Tests of hypotheses on the β relations

A graduate course in the Cointegrated VAR model: Special topics in Rome

Katarina Juselius

University of Copenhagen

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Formulating hypotheses as restrictions on the beta vectors

Hypotheses on the cointegration vectors can be formulated in two alternative ways: either by specifying the s_i free parameters, or alternatively the m_i restrictions of each β_i vector. Constraining β_i by the design matrix H_i defining the s_i free parameters:

$$\beta = (\beta_1, ..., \beta_r) = (H_1\varphi_1, ..., H_r\varphi_r), \tag{1}$$

where φ_i is a $(s_i \times 1)$ coefficient matrix, H_i is a $(p1 \times s_i)$ design matrix, $p1$ is the dimension of $\tilde{x}_{t-1}$ in the VAR model, and $i = 1, ..., r$. In this case, the design matrices define the s_i free parameters in each cointegration vector.

Constraining β_i by the restriction matrices R_i $(p1 \times m_i)$ defining the $m_i = p - s_i$ restrictions on β_i:

$$\begin{aligned} R_1'\beta_1 &= 0 \\ &\vdots \\ R_r'\beta_r &= 0 \end{aligned}$$

Same restrictions on all beta vectors

Typical examples are (1) tests of long-run exclusion of a variable, i.e. a zero row restriction on β, (2) tests of long-run price homogeneity
These are testable hypothese but they are not identifying as they impose identical restrictions on all cointegration relations.
All H_i (or R_i), $i = 1, ...r$, are identical and we can formulate the hypothesis as:

$$H^c(r) : \beta^c = (H\varphi_1, ..., H\varphi_r) = H\varphi \tag{2}$$

where β^c is $p1 \times r$, H is $p1 \times s$, φ is $s \times r$ and s is the number of unrestricted coefficients in each vector, or alternatively as:

$$H^c(r) : R'\beta = 0$$

where R is $p1 \times m$ and m is the number of restrictions imposed on each vector.

The hypothesis $H^c(r)$ is tested against $H(r) : \beta$ unrestricted, i.e. we test the following restricted model:

$$\Delta x_t = \alpha \varphi' H' \tilde{x}_{t-1} + \sum_{i=1}^{k-1} \Gamma_i \Delta x_{t-i} + \varepsilon_t \tag{3}$$

Example: the test of long-run proportionality between m^r and y^r in all cointegration relations:

$$H' = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \ \boldsymbol{\varphi} = \begin{bmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ \varphi_{31} & \varphi_{32} & \varphi_{33} \\ \varphi_{41} & \varphi_{42} & \varphi_{43} \\ \varphi_{51} & \varphi_{52} & \varphi_{53} \end{bmatrix}$$

The transformed data vector becomes:

$$H' x_t = \begin{bmatrix} (m^r - y^r)_t \\ \Delta p_t \\ R_{m,t} \\ R_{b,t} \\ D_s 831_t \end{bmatrix}.$$

Illustrations

Example 1: A test of long-run exclusion of a linear trend in the cointegration relations for $\tilde{x}_t' = [m_t^r, y_t^r, \Delta p_t, R_{m,t}, R_{b,t}, D_s831_t, t]$.

$$\mathcal{H}_1 : \beta^c = H\varphi \text{ or } R'\beta = 0$$

where

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \varphi = \begin{bmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ \varphi_{31} & \varphi_{32} & \varphi_{33} \\ \varphi_{41} & \varphi_{42} & \varphi_{43} \\ \varphi_{51} & \varphi_{52} & \varphi_{53} \\ \varphi_{61} & \varphi_{62} & \varphi_{63} \end{bmatrix}$$

and

$$R' = [0, 0, 0, 0, 0, 0, 1].$$

Example 2. A test of long-run homogeneity between m^r and y^r in all cointegrating relations for $\tilde{x}_t' = [m_t^r, y_t^r, \Delta p_t, R_{m,t}, R_{b,t}, D_s831_t, t]$. The design matrices H and R have the following form:

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \; R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Table: Tests of same restriction on all cointegration relations

	m^r	y^r	Δp	R_m	R_b	D_s831	*trend*
				Unrestricted estimates			
β_1'	0.06	−0.03	1.00	−0.29	0.57	−0.002	0.000
β_2'	1.00	−1.02	−3.45	−8.51	8.00	−0.24	−0.001
β_3'	−0.00	0.02	0.02	1.00	−0.62	−0.01	−0.000
				$\mathcal{H}_1 : \beta_{7.} = 0,\ \chi^2(3) = 0.92[0.82]$			
$\beta_1^{c\prime}$	0.07	−0.03	1.00	−0.29	0.59	−0.002	**0.0**
$\beta_2^{c\prime}$	1.00	−1.22	−3.71	−10.39	8.82	−0.25	**0.0**
$\beta_3^{c\prime}$	−0.00	0.02	0.01	1.00	−0.64	−0.00	**0.0**
				$\mathcal{H}_2 : \beta_{6.} = 0,\ \chi^2(3) = 19.08[0.00]$			
$\beta_1^{c\prime}$	0.06	−0.03	1.00	−0.32	0.56	**0.0**	0.00
$\beta_2^{c\prime}$	1.00	−1.72	−3.41	−42.2	28.6	**0.0**	−0.00
$\beta_3^{c\prime}$	0.08	0.14	0.09	1.00	2.11	**0.0**	−0.00

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β_2'	1.00	−1.02	−3.45	−8.51	8.00	−0.24	−0.001
β_3'	−0.00	0.02	0.02	1.00	−0.62	−0.01	−0.000
			$\mathcal{H}_3 : \beta_{1.} = -\beta_{2.},\ \chi^2(3) = 3.36[0.34]$				
$\beta_1^{c\prime}$	**0.06**	**−0.06**	1.00	−0.57	0.59	−0.00	0.000
$\beta_2^{c\prime}$	**1.00**	**−1.00**	−3.46	−8.07	7.82	−0.25	−0.001
$\beta_3^{c\prime}$	**−0.01**	**0.01**	0.03	1.00	−0.76	−0.01	0.000
			$\mathcal{H}_4 : \beta_{4.} = -\beta_{5.},\ \chi^2(3) = 4.90[0.18]$				
$\beta_1^{c\prime}$	0.06	−0.05	1.00	**−0.64**	**0.64**	−0.00	0.000
$\beta_2^{c\prime}$	1.00	−1.00	−3.45	**−7.61**	**7.61**	−0.25	−0.001
$\beta_3^{c\prime}$	−0.02	0.02	0.03	**1.00**	**−1.00**	−0.01	0.000

Table: Tests of same restriction on all cointegration relations

	m^r	y^r	Δp	R_m	R_b	D_s831	$trend$
				Unrestricted estimates			
β_1'	0.06	-0.03	1.00	-0.29	0.57	-0.002	0.000
β_2'	1.00	-1.02	-3.45	-8.51	8.00	-0.24	-0.001
β_3'	-0.00	0.02	0.02	1.00	-0.62	-0.01	-0.000
	$\mathcal{H}_5 : \beta_{1.} = -\beta_{2.}$ and $\beta_{7.} = 0,\ \chi^2(6) = 9.36[0.15]$						
$\beta_1^{c\prime}$	**0.08**	**$-$0.08**	1.00	-1.30	0.91	0.001	**0.0**
$\beta_2^{c\prime}$	**1.00**	**$-$1.00**	-4.56	-1.81	5.21	-0.315	**0.0**
$\beta_3^{c\prime}$	**$-$0.01**	**0.01**	0.07	1.00	-0.87	-0.004	**0.0**
	$\mathcal{H}_6 : \beta_{1.} = -\beta_{2.}, \beta_{4.} = -\beta_{5.}$ and $\beta_{7.} = 0,\ \chi^2(9) = 16.52[0.05]$						
$\beta_1^{c\prime}$	**0.12**	**$-$0.12**	1.00	**$-$0.89**	**0.89**	-0.008	**0.0**
$\beta_2^{c\prime}$	**1.00**	**$-$1.00**	-7.27	**$-$11.36**	**11.36**	-0.323	**0.0**
$\beta_3^{c\prime}$	**$-$0.02**	**0.02**	-0.03	**1.00**	**$-$1.00**	-0.009	**0.0**

Some beta vectors are known

This test is useful when we want to test whether a hypothetical known vector is stationary. For example, we might be interested in whether the real interest rate defined as $R - \Delta p$ is stationary, whether Δp is stationary by itself, and whether the income velocity of money, $m^r - y^r$ is stationary. To formulate this hypothesis, it is convenient to decompose the r cointegrating relations into n_k known vectors b (in most cases $n_k = 1$) and $r - n_k$ unrestricted vectors φ:

$$\mathcal{H}^c(r) : \beta^c = (b, \varphi), \tag{4}$$

where b is a $p1 \times n_k$, and φ is a $p1 \times (r - n_k)$ vector. We partition

$$\alpha = (\alpha_1, \alpha_2), \tag{5}$$

where α_1 are the adjustment coefficients to b, and α_2 to φ. The cointegrated VAR model can now be written as:

$$\Delta x_t = \alpha_1 b' x_{t-1} + \alpha_2 \varphi' x_{t-1} + \Gamma_1 \Delta x_{t-1} + \Phi D_t + \varepsilon_t. \tag{6}$$

Illustrations

We would like to know whether the inflation rate, nominal and real interest rates, and the interest rate spread are stationary by themselves. Note that the null is stationarity in this case. For example, the test that $\Delta p_t \sim I(0)$ is formulated as:

$$\mathcal{H}_6 : \beta = (b, \varphi),$$

where

$$b' = \left[\begin{array}{cccccc} 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

i.e., b is a unit vector that picks up the inflation rate. The remaining r-1 $=$ 2 vectors are unrestricted and described by the matrix φ of dimension $p1 \times r - 1 = 6 \times 2$. The coefficients φ_{ij} are uniquely determined based on the ordering of eigenvalues and the normalization $\varphi' S_{11.\mathbf{b}} \varphi = I$.

Table: Testing the stationarity of single relations

	m_t^r	y_t^r	Δp_t	$R_{m,t}$	$R_{b,t}$	$D_s 831_t$	$\chi^2(v)$	*p-value*
				Tests of a known β vector				
$\mathcal{H}_7$	0	0	1	0	0	0	20.9 (3)	0.00
$\mathcal{H}_8$	0	0	0	1	0	0	21.1(3)	0.00
$\mathcal{H}_9$	0	0	0	0	1	0	24.2(3)	0.00
$\mathcal{H}_{10}$	0	0	1	-1	0	0	19.4 (3)	0.00
$\mathcal{H}_{11}$	0	0	1	0	-1	0	17.8 (3)	0.00
$\mathcal{H}_{12}$	0	0	0	1	-1	0	24.9 (3)	0.00

When testing the stationarity of a variable there are two important caveats:

1. The test results are not invariant to the choice of rank r^*. If a conservative value of r is chosen (a small r^*) then stationarity will be more difficult to accept than for a choice of r larger than r^*. Therefore, the test results are crucially dependent on the specific value of cointegration rank being chosen. For a given r^*, we can then ask whether any of the variables is a unit vector in the cointegration space. Thus if, instead, we had chosen $r = 4$, the strong rejection of stationarity of all six hypotheses might have been reversed.
2. If we have included deterministic variables, for example shift dummies, in the cointegration space then a more appropriate hypothesis might be stationarity when allowing for a shift in the mean. Thus, the strong rejection of stationarity might also be related to the shift dummy D_s831_t. For example, if the interest rate spread is stationary around one level before 1983 and another level after that date, then $\mathcal{H}_{12}$ would probably be rejected as a consequence of imposing a zero restriction on D_s831_t. In this case, it would be more relevant to ask whether $(R_m - R_b - b_1 D_s831)_t \sim I(0)$ rather than $(R_m - R_b)_t \sim I(0)$, where $b_1 D_s831_t$ is the estimated shift in the level

Testing stationarity when some, but not all, coefficients are known

In the general case, we formulate the hypothesis as $\beta = \{H_1\varphi_1, H_2\varphi_2, ..., H_r\varphi_r\}$, where H_i is a $(p1 \times s_i)$ matrix, $i = 1, .., r$. In this case the cointegration structure needs to be identified. If instead we would like to focus on just a few cointegration relation we divide the r cointegration relations into two groups containing r_1 and r_2 vectors each. We will here focus on the special case $\beta = \{H_1\varphi_1, \psi\}$, where H_1 is $p1 \times s_1$, φ_1 is $s_1 \times 1$ and ψ is $p1 \times r - 1$ and formulate the hypothesis: the null hypothesis:

$$\mathcal{H}_0 : \beta = (\beta_1, \beta_2) = (H_1\varphi_1, \psi). \tag{7}$$

As before, we partition α so that it corresponds to the partitioning of β:

$$\alpha = (\alpha_1, \alpha_2).$$

The model can be written as:

Table: Testing the stationarity of single relations

	m_t^r	y_t^r	Δp_t	$R_{m,t}$	$R_{b,t}$	D_s831_t	$\chi^2(v)$	*p-value*
				Tests of liquidity ratio relations				
$\mathcal{H}_{13}$	1	-1	0	0	0	-0.34	1.5 (2)	**0.47**
$\mathcal{H}_{14}$	1	-1	1.66	0	0	-0.30	1.3 (1)	**0.25**
$\mathcal{H}_{15}$	1	-1	0	-9.02	9.02	-0.21	0.2 (1)	**0.64**
$\mathcal{H}_{16}$	1	-1	1.39	-8.92	8.92	-0.19	–	–
				Test of real income relations				
$\mathcal{H}_{17}$	0	1	34.1	0	0	0.43	0.8 (1)	**0.36**
$\mathcal{H}_{18}$	0	1	-54.1	54.1	0	-0.82	8.5 (1)	0.00
$\mathcal{H}_{19}$	0	1	-44.7	0	44.7	0.10	12.7(1)	0.00
$\mathcal{H}_{20}$	0	1	0	-161	161	2.0	3.5 (1)	0.06

Table: Testing the stationarity of single relations

	m_t^r	y_t^r	Δp_t	$R_{m,t}$	$R_{b,t}$	D_s831_t	$\chi^2(v)$	p-value
				Tests of inflation, real interest rates and the spread				
$\mathcal{H}_{21}$	0	0	1	0	0	0.020	9.0 (2)	0.01
$\mathcal{H}_{22}$	0	0	1	-1	0	0.011	10.7 (2)	0.00
$\mathcal{H}_{23}$	0	0	1	0	-1	-0.009	14.3 (2)	0.00
$\mathcal{H}_{24}$	0	0	0	1	-1	-0.014	4.2 (2)	**0.12**
				Tests of combinations of interest rates and inflation rates				
$\mathcal{H}_{25}$	0	0	1	-0.44	0	0.015	4.9 (1)	0.03
$\mathcal{H}_{26}$	0	0	1	0	-0.29	0.013	6.7 (1)	0.01
$\mathcal{H}_{27}$	0	0	0	1	-0.81	-0.009	1.7 (1)	**0.19**
				Tests of homogeneity between inflation and the interest rates				
$\mathcal{H}_{28}$	0	0	-0.30	1	-0.70	-0.012	0.02 (1)	**0.90**
$\mathcal{H}_{29}$	0	0	0.12	1	-1	-0.012	4.1 (1)	0.04

Do the result support the theory consistent CVAR scenario?

$$
\begin{aligned}
&(m - p - y^r)_t \sim I(0), \\
&(R_b - R_m)_t \sim I(0), \\
&(R_m - \Delta p)_t \sim I(0), \\
&(R_b - \Delta p)_t \sim I(0).
\end{aligned}
$$

How should we revise the theory in the light of the empirical results?