perp \Gamma \beta_\perp)^{-1}$. The matrices $\beta_\perp$ and $\alpha_\perp$ can be directly calculated for given estimates of α, β.

The decomposition of $C = \widetilde{\beta}_{\perp}\alpha'_{\perp}$ resembles the decomposition $\Pi = \alpha\beta'$ but with the important difference that $\widetilde{\beta}_{\perp}$ is a function not only of $\beta_{\perp}$, but also of $\alpha_{\perp}$. Similar to α and β, one can transform $\widetilde{\beta}_{\perp}$ and $\alpha_{\perp}$ by a nonsingular $(p-r) \times (p-r)$ matrix Q

$$C = \widetilde{\beta}_{\perp} Q Q^{-1} \alpha'_{\perp} = \widetilde{\beta}^{c}_{\perp} (\alpha^{c}_{\perp})' \tag{5}$$

without changing the value of the likelihood function. The Q transformation leads to just-identified common trends for which no testing is involved. Additional restrictions on $\widetilde{\beta}_{\perp}$ and $\alpha_{\perp}$ constrain the likelihood function and, hence, are testable. Tests of such overidentifying restrictions on the common trends are in general highly nonlinear and, therefore, difficult to test. There are, however, a few special cases of overidentifying restrictions on $\alpha_{\perp}$ and $\beta_{\perp}$ which can be expressed as testable restrictions on α and β

Case 1: Long-run homogeneity

$$\beta = \begin{bmatrix} a & b & c \\ -\omega_1 a & -\omega_2 b & -\omega_3 c \\ -(1-\omega_1)a & -(1-\omega_2)b & -(1-\omega_3)c \\ * & * & * \\ * & * & * \end{bmatrix} \rightarrow \beta_\perp = \begin{bmatrix} 1 & * \\ 1 & * \\ 1 & * \\ 0 & * \\ 0 & * \end{bmatrix},$$

i.e. one of the stochastic trends affects the homogeneously related variables with equal weights.

Case 2: A stationary variable in beta

$$\beta = \begin{bmatrix} 0 & * & * \\ 1 & * & * \\ 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \rightarrow \beta_{\perp} = \begin{bmatrix} * & * \\ 0 & 0 \\ * & * \\ * & * \\ * & * \end{bmatrix},$$

corresponds to a zero row in $\beta_{\perp}$ and the C-matrix.

Case 3: A column in alpha is proportional to a unit vector

$$\alpha = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \rightarrow \alpha_\perp = \begin{bmatrix} 0 & 0 \\ * & * \\ * & * \\ * & * \\ * & * \end{bmatrix},$$

corresponds to a zero column in the C matrix.

Case 4: A row in alpha is equal to zero

$$\alpha = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ 0 & 0 & 0 \end{bmatrix} \to \alpha_\perp = \begin{bmatrix} * & 0 \\ * & 0 \\ * & 0 \\ * & 0 \\ * & 1 \end{bmatrix},$$

i.e. cumulated shocks to the zero row variable is a common driving trend and the variable is weakly exogenous for the long-run parameters.

	$\hat{\varepsilon}_{m^r}$	$\hat{\varepsilon}_{y^r}$	$\hat{\varepsilon}_{\Delta p}$	$\hat{\varepsilon}_{R_m}$	$\hat{\varepsilon}_{R_b}$
σ_{ε_i}	0.0231	0.0138	0.0080	0.0010	0.0014
The unrestricted estimates					
$\hat{\alpha}'_{\perp,1}$	0.05	0.17	-0.07	0.10	1.00
$\hat{\alpha}'_{\perp,2}$	0.06	-0.18	0.13	1.00	-0.06
	m^r	y^r	Δp	R_m	R_b
$\widetilde{\hat{\beta}}'_{\perp,1}$	-5.12	-2.01	-0.13	0.69	1.06
$\widetilde{\hat{\beta}}'_{\perp,2}$	-8.76	-6.62	0.10	0.61	0.84
A just identified representation					
$\hat{\alpha}^{c\prime}_{\perp,1}$	0.00 [0.01]	0.00	0.06 [1.08]	1.11 [1.45]	**1.00**
$\hat{\alpha}^{c\prime}_{\perp,2}$	0.32 [1.63]	**1.00**	-0.78 [-1.49]	-5.94 [-0.84]	0.00
	m^r	y^r	Δp	R_m	R_b
$\widetilde{\hat{\beta}}^{c\prime}_{\perp,1}$	$\mathbf{-4.46}$ [-2.06]	-1.56 [-0.79]	-0.13 [-1.75]	**0.64** [4.38]	**0.99** [4.56]
$\widetilde{\hat{\beta}}^{c\prime}_{\perp,2}$	**0.69** [3.62]	**0.83** [4.73]	$\mathbf{-0.04}$ [-5.69]	0.01 [0.57]	0.03 [1.52]

Table: The MA representation for unrestricted α and β

	$\hat{\varepsilon}_{m^r}$	$\hat{\varepsilon}_{y^r}$	$\hat{\varepsilon}_{\Delta p}$	$\hat{\varepsilon}_{R_m}$	$\hat{\varepsilon}_{R_b}$	
σ_{ε_i}	0.0231	0.0138	0.0080	0.0010	0.0014	
The **C** *matrix*						
	$\hat{\varepsilon}_{m^r}$	$\hat{\varepsilon}_{y^r}$	$\hat{\varepsilon}_{\Delta p}$	$\hat{\varepsilon}_{R_m}$	$\hat{\varepsilon}_{R_b}$	$t \times \gamma_{0,i}$
m^r	0.22 (1.5)	**0.69** (3.6)	$-$**0.81** ($-$1.8)	$-$**9.04** ($-$1.8)	$-$**4.46** ($-$2.1)	0.0038
y^r	**0.26** (1.9)	**0.83** (4.7)	$-$**0.74** ($-$1.8)	$-$6.65 ($-$1.4)	$-$1.56 ($-$0.8)	0.0039
Δp	$-$**0.01** ($-$2.3)	$-$**0.04** ($-$5.7)	0.02 (1.3)	0.08 (0.4)	$-$**0.13** ($-$1.8)	$-$0.0001
R_m	0.00 (0.2)	0.01 (0.6)	0.03 (1.1)	**0.67** (2.0)	**0.64** (4.4)	$-$0.0001
R_b	0.01 (0.6)	0.03 (1.5)	0.04 (0.8)	**0.92** (1.8)	**0.99** (4.6)	$-$0.0001
	(t-ratios in parentheses)					

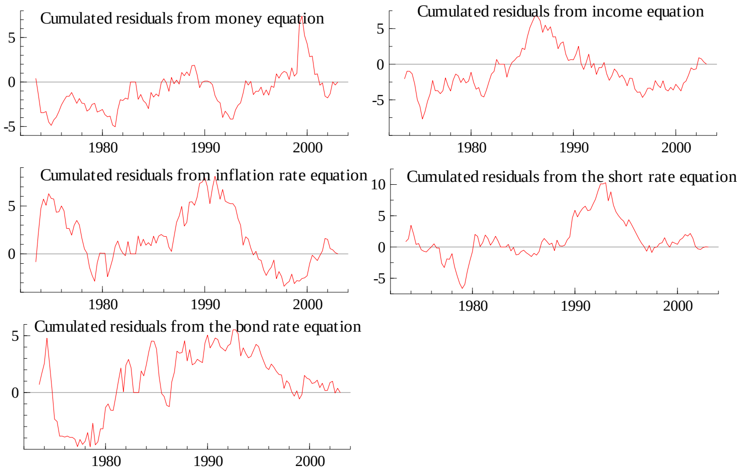

Figure 14.1. Cumulated residuals from each equation of the VAR system.

The two (unrestricted) common trends are defined by:

$$\begin{aligned} \sum\nolimits_{i=1}^{t} u_{1,i} &= \hat{\alpha}'_{\perp,1} \sum\nolimits_{i=1}^{t} \hat{\varepsilon}_i, \\ \sum\nolimits_{i=1}^{t} u_{2,i} &= \hat{\alpha}'_{\perp,2} \sum\nolimits_{i=1}^{t} \hat{\varepsilon}_i, \end{aligned}$$

where $\hat{\alpha}_{\perp,1}$ and $\hat{\alpha}_{\perp,2}$ are given by the just-identified estimates.

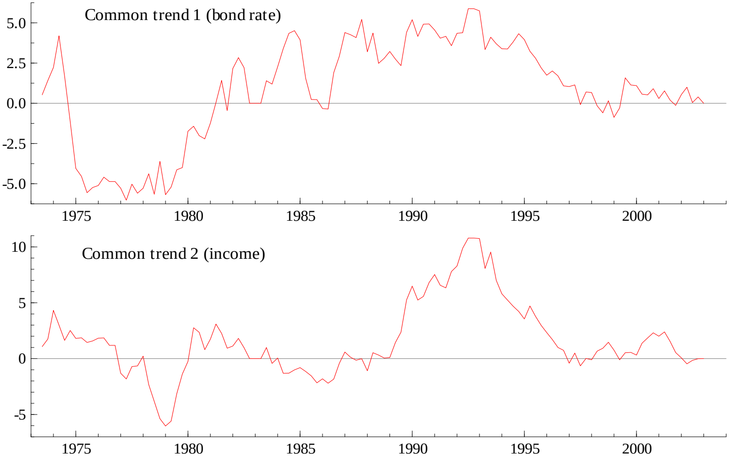

Figure 14.2. The common trends based on the estimates in Table 14.1.

Treating the bond rate as long-run exogenous

Under this assumption, the bond rate corresponds to a unit vector in $\alpha_\perp$:

$$\alpha'_\perp = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ * & 1 & * & * & 0 \end{bmatrix},$$

and one of the two common trends is now identified as $\sum_{i=1}^{t} \hat{\varepsilon}_{Rb,i}$. The previous (0, 1) restriction for $\hat{\varepsilon}_{y^r,i}$ and the (1, 0) restriction for $\hat{\varepsilon}_{R_b,i}$ are just identifying and the remaining three restrictions are overidentifying. These restrictions correspond to the three degrees of freedom of the weak exogeneity test of the bond rate

Is the bond rate a common stochastic trend in this case?

Long-run (weak) exogeneity of a variable implies that its *cumulated residuals* are a common stochastic trend.

It does not imply that the *variable* itself is a common trend. For this to be the case the variable need to be strongly exogenous and the rows of the `$_i$ matrices associated with the exogenous variable have to be zero.

For example, if $\Delta x_{j,t} = \varepsilon_{j,t}$ then $x_{j,t} = \sum_{i=1}^{t} \varepsilon_{j,i}$ and the common stochastic trend coincides with the variable itself.

Both the bond rate and the real income variable exhibited significant effects from lagged changes of the vector process, so neither of them satisfies the condition for being strongly exogenous.

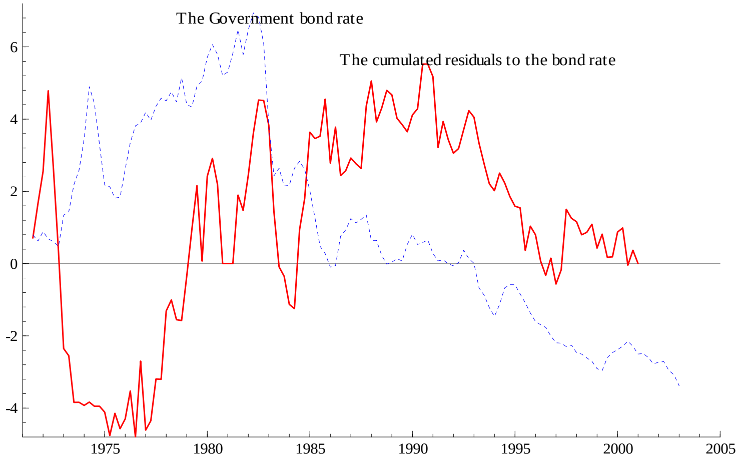

Figure 14.3. The cumulated residuals from the bond rate equation compared to the bond rate.

Table: The MA representation when the bond rate is assumed weakly exogenous

	$\hat{\varepsilon}_{m^r}$	$\hat{\varepsilon}_{y^r}$	$\hat{\varepsilon}_{\Delta p}$	$\hat{\varepsilon}_{R_m}$	$\hat{\varepsilon}_{R_b}$
$\hat{\sigma}_{\varepsilon_i}$	0.0231	0.0138	0.0080	0.0010	0.0014
$\hat{\alpha}'_{\perp,1}{}^{c}$	0.00	0.00	0.00	0.00	**1.00**
$\hat{\alpha}'_{\perp,2}{}^{c}$	0.33 [1.61]	**1.00**	-0.82 [-1.61]	-7.32 [-1.28]	0.00
	m^r	y^r	Δp	R_m	R_b
$\widetilde{\hat{\beta}}'_{\perp,1}{}^{c}$	**-9.42** [-5.21]	-2.45 [-1.40]	**-0.14** [-2.00]	**0.78** [7.30]	**1.48** [7.30]
$\widetilde{\hat{\beta}}'_{\perp,2}{}^{c}$	**0.53** [3.24]	**0.79** [5.00]	**-0.03** [-5.20]	-0.00 [-0.22]	0.03 [1.48]

Table: The MA representation when the bond rate is assumed weakly exogenous

	$\hat{\varepsilon}_{m^r}$	$\hat{\varepsilon}_{y^r}$	$\hat{\varepsilon}_{\Delta p}$	$\hat{\varepsilon}_{R_m}$	$\hat{\varepsilon}_{R_b}$	
$\hat{\sigma}_{\varepsilon_i}$	0.0231	0.0138	0.0080	0.0010	0.0014	
			The **C** *matrix*			
	$\hat{\varepsilon}_{m^r}$	$\hat{\varepsilon}_{y^r}$	$\hat{\varepsilon}_{\Delta p}$	$\hat{\varepsilon}_{R_m}$	$\hat{\varepsilon}_{R_b}$	$t \times \gamma_{0,i}$
m^r	0.18	**0.53**	−0.43	−3.88	−**9.42**	0.0033
	(1.3)	(3.2)	(−1.1)	(−0.7)	(−5.2)	
y^r	**0.26**	**0.79**	−0.65	−5.78	−2.45	0.0038
	(1.9)	(5.0)	(−1.6)	(−1.3)	(−1.4)	
Δp	−**0.01**	−**0.03**	0.03	0.25	−**0.14**	−0.0001
	(−2.0)	(−5.2)	(1.7)	(1.4)	(−2.0)	
R_m	−0.00	−0.00	0.00	0.01	**0.78**	−0.0001
	(−0.1)	(−0.2)	(0.1)	(0.1)	(7.3)	
R_b	0.00	0.03	−0.02	−0.20	**1.48**	−0.0000
	(0.6)	(1.5)	(−0.5)	(−0.4)	(7.3)	

t-ratios in parentheses

Assuming that both the bond rate and the real income are eogenous

The two zero row restrictions on α (which were rejected based on p-value of 0.02) imply six overidentifying restrictions on the common trends, $\alpha_{\perp}^{c\prime}\sum_{i=1}^{t}\hat{\varepsilon}_i$ with

$$\alpha_{\perp}^{c\prime}=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

In this case, the underlying exogenous shocks are $\hat{u}_{1,t}=\sum_{i=1}^{t}\hat{\varepsilon}_{y^r,i}$ and $\hat{u}_{2,t}=\sum_{i=1}^{t}\hat{\varepsilon}_{R_b,i}$.

Table: The MA representation when β is restricted to HS.4 and both the bond and the real income is assumed weakly exogenous

	$\hat{\varepsilon}_{m^r}$	$\hat{\varepsilon}_{y^r}$	$\hat{\varepsilon}_{\Delta p}$	$\hat{\varepsilon}_{R_m}$	$\hat{\varepsilon}_{R_b}$	
$\hat{\alpha}_{\perp,1}$	0	0	0	0	**1**	
$\hat{\alpha}_{\perp,2}$	0	**1**	0	0	0	
The C matrix						
	$\hat{\varepsilon}_{m^r}$	$\hat{\varepsilon}_{y^r}$ $(\widetilde{\hat{\beta}}_{\perp,2})$	$\hat{\varepsilon}_{\Delta p}$	$\hat{\varepsilon}_{R_m}$	$\hat{\varepsilon}_{R_b}$ $(\widetilde{\hat{\beta}}_{\perp,1})$	$t \times \gamma_{0,i}$
m^r	0	1.02 [4.17]	0	0	−10.12 [−3.65]	0.0031
y^r	0	1.14 [5.24]	0	0	−5.42 [−2.20]	0.0029
Δp	0	−0.03 [−5.24]	0	0	0.16 [2.20]	−0.0001
R_m	0	0.02 [1.32]	0	0	1.00 [4.72]	−0.0000
R_b	0	0.03 [1.32]	0	0	1.35 [4.72]	−0.0001