Distributional Time Series

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17 March 2016

References

Main Contents: Theory

- Park and Qian (2012), Functional Regression of Continuous State Distributions, *Journal of Econometrics*, 167, 397-412.
- Chang, Kim and Park (2016), Nonstationarity in Time Series of State Densities, *Journal of Econometrics*, 192, 152-167.
- Chang, Kim and Park (2015), Common Trends in Time Series of Cross Sectional Distributions.
- Hu, Park and Qian (2015), Functional Autoregressive Model for Time Series of State Distributions.
- Chang, Hu and Park (2016), On the Error Correction Model for Functional Time Series with Unit Roots.

References

Main Contents: Applications

- Chang, Kim, Miller, Park and Park (2015), Time Series Analysis of Global Temperature Distributions: Identifying and Estimating Persistent Features in Temperature Anomalies.
- Chang, Hu and Park (2015), A Study of Distributional Income Dynamics.
- Chang, Hong, Kim and Park (2015), An Empirical Analysis of World Income Distributions.

Background Material

- Bosq (2000), Linear Processes in Function Spaces.

Outline

I. Basic Framework

II. Distributional Autoregression

III. Distributional Unit Roots

IV. Distributional Cointegration

I. Basic Framework

Objective

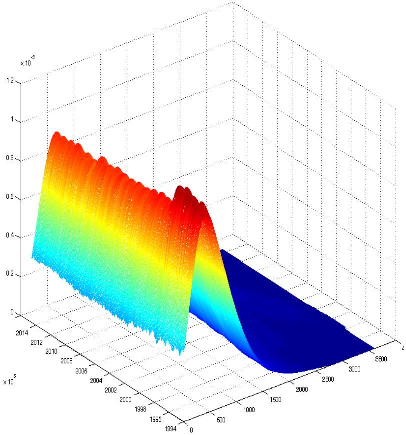
Develop a new framework and methodology to analyze the time series of cross-sectional distributions such as

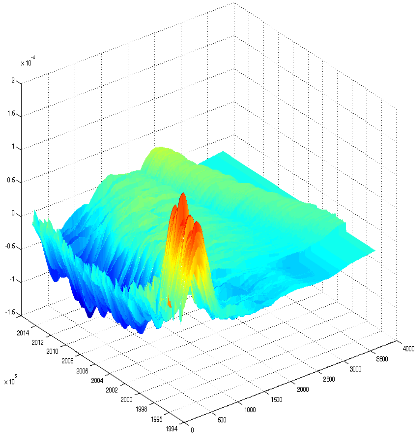
- individual earnings
- household income and expenditures
- NYSE stock returns
- global temperatures

or the time series of intra-period distributions such as

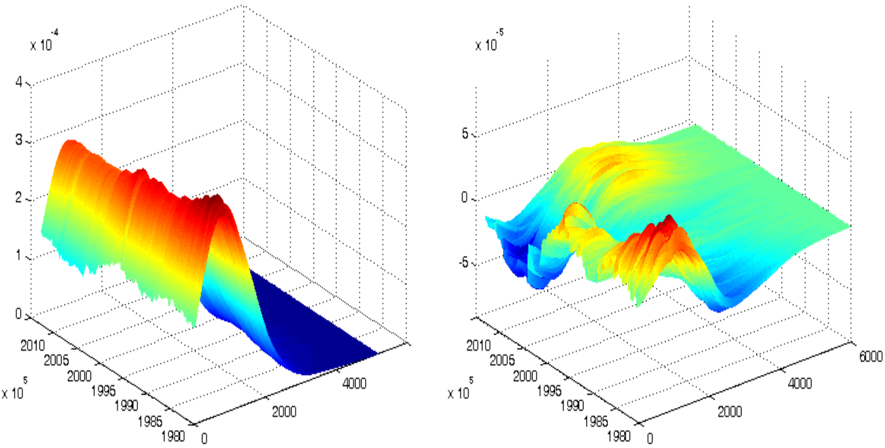
- stock returns
- exchange rate returns

Distributions of Individual Earnings

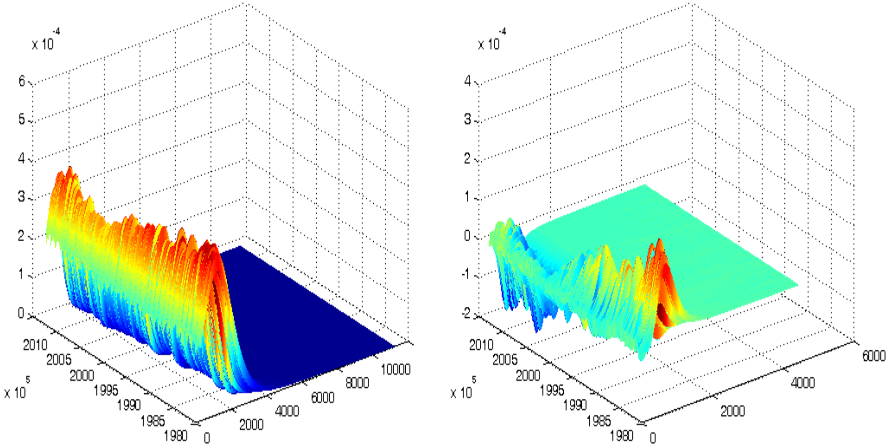




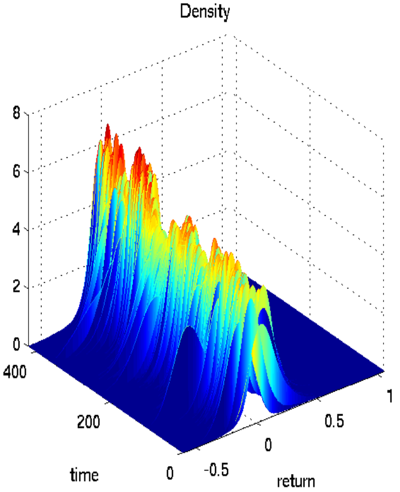
Distributions of Household Income

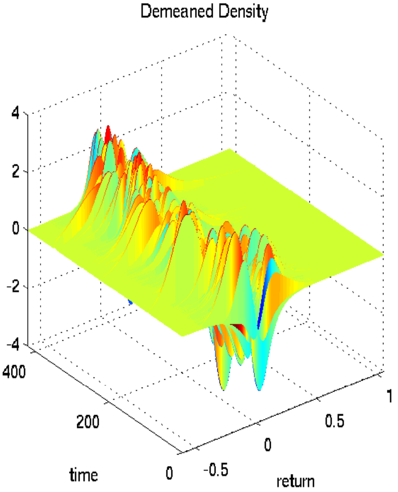


Distributions of Household Expenditures

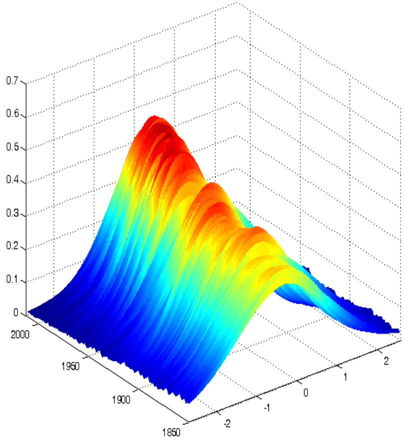


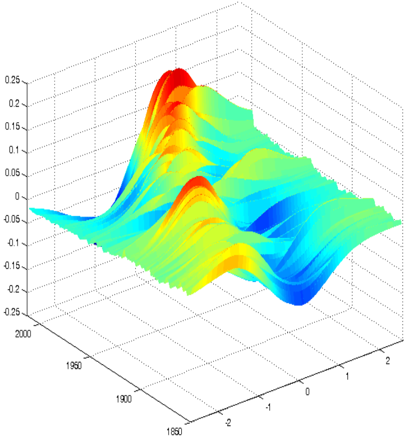
Distributions of NYSE Stock Returns



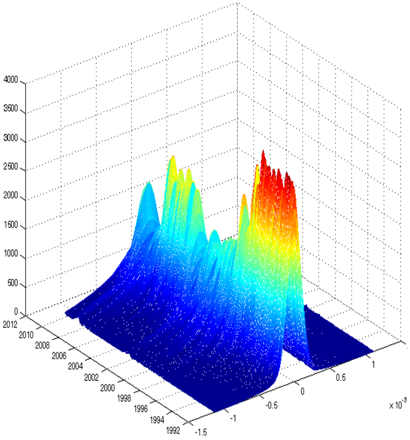


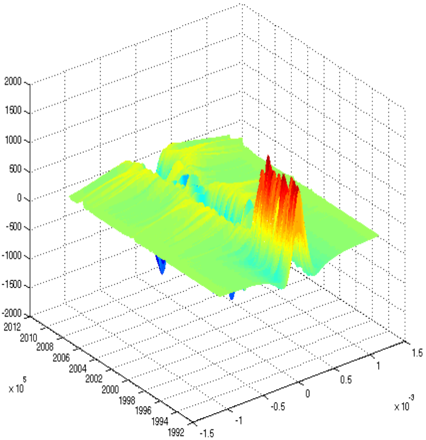
Global Temperature Distributions



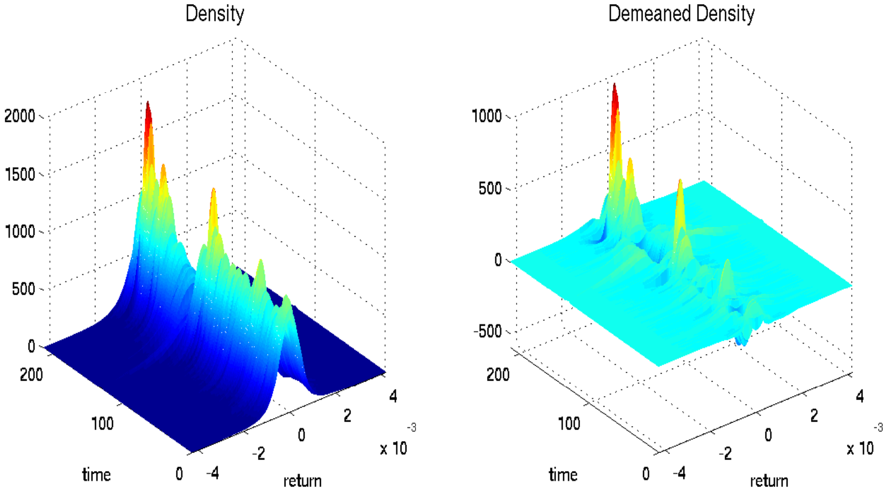


Intra-Month Distributions of S&P 500 Returns





Intra-Month Distributions of GBP/USD Ex Returns



Technical Background

Hilbert-Valued Random Variables

Let

$$w : \ \Omega \to H$$

where H is a Hilbert space.

Hilbert-valued random variables include

- Real random variables: $H = \mathbb{R}$
- Vector-valued random variables: $H = \mathbb{R}^N$
- Function-valued random variables: $H = L^2(\mathbb{R})$

as special cases.

Mean and Variance Operator

The mean $\mathbb{E}w$ of a random variable in H is defined as a vector in H satisfying

$$\langle v, \mathbb{E}w \rangle = \mathbb{E}\langle v, w \rangle$$

for all $v \in H$, which exists if $\mathbb{E}\|w\| < \infty$.

For w such that $\mathbb{E}w = 0$, the variance $\mathbb{E}(w \otimes w)$ of w is given by an operator for which

$$\mathbb{E}\langle u, w \rangle \langle w, v \rangle = \langle u, \mathbb{E}(w \otimes w) v \rangle$$

for all $u, v \in H$, which exists if $\mathbb{E}\|w\|^2 < \infty$.

- For a finite dimensional w, $w \otimes w$ reduces to ww', and $\mathbb{E}(w \otimes w)$ reduces to $\mathbb{E}ww'$.
- For an operator A with its adjoint A^*, we may easily deduce that $\mathbb{E}(Aw \otimes Aw) = A\big[\mathbb{E}(w \otimes w)\big]A^*$.

Model for Functional Data

For each time $t = 1, 2, \ldots$, suppose there is a distribution represented by a probability density f_t, whose value at ordinate $x \in \mathbb{R}$ is denoted by $f_t(x)$.

Denote by

$$w_t = f_t - \mathbb{E} f_t$$

a demeaned density function and treat w_t as functional data taking values in Hilbert space H.

We define H to be the set of functions on a compact subset K of $\mathbb{R}$ that have vanishing integrals and are square integrable, i.e.,

$$H = \left\{ v \,\middle|\, \int_K v(x)dx = 0, \int_K v^2(x)dx < \infty \right\}$$

with inner product $\langle u, v \rangle = \int_K u(x)v(x)dx$ for $u, v \in H$.

Moment and Coordinate Process

For a random variable w taking values in H, we define its v-moment as

$$\langle v, w \rangle,$$

which reduces to the usual k-th moment if we choose $v = \iota_k$ with $\iota_k(x) = x^k$ normalized properly so that $\iota_k \in H$.

Since H is separable, we may write (w_t) as

$$w_t = \sum_{i=1}^{\infty} \langle v_i, w_t \rangle v_i$$

for each t, where (v_i) is an orthonormal basis of H. In this context, we call

$$\langle v_i, w_t \rangle$$

the i-th coordinate process.

Coordinate Time Series

In general, time series properties of coordinate processes are different on different coordinates.

- (w_t) is stationary if $\big(\langle v, w_t\rangle\big)$ is stationary for all $v \in H$. Mean reversion in all directions. Deviates from mean only temporarily, and randomly fluctuates around the mean in all directions.
- (w_t) has a unit root in the direction of v if $\big(\langle v, w_t\rangle\big)$ is a unit root process. Persistent, and non mean reverting due to the presence a stochastic trend with no mean reversion in the direction of v.
- (w_t) is explosive in the direction of v if $\big(\langle v, w_t\rangle\big)$ has an explosive root. No mean reversion in the direction of v.

We provide a mathematical framework to more explicitly identify and analyze the unit root and cointegration directions in the function space of state densities.

II. Distributional Autoregression

FAR(1) Model

FAR(1) model can be represented as

$$\begin{aligned} w_t &= A w_{t-1} + \varepsilon_t \\ &= \sum_{i=1}^{\infty} \lambda_i (u_i \otimes v_i)(w_{t-1}) + \varepsilon_t \\ &= \sum_{i=1}^{\infty} \lambda_i \langle v_i, w_{t-1} \rangle u_i + \varepsilon_t, \end{aligned}$$

and we call

- v_i's progressive features
- u_i's regressive features

respectively.

Estimation

Though (f_t) are not directly observable, we may consistently estimate them from cross-sectional or intra-period observations. If the size N of cross-sectional or intra-period observations is large enough relative to the time span T, the use of estimated densities will not affect our analysis asymptotically.

We let

$$P = \mathbb{E}(w_t \otimes w_{t-1}) \quad \text{and} \quad Q = \mathbb{E}(w_t \otimes w_t),$$

which are estimated respectively by

$$\hat{P} = \frac{1}{T}\sum_{t=1}^{T}(w_t \otimes w_{t-1}) \quad \text{and} \quad \hat{Q} = \frac{1}{T}\sum_{t=1}^{T}(w_t \otimes w_t)$$

from the sample (w_t) of size T.

Estimation Strategy

We should not estimate A by

$$A = PQ^{-1},$$

since $A = PQ^{-1}$ is defined only on $\mathcal{R}(Q) \subsetneq H$ and we have an ill-posed inverse problem.

We use the spectral representation $Q = \sum_{i=1}^{\infty} \lambda_i (v_i \otimes v_i)$ with $\lambda_1 > \lambda_2 > \cdots$, and approximate Q^{-1} by

$$Q_K^{+} = \sum_{i=1}^{K} \frac{1}{\lambda_i} (v_i \otimes v_i)$$

and define

$$A_K = PQ_K^{+}.$$

which we may estimate using $\hat{P}$ and $\hat{Q}$.

Orthonormal Moment Basis

We define a basis (ι_κ°) of H such that

- ι_κ° is a κ-th order polynomial
- (ι_κ°) is an orthonormal basis of H with respect to the inner product $\langle \cdot, Q \cdot \rangle$

We call such a basis an orthonormal moment basis. An orthonormal moment basis of H may be obtained through the Gram-Schmidt orthogonalization process.

Response Function to Moment Basis

From

$$\begin{aligned}\langle \iota_\kappa^\circ, w_t\rangle &= \langle \iota_\kappa^\circ, Aw_{t-1}\rangle + \langle \iota_\kappa^\circ, \varepsilon_t\rangle \\ &= \langle A^*\iota_\kappa^\circ, w_{t-1}\rangle + \langle \iota_\kappa^\circ, \varepsilon_t\rangle,\end{aligned}$$

we define

$$A^*\iota_\kappa^\circ$$

to be the response function for the κ-th moment of (w_t).

Moment Dynamics of State Distributions

It follows from $\langle v, w_t \rangle = \langle v, Aw_{t-1} \rangle + \langle v, \varepsilon_t \rangle$ that

$$\begin{aligned}\mathbb{E}\langle v, w_t \rangle^2 &= \mathbb{E}\langle A^* v, w_{t-1} \rangle^2 + \mathbb{E}\langle v, \varepsilon_t \rangle^2 \\ &= \sum_{\kappa=1}^{\infty} \langle v, AQ\iota_\kappa^\circ \rangle^2 \mathbb{E}\langle \iota_\kappa^\circ, w_{t-1} \rangle^2 + \mathbb{E}\langle v, \varepsilon_t \rangle^2.\end{aligned}$$

We define

$$R_v^2 = 1 - \frac{\mathbb{E}\langle v, \varepsilon_t \rangle^2}{\mathbb{E}\langle v, w_t \rangle^2} = 1 - \frac{\langle v, \Sigma v \rangle}{\langle v, Qv \rangle}$$

and

$$\pi_v(\kappa) = \frac{\langle v, AQ\iota_\kappa^\circ \rangle^2}{\mathbb{E}\langle v, w_t \rangle^2} = \frac{\langle v, AQ\iota_\kappa^\circ \rangle^2}{\langle v, Qv \rangle},$$

which is the proportion of variance in $\langle v, w_t \rangle$ that comes from the variance of the past κ-th moment.

Empirical Illustrations

Intra-Month GBP/USD Ex Returns

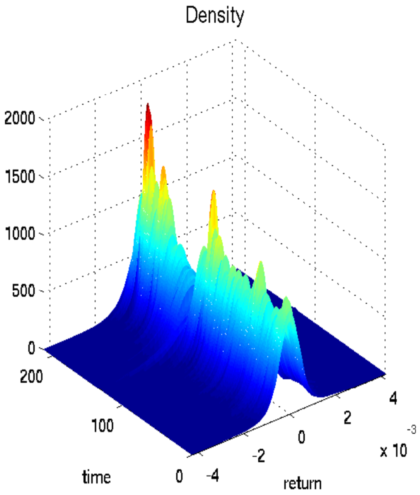
Data Description

- 15-minute log returns of the UK Pound/US Dollar exchange rate
- Jan 1999 - April 2015
- Every 4 weeks as a period, 212 periods
- The number of observations each period is $1550 \sim 1904$ (mean 1880)

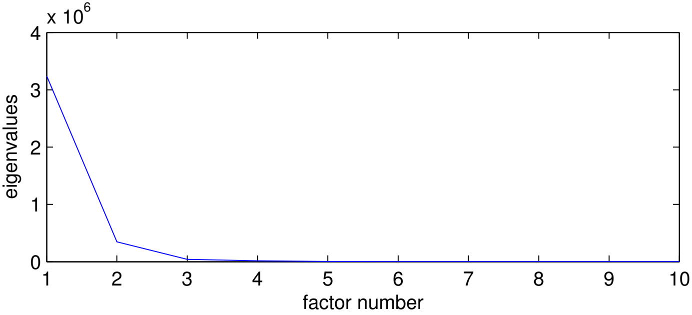
Densities are estimated by the kernel method using

- Support $[-0.0043, 0.0043]$
- Epanechnikov kernel
- Optimal feasible bandwidth given by $h_t = 2.3449\hat{\sigma}_t N_t^{-1/5}$
- Represent density with Daubechies wavelets using 1037 basis functions

Data Plot

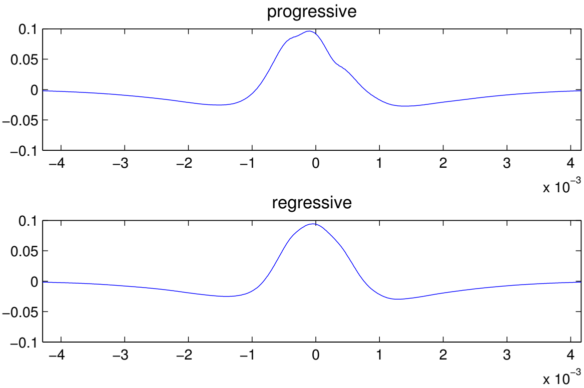

Density
2000
1500
1000
500
0
200
100
time
0
-4
-2
0
2
4
return
x 10^-3
Demeaned Density
1000
500
0
-500
200
100
time
0
-4
-2
0
2
4
return
x 10^-3

Scree Plot



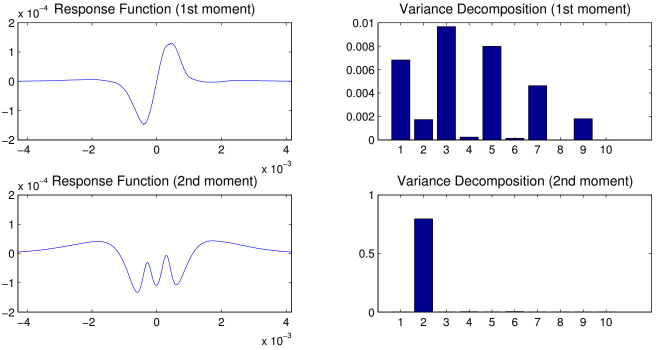
We set $K = 4$ to get the best prediction performance, and the first 4 principal components explain 99.7% of variance in density process.

Progressive and Regressive Features



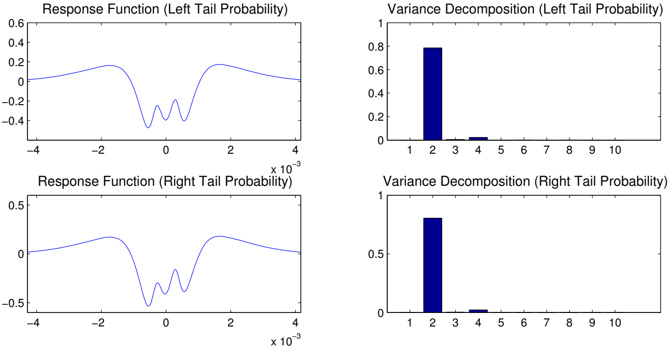
Our principal progressive and regressive features show that normal returns near the origin play most important roles both progressively and regressively. Tail returns do not generate any major dynamics neither in the forward nor in the backward.

Dynamic Analysis in Moments



The response functions and the variance decompositions for the first two moments of GBP/USD exchange rate log returns.

Dynamic Analysis in Tail Probabilities



The response functions and the variance decompositions for the tail probabilities of GBP/USD exchange rate log returns

NYSE Stock Returns

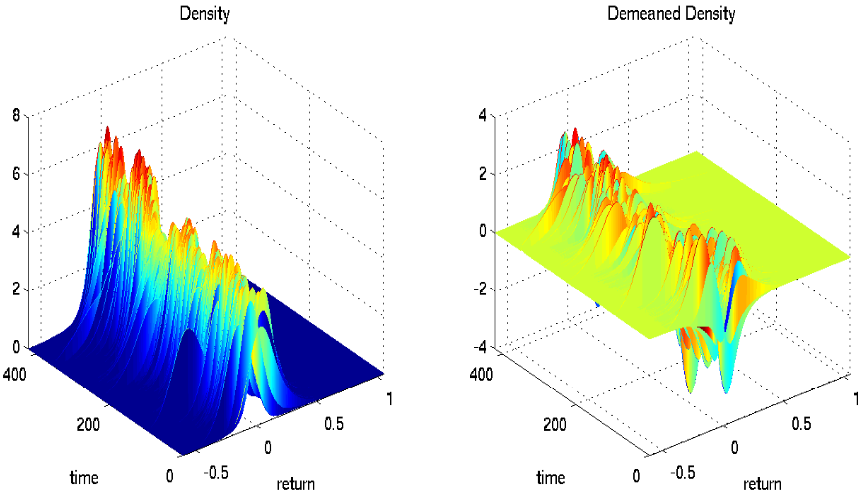
Data Description

- Monthly returns of stocks listed on NYSE
- Jan 1980 - Dec 2014
- One month as a period, 420 periods
- The number of observations each period is $1926 \sim 3076$ (mean 2464)

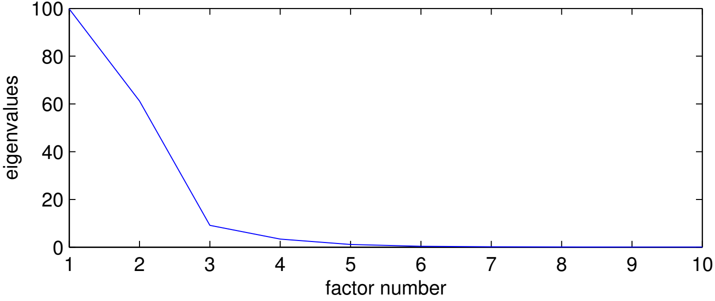
Densities are estimated by the kernel method using

- Support $[-0.6071, 0.1.0548]$
- Epanechnikov kernel
- Optimal feasible bandwidth given by $h_t = 2.3449\hat{\sigma}_t N_t^{-1/5}$
- Represent density with Daubechies wavelets using 1037 basis functions

Data Plot

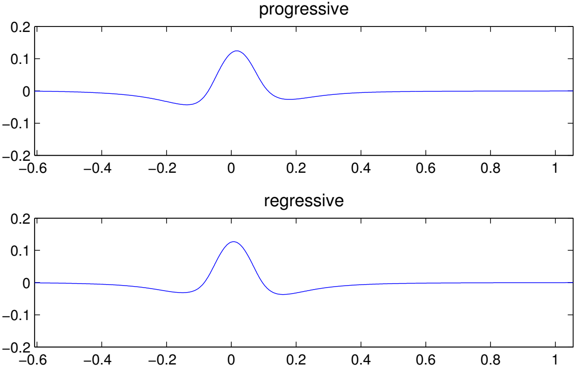

Density
8
6
4
2
0
400
200
time
0
-0.5
0
0.5
1
return
Demeaned Density
4
2
0
-2
-4
400
200
time
0
-0.5
0
0.5
1
return

Scree Plot



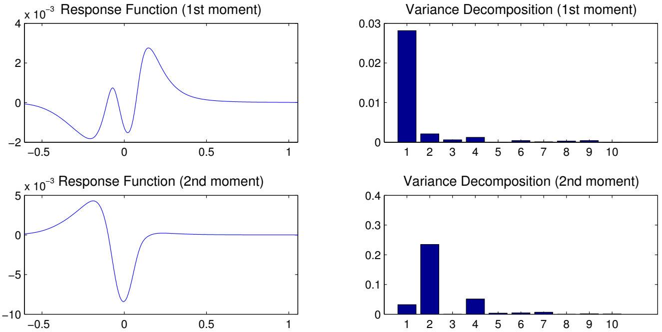
We set $K = 3$ to get the best prediction performance, and the first 3 principal components explain 97% of variance in density process.

Progressive and Regressive Features



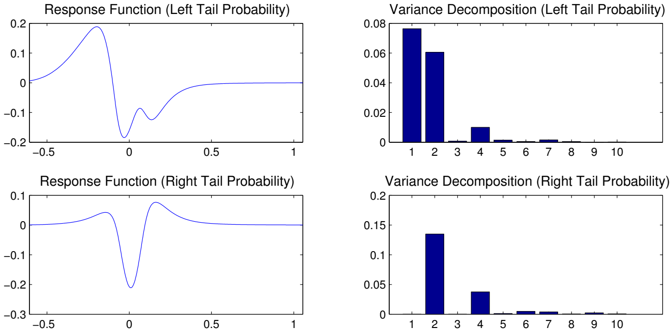
Again, our principal progressive and regressive features show that normal returns near the origin play most important roles both progressively and regressively. Tail returns do not generate any major dynamics neither in the forward nor in the backward.

Dynamic Analysis in Moments



The response functions and the variance decompositions for the first two moments of the NYSE stocks monthly returns.

Dynamic Analysis in Tail Probabilities



The response functions and the variance decompositions for the tail probabilities of the NYSE stocks monthly returns.

II. Distributional Unit Roots

Unit Root and Stationarity Subspaces

Using the symbol $\bigvee$ to denote span, we let

$$H_N = \bigvee_{i=1}^{n} v_i \quad \text{and} \quad H_S = \bigvee_{i=n+1}^{\infty} v_i$$

so that $H = H_N \oplus H_S$. In what follows, H_N and H_S will respectively be referred to as the unit root and stationarity subspaces of H.

We also let Π_N and Π_S be the projections on H_N and H_S, respectively. Moreover, we define

$$w_t^N = \Pi_N w_t \quad \text{and} \quad w_t^S = \Pi_S w_t$$

Note that $\Pi_N + \Pi_S = 1$ (the identity operator on H), so in particular we have

$$w_t = w_t^N + w_t^S$$

Unit Root and Stationary Processes

When $u_t = \Delta w_t = \Phi(L)\varepsilon_t$, it follows that

$$w_t^N = \Pi_N w_t = \Pi_N \Phi(1) \sum_{i=1}^{t} \varepsilon_i - \Pi_N \bar{u}_t$$

and

$$w_t^S = \Pi_S w_t = -\Pi_S \bar{u}_t$$

Clearly, (w_t^N) is an integrated process, while (w_t^S) is stationary.

The unit root dimension n is unknown in practical applications.

We will explain how to

- Determine n statistically
- Estimate the subspaces H_S and H_N

Sample Variance Operator

Our test for unit roots in (w_t) is based on the sample variance operator

$$M^T = \sum_{t=1}^{T} w_t \otimes w_t,$$

whose quadratic form is given by

$$\langle v, M^T v \rangle = \sum_{t=1}^{T} \langle v, w_t \rangle^2$$

for $v \in H$.

Asymptotic behavior of the quadratic form of sample variance operator depends crucially on whether v is in H_N or in H_S.

Stationarity-Nonstationarity of Coordinate Processes

For $v \in H_S$, the coordinate process $(\langle v, w_t \rangle)$ becomes stationary and we expect that

$$T^{-1} \sum_{t=1}^{T} \langle v, w_t \rangle^2 \to_p \mathbb{E} \langle v, w_t \rangle^2$$

as long as the expectation exists.

On the other hand, if $v \in H_N$ and the coordinate process $(\langle v, w_t \rangle)$ is integrated, it follows under a very mild condition that

$$T^{-2} \sum_{t=1}^{T} \langle v, w_t \rangle^2 \to_d \int_0^1 V(r)^2 dr - \left(\int_0^1 V(r) dr \right)^2,$$

where V is a Brownian motion.

Therefore, the quadratic form has different orders of magnitude, i.e., $O_p(T)$ and $O_p(T^2)$, depending upon whether the coordinate process $(\langle v, w_t \rangle)$ is stationary or integrated.

Nonstationarity and Stationarity Subspaces

We let H_N be n-dimensional.

Denote by $v_1^T, v_2^T, \ldots$ the orthonormal eigenvectors of the sample variance operator M^T.

It is shown that

$$v_i^T \to_p v_i$$

for $i = 1, 2, \ldots$, as $T \to \infty$.

Estimation of Nonstationarity Subspace

Once we determine the number of unit roots n in (w_t), we may estimate the nonstationarity subspace H_N by

$$H_N^T = \bigvee_{i=1}^{n} v_i^T,$$

i.e., the span of the n orthonormal eigenvectors of the sample variance operator M^T associated with n largest eigenvalues of M_T.

Recall

$$H_N = \bigvee_{i=1}^{n} v_i \quad \text{and} \quad H_S = \bigvee_{i=n+1}^{\infty} v_i.$$

We establish the consistency of H_N^T for H_N.

Functional Principal Component Analysis

If we define $\lambda_1^T \geq \lambda_2^T \geq \cdots$ to be the eigenvalues of M^T associated with the eigenvectors $v_1^T, v_2^T, \ldots$, then we have

$$\lambda_i^T = \langle v_i^T, M^T v_i^T \rangle = \sum_{t=1}^{T} \langle v_i^T, w_t \rangle^2$$

for $i = 1, 2, \ldots$.

Therefore, it follows that

$$\lambda_i^T = \begin{cases} O_p(T^2) & \text{for } i = 1, \ldots, n \\ O_p(T) & \text{for } i = n+1, \ldots \end{cases},$$

Onto Testing for Distributional Unit Roots

To determine the number of unit roots in (w_t), we consider the test of the null hypothesis

$$\mathsf{H}_0 : \ \text{dim}\,(H_N) = n$$

against the alternative hypothesis

$$\mathsf{H}_1 : \ \text{dim}\,(H_N) \leq n - 1$$

successively downward.

More precisely, we start testing the null with $n = n_{\max}$, where $n_{\max}$ is large enough so that dim $(H_N) \leq n_{\max}$.

Continue with $n = n_{\max} - 1$ if the null is rejected in favor of the alternative. If, for any n, dim $(H_N) \leq n$ and the null is not rejected, then we may conclude that dim $(H_N) = n$.

Therefore, we may estimate the number of unit roots in (w_t) by the smallest value of n for which we fail to reject the null.

Intuitive but Infeasible Test

We expect that the eigenvalue λ_n^T would have a discriminatory power for the test of null against the alternative, since it has different orders of stochastic magnitudes under the null and alternative hypotheses.

However, it cannot be used directly as a test statistic, since its limit distribution is dependent upon nuisance parameters.

Therefore, we need to modify it appropriately to get rid of its nuisance parameter dependency problem.

A Feasible Test for Unit Root Dimension

To introduce our test, define (z_t^T) for $t = 1, \ldots, T$ by

$$z_t^T = (\langle v_1^T, w_t \rangle, \ldots, \langle v_n^T, w_t \rangle)'$$

Also define the product sample moment $M_n^T = \sum_{t=1}^T z_t^T z_t^{T\prime}$ (sample variance in the unit root subspace), and the long-run variance estimator $\Omega_n^T = \sum_{|k| \leq \ell} \varpi_\ell(k) \Gamma_T(k)$ of (z_t^T), where ϖ_ℓ is the weight function with bandwidth parameter ℓ and Γ_T is the sample autocovariance function defined as
$\Gamma_T(k) = T^{-1} \sum_t \Delta z_t^T \Delta z_{t-k}^{T\prime}$.

Our test statistic is defined as

$$\tau_n^T = T^{-2} \lambda_{\min} \left(M_n^T, \Omega_n^T \right),$$

where $\lambda_{\min} \left(M_n^T, \Omega_n^T \right)$ is the smallest generalized eigenvalue of M_n^T with respect to Ω_n^T.

Asymptotics for Distributional Unit Root Test

Under very general conditions, we show that

$$\tau_n^T \to_d \lambda_{\min}\left(\int_0^1 W_n(r)W_n(r)'dr - \int_0^1 W_n(r)dr\int_0^1 W_n(r)'dr\right)$$

under the null, as $T \to \infty$, where W_n is n-dimensional standard vector Brownian motion and $\lambda_{\min}(\cdot)$ denotes the smallest eigenvalue of its matrix argument.

On the other hand, we have $\tau_n^T \to_p 0$ under the alternative as $T \to \infty$.

Therefore, we reject the null in favor of the alternative if the test statistic τ_n^T takes small values.

Critical Values for Distributional Unit Root Test τ_n^T

Critical values for the tests are obtained based on τ_n^T for $n = 1, \ldots, 5$, by simulations.

For simulations, BM is approximated by standardized partial sum of mean zero i.i.d. normal random variates with sample size 10,000, and actual critical values are computed using 100,000 iterations.

n	1	2	3	4	5
1%	0.0274	0.0175	0.0118	0.0103	0.0085
5%	0.0385	0.0223	0.0154	0.0127	0.0101
10%	0.0478	0.0267	0.0175	0.0139	0.0111

Degree of Persistency in Moments

We may now find how much nonstationarity proportion exists in each cross-sectional moment.

In what follows, we redefine ι_κ as $\iota_\kappa - \frac{1}{|K|}\int_K \iota_\kappa(x)dx$, so that we may regard it as an element in H.

We may decompose ι_κ as $\iota_\kappa = \Pi_N \iota_\kappa + \Pi_S \iota_\kappa$, from which it follows that

$$\|\iota_\kappa\|^2 = \|\Pi_N \iota_\kappa\|^2 + \|\Pi_S \iota_\kappa\|^2 = \sum_{i=1}^{n} \langle \iota_\kappa, v_i \rangle^2 + \sum_{i=n+1}^{\infty} \langle \iota_\kappa, v_i \rangle^2,$$

where (v_i), $i = 1, 2, \ldots$, is an orthonormal basis of H such that $(v_i)_{1 \le i \le n}$ and $(v_i)_{i \ge n+1}$ span H_N and H_S, respectively.

Nonstationarity Proportion in Moments

To measure the proportion of ι_κ lying in H_N, we define

$$\pi_\kappa = \frac{\|\Pi_N \iota_\kappa\|}{\|\iota_\kappa\|} = \sqrt{\frac{\sum_{i=1}^{n} \langle \iota_\kappa, v_i \rangle^2}{\sum_{i=1}^{\infty} \langle \iota_\kappa, v_i \rangle^2}}.$$

If ι_κ is entirely in H_N and H_S, we have $\pi_\kappa = 1$ and $\pi_\kappa = 0$, respectively,

Therefore, we may use π_κ to represent the proportion of nonstationary component in the κ-th cross-sectional moment of (w_t).

The κ-th cross-sectional moment of (w_t) has more dominant unit root component as π_κ tends to unity, whereas it becomes more stationary as π_κ approaches to zero.

Sample Nonstationarity Proportion

The nonstationarity proportion π_κ of the κ-th cross-sectional moment is not directly applicable, since H_N and H_S are unknown.

However, we may use its sample version

$$\pi_\kappa^T = \sqrt{\frac{\sum_{i=1}^{n} \langle \iota_\kappa, v_i^T \rangle^2}{\sum_{i=1}^{T} \langle \iota_\kappa, v_i^T \rangle^2}}.$$

The sample version π_κ^T of π_κ will be referred to as the *sample* nonstationarity proportion of the κ-th cross-sectional moment of (w_t).

We show that the sample version π_κ^T is a consistent estimator for the original π_κ.

Empirical Illustrations

Overview

We demonstrate how to define and estimate the state densities, and test for unit roots in the time series of densities representing cross-sectional or intra-period distributions of economic variables.

State densities are estimated by standard kernel density estimation method on cross-sectional or intra-period observations, and their nonstationarities are analyzed using the test $\hat{\tau}_T^n$.

Unit root dimension n of state densities is determined by applying $\hat{\tau}_T^n$ successively downward starting from $n = n_{\max}$.

Unit root space H_N is then estimated and the unit root proportion (π_κ) is computed for the first several moments. π_κ provides the proportion of nonstationary fluctuation in the κ-th moment of the state distribution.

Representation of Functions as Numerical Vectors

For the representation of infinite dimensional functions in Hilbert space as finite dimensional numerical vectors, we use a Daubechies wavelet basis.

Wavelets are two dimensional arrays in location and resolutions, and hence they provide more flexibilities in fitting the state densities in our applications, some of which have severe asymmetry and time-varying support. The wavelet basis in general yields a much better fit than the trigonometric basis.

The Daubechies wavelet is implemented with 1037 basis functions.

Cross-Sectional Distributions of Individual Earnings

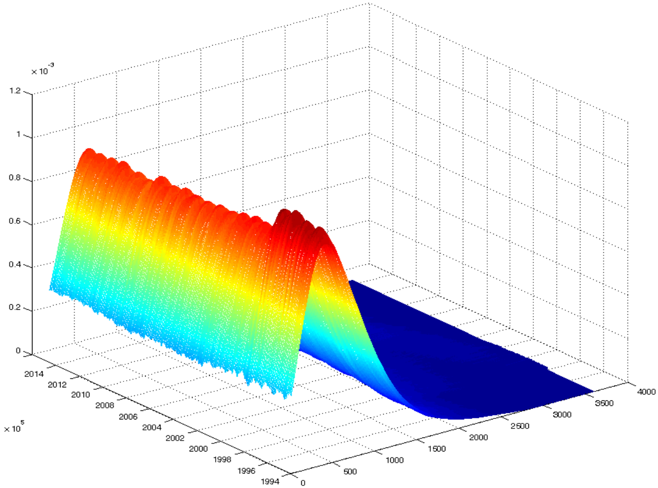
Distributions of Individual Earnings

The cross-sectional observations of individual weekly earnings are obtained at monthly frequency from Current Population Survey (CPS) data set. The individual weekly earnings are deflated by consumer price index with base year 2005.

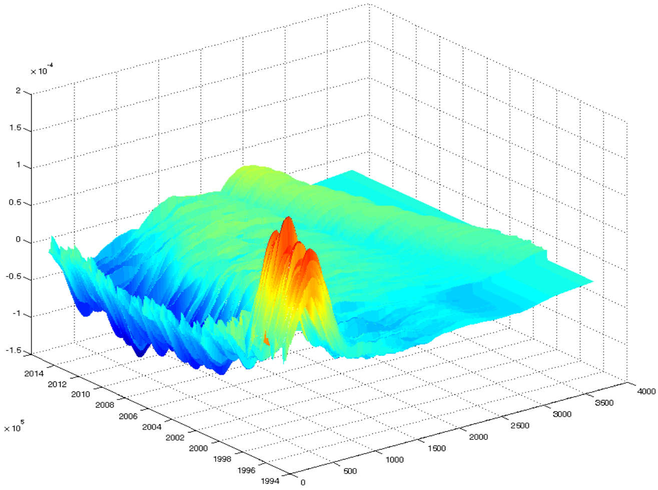
The data set provides 247 time series observations spanning from January 1994 to July 2014, and the number of cross-sectional observations for each month ranges from 12,180 (April 1996) to 15,826 (October 2001).

For confidentiality reasons, individual earnings are topcoded above a certain level. Top code value was revised in 1998 up to $2,885 from $1,923. We drop all topcoded individual earnings as well as zero earnings as in Liu (2011) and Shin and Solon (2011).

Densities of Weekly Individual Earnings



Demeaned Densities of Weekly Individual Earnings



Unit Root Dimension - Individual Earnings

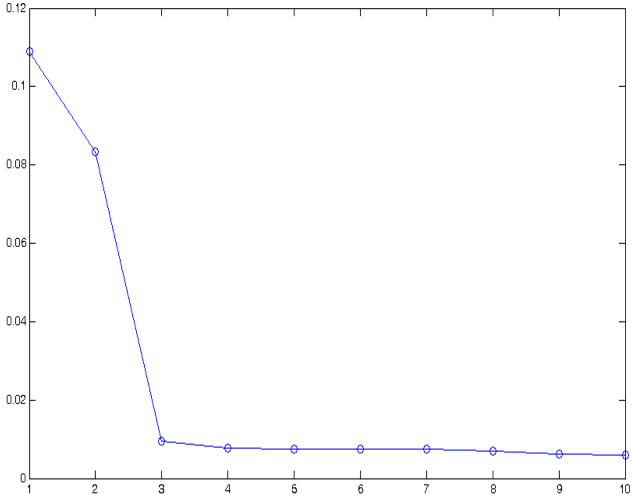
To determine the unit root dimension n in the time series of cross-sectional distributions of individual earnings, we use the statistic $\hat{\tau}_n^T$ to test for the null hypothesis $\mathsf{H}_0 : \dim(H_N) = n$ against the alternative $\mathsf{H}_1 : \dim(H_N) \leq n - 1$ with $n = 1, \ldots, 5$.

M	1	2	3	4	5
$\hat{\tau}_n^T$	0.1090	0.0834	0.0094	0.0078	0.0075

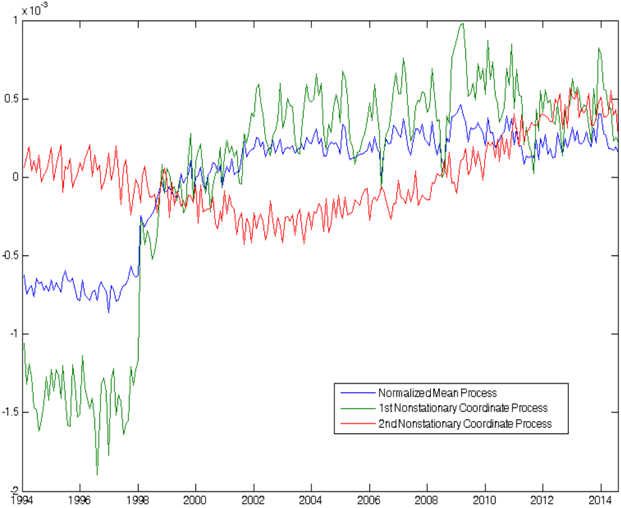
Our test, strongly and unambiguously, rejects H_0 against H_1 successively for $n = 5, 4, 3$. Clearly, however, the test cannot reject H_0 in favor of H_1 for $n = 2$.

We conclude that there exists two-dimensional unit root, and set $\hat{n}_T = 2$.

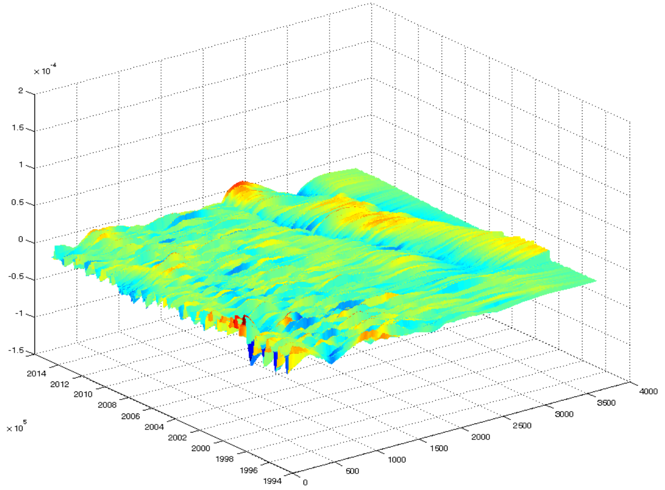
Scree Plot of Eigenvalues - Individual Earnings

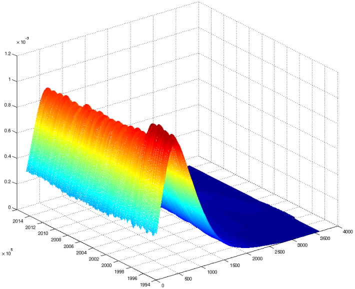


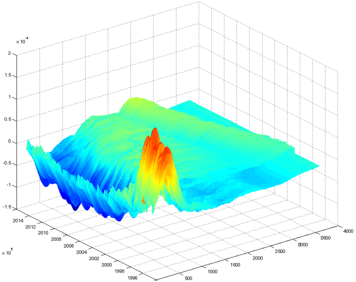
Integrated Coordinate Processes - Individual Earnings

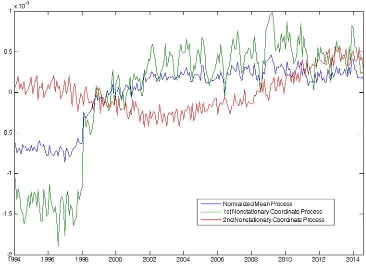


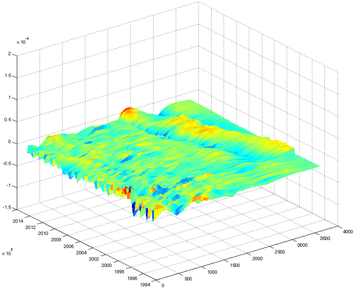
Stationary Distributions - Individual Earnings








Normalized Mean Process
1st Nonstationary Coordinate Process
2nd Nonstationary Coordinate Process



UR Proportions in Moments - Individual Earnings

We compute the estimates $\hat{\pi}_{\kappa}^{T}$ of the unit root proportions π_{κ} with $\hat{n}_T = 2$ for the first four moments.

$\hat{\pi}_1^T$	$\hat{\pi}_2^T$	$\hat{\pi}_3^T$	$\hat{\pi}_4^T$
0.5280	0.3388	0.2377	0.1822

The unit root proportions for the first four moments are all nonnegligibly large. In particular, the unit root proportions for the first two moments are quite substantial.

The presence of a substantial unit root proportion in the second moment explains the recent empirical findings on changes in volatilities of individual earnings. Dynan *et al* (2008) and others.

Nonstationarity in time series of individual earnings distributions would certainly make their volatilities more persistent.

Intra-month Distributions of Stock Returns

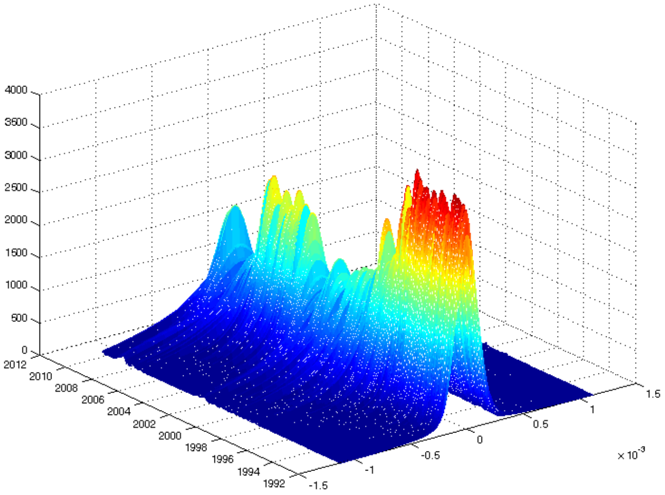
Intra-month S&P 500 Return Distributions

For each month during January 1992 to June 2010, we use S&P 500 index returns at one-minute frequency to estimate 222 densities for the intra-month distributions. The one-minute returns of S&P 500 index are obtained from Tick Data Inc. The number of intra-month observations varies from 7211 to 9177, except for September 2001, for which we only have 5982 observations.

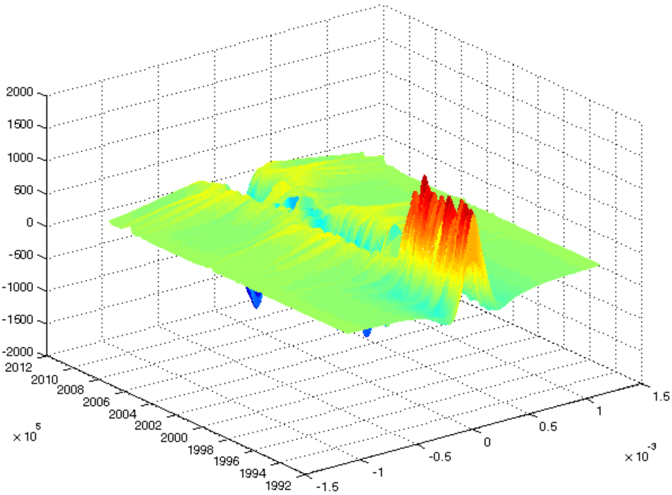
The intra-month observations are truncated at 0.50% and 99.5% percentiles before we estimate the state densities.

To avoid micro-structure noise, we also use the five-minute observations to estimate the intra-month observations. Our empirical results are, however, virtually unchanged.

Intra-month S&P 500 Returns



Demeaned Intra-Month S&P 500 Returns



Unit Root Dimension - S&P 500 Returns

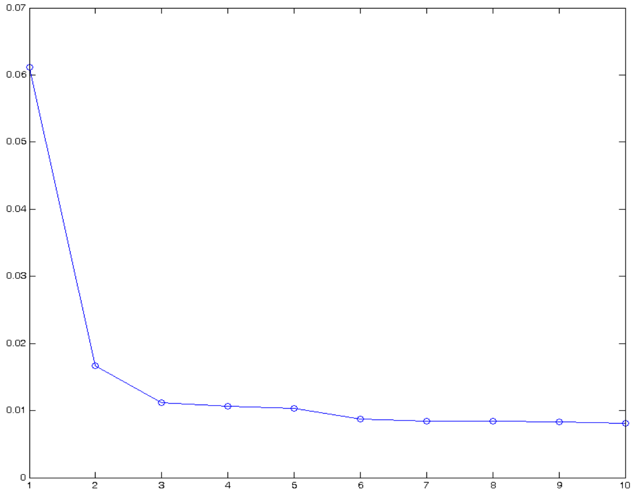
To test for existence of nonstationarity in time series of intra-month S&P 500 return distributions, we use $\hat{\tau}_n^T$ to test $\mathrm{H}_0 : \dim(H_N) = n$ against $\mathrm{H}_1 : \dim(H_N) \leq n - 1$ with $n = 1, \ldots, 5$.

M	1	2	3	4	5
$\hat{\tau}_n^T$	0.0612	0.0167	0.0112	0.0107	0.00104

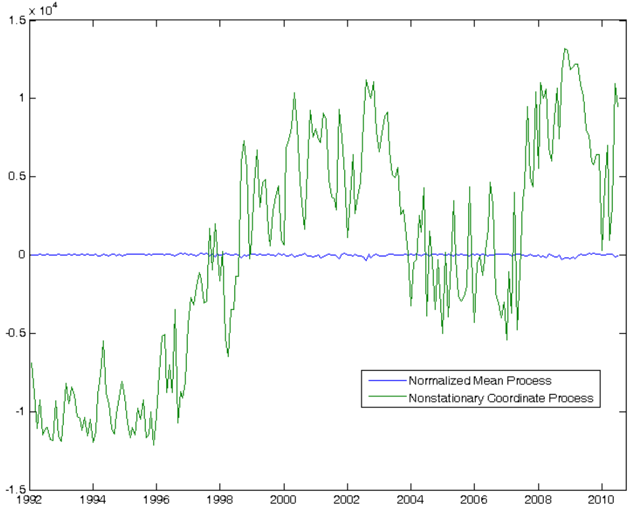
Our test successively rejects H_0 against H_1 for $n = 5, 4, 3, 2$.

However, at 5% level, the test cannot reject H_0 in favor of H_1 for $n = 1$. Our test result implies that there exists one-dimensional unit root, i.e., $\hat{n}_T = 1$.

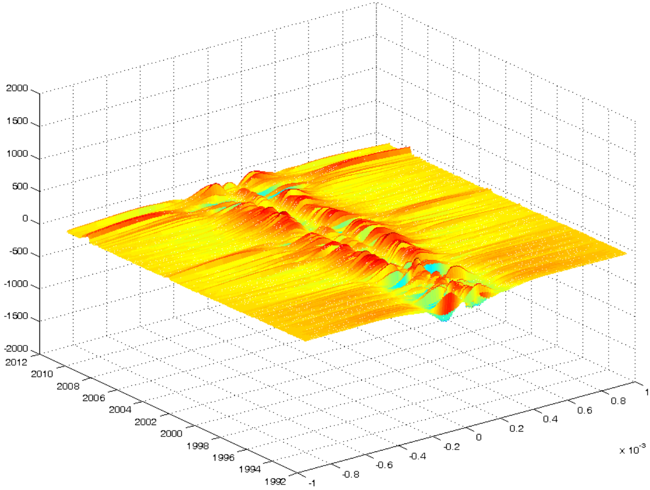
Scree Plot of Eigenvalues - S&P 500 Returns

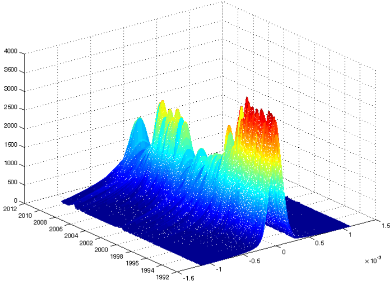


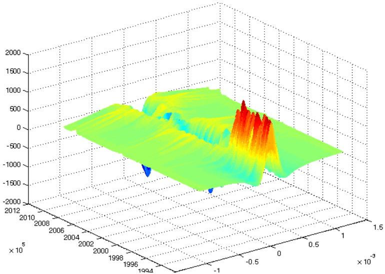
Integrated Coordinate Processes - S&P 500 Returns

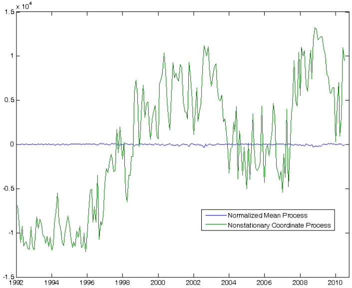


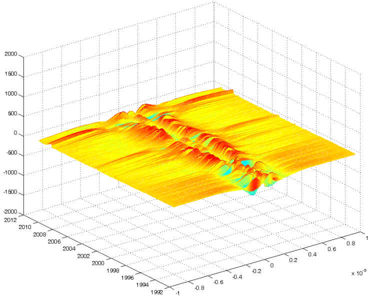
Stationary Components - S&P 500 Returns




4000
3500
3000
2500
2000
1500
1000
500
0
2012
2010
2008
2006
2004
2002
2000
1998
1996
1994
1992
-1.5
-1
-0.5
0
0.5
1
1.5
x 10^-3


2000
1500
1000
500
0
-500
-1000
-1500
-2000
x 10^5
2012
2010
2008
2006
2004
2002
2000
1998
1996
1994
-1
-0.5
0
0.5
1
1.5
x 10^-3


x 10^4
1.5
1
0.5
0
-0.5
-1
-1.5
1992
1994
1996
1998
2000
2002
2004
2006
2008
2010
Normalized Mean Process
Nonstationary Coordinate Process


2000
1500
1000
500
0
-500
-1000
-1500
-2000
2012
2010
2008
2006
2004
2002
2000
1998
1996
1994
1992
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
x 10^-3

UR Proportions in Moments - S&P 500 Returns

Compute the estimates $\hat{\pi}_\kappa^T$ of the unit root proportions π_κ for the first four moments, with $\hat{n}_T = 1$.

$\hat{\pi}_1^T$	$\hat{\pi}_2^T$	$\hat{\pi}_3^T$	$\hat{\pi}_4^T$
0.0047	0.2087	0.0039	0.0958

The nonstationarity is more concentrated in the second and fourth moments, with the unit root proportion of the second moment being the largest.

The unit root proportion of the first and third moments are almost negligible. This is well expected, since for many financial time series strong persistency is observed mainly in volatility and kurtosis.

III. Distributional Cointegration

Common Trends in Distributional Time Series

Introduce the notion of distributional cointegration between two time series of densities representing cross-sectional distributions of some economic variables

Explain how to estimate and test for such cointegrating relationships.

A New Framework

To analyze time series of densities representing cross-sectional distributions allowing for unit root type of nonstationarity

To analyze possible cointegration among cross-sectional distributions

To learn and interpret both longrun and shortrun relationships between two time series of cross-sectional distributions

Model and Methodology

Distributional Time Series

Let (f_t) and (g_t) be two time series of densities representing cross-sectional distributions of some economic variables, which we call distributional time series for short.

We regard the densities (f_t) and (g_t) as random elements taking values on the Hilbert space H of square integrable functions on $\mathbb{R}$.

For the main application in the paper, we designate (f_t) and (g_t) respectively to be the monthly time series of densities for income and consumption distributions. They are of course not directly observable and should be estimated using cross-sectional observations on household income and consumption.

However, to present our framework and methodology more effectively, we tentatively assume that they are observable.

Coordinate Processes

For the time series of densities (f_t) and (g_t), we define

$$\big(\langle v, f_t\rangle\big) \quad \text{and} \quad \big(\langle w, g_t\rangle\big)$$

to be the coordinate processes of (f_t) and (g_t) respectively in the directions of v and w for any $v, w \in H$.

Cross-Sectional Moments

The coordinate processes of (f_t) and (g_t) in the direction of ι_κ, where

$$\iota_\kappa(s) = s^\kappa,$$

are particularly important, since we have

$$\langle \iota_\kappa, f_t \rangle = \int s^\kappa f_t(s) ds \quad \text{and} \quad \langle \iota_\kappa, g_t \rangle = \int s^\kappa g_t(s) ds,$$

which represent the κ-th moments of the distributions represented by f_t and g_t for each $t = 1, \ldots, T$.

They will be referred subsequently to as the κ-th cross-sectional moments of (f_t) and (g_t) respectively.

Distributional Regression

We consider the distributional regression

$$g_t = \mu + Af_t + e_t$$

for $t = 1, \ldots, T$, where regressand and regressor are time series of densities for cross-sectional distributions, μ and A are function and operator parameters, and (e_t) is a function-valued error process.

Operator A generalizes regression coefficient in finite-dimensional regression, and may be called the regression operator.

We allow for nonstationarity in both (f_t) and (g_t). In particular, we let some of their coordinate processes $(\langle v, f_t \rangle)$ and $(\langle w, g_t \rangle)$ have unit roots and cointegration, which will be referred to as the distributional unit roots and cointegration.

We assume that (e_t) is stationary and mean zero, i.e., $\mathbb{E}e_t = 0$ for all $t = 1, \ldots, T$, and impose some exogeneity condition for (f_t).

Coordinate Regression

Coordinate regression of (g_t) in any direction $w \in H$ can be readily obtained from our distributional regression as

$$\begin{aligned}\langle w, g_t \rangle &= \langle w, \mu \rangle + \langle w, Af_t \rangle + \langle w, e_t \rangle \\ &= \langle w, \mu \rangle + \langle A^* w, f_t \rangle + \langle w, e_t \rangle\end{aligned}$$

for any $w \in H$, where A^* is the adjoint operator of A and $t = 1, \ldots, T$.

Represents a relationship between particular coordinate processes of (g_t) and (f_t).

May be interpreted as the usual bivariate regression of the coordinate process $(\langle w, g_t \rangle)$ of (g_t) on the coordinate process $(\langle v, f_t \rangle)$ of (f_t) with $v = A^* w$ for any $w \in H$.

Reveals the effect of the distribution represented by (f_t) on the coordinate process $(\langle w, g_t \rangle)$ of distribution (g_t) for $w \in H$.

More on Coordinate Regression

The coordinate regression of (g_t) in any direction $w \in H$ is given as

$$\langle w, g_t \rangle = \langle w, \mu \rangle + \langle A^* w, f_t \rangle + \langle w, e_t \rangle$$

The effect of the distribution represented by (f_t) on the coordinate process $(\langle w, g_t \rangle)$ is summarized by $v = A^* w$, which we call the response function of (f_t) to the coordinate process $(\langle w, g_t \rangle)$.

If we set $w = \iota_\kappa$, the coordinate regression reveals how the κ-th cross- moment of (g_t) is affected by the distribution represented by (f_t), and the response function $v = A^* w = A^* \iota_\kappa$ measures the effect of (f_t) on the κ-th cross-sectional moments of (g_t).

We analyze the coordinate regression separately for stationary and nonstationary components of (f_t) and (g_t).

Regression in a Demeaned Form

We may consider the dist regression in a demeaned form as

$$y_t = Ax_t + \varepsilon_t,$$

where

$$x_t = f_t - \frac{1}{T}\sum_{t=1}^{T} f_t, \qquad y_t = g_t - \frac{1}{T}\sum_{t=1}^{T} g_t$$

and $\varepsilon_t = e_t - T^{-1}\sum_{t=1}^{T} e_t$ for $t = 1, \ldots, T$.

Note that $\varepsilon_t \approx e_t - \mathbb{E}e_t = e_t$ for large T, since we assume that (e_t) is stationary and has mean zero.

However, in general, (x_t) and (y_t) do not behave the same as $(f_t - \mathbb{E}f_t)$ and $(g_t - \mathbb{E}g_t)$ even asymptotically, since (f_t) and (g_t) are nonstationary.

We mainly deal with the demeaned densities (x_t) and (y_t) in our statistical analysis.

Demeaned Densities and Moment Functions

We assume that the densities (f_t) and (g_t) all have supports included in a compact subset K of $\mathbb{R}$, for $t = 1, \ldots, T$.

Then the demeaned densities (x_t) and (y_t) take values in

$$L_0^2(K) = \left\{ w \in H \,\middle|\, \int_K w(s)ds = 0, \int_K w^2(s)ds < \infty \right\},$$

which is a subspace of the Hilbert space $L^2(\mathbb{R})$ of square integrable functions on $\mathbb{R}$ endowed with the usual inner product.

The moment functions ι_κ are redefined as

$$\iota_\kappa(s) = s^\kappa - \frac{1}{|K|} \int_K s^\kappa ds,$$

where $|K|$ denotes the length of K, so that they belong to $L_0^2(K)$.

For all our actual computations, we use an approximate one-to-one correspondence between $L_0^2(K)$ and $\mathbb{R}^M$ for some large M using a Wavelet basis in $L_0^2(K)$.

Stationarity and Nonstationarity Subspaces

We allow for nonstationarity in (f_t) and (g_t). More precisely, the coordinate processes $(\langle v, f_t\rangle)$ and $(\langle w, g_t\rangle)$ are allowed to have unit roots in the directions of some v and w for $v, w \in H$.

Stationarity subspaces F_S and G_S of (f_t) and (g_t) are defined as the subspaces of H defined as

$$F_S = \{v \in H \big| \langle v, f_t\rangle \text{ is stationary}\}$$
$$G_S = \{w \in H \big| \langle w, g_t\rangle \text{ is stationary}\},$$

Nonstationarity subspaces F_N and G_N of (f_t) and (g_t) are defined as orthogonal complements of F_S and G_S, so that $H = F_N \oplus F_S = G_N \oplus G_S$.

We only consider the unit root type nonstationarity in (f_t) and (g_t), and therefore the time series $(\langle v, f_t\rangle)$ and $(\langle w, g_t\rangle)$ are unit root processes for all $v \in F_N$ and $w \in G_N$.

Distributional Cointegration

If (f_t) and (g_t) have the unit root type nonstationarity, it is natural to consider the possibility that some of their coordinate processes are cointegrated.

That is, for some $v \in F_N$ and $w \in G_N$, we may have

$$\langle w, g_t \rangle = \pi + \langle v, f_t \rangle + u_t$$

with some constant π, where (u_t) is a general stationary process with mean zero.

Distributional Cointegrating Function

Assume F_N and G_N are p- and q-dimensional and there are p- and q-unit roots in (f_t) and (g_t), respectively.

Therefore, we have $v_1, \ldots, v_p$ and $w_1, \ldots, w_q$, which are linearly independent and span F_N and G_N, such that $\langle v_i, f_t \rangle$ and $\langle w_j, g_t \rangle$ are unit root processes for $i = 1, \ldots, p$ and $j = 1, \ldots, q$. If the $(p+q)$-dimensional unit root process (z_t) defined as

$$z_t = \big(\langle v_1, f_t \rangle, \ldots, \langle v_p, f_t \rangle, \langle w_1, g_t \rangle, \ldots, \langle w_q, g_t \rangle \big)'$$

is cointegrated with the cointegrating vector

$$c = (-a_1, \ldots, -a_p, b_1, \ldots, b_q)',$$

then the distributional cointegration holds with

$$v = a_1 v_1 + \cdots + a_p v_p \quad \text{and} \quad w = b_1 w_1 + \cdots + b_q w_q.$$

The pair of functions v and w are called distributional cointegrating functions of two time series (f_t) and (g_t) of densities.

Longrun Response Function

Denote the distributional cointegrating functions by

$$v^C = a_1 v_1 + \cdots + a_p v_p$$
$$w^C = b_1 w_1 + \cdots + b_q w_q$$

The distributional cointegrating function (v^C, w^C) of (f_t) and (g_t) measures the longrun response v^C of the time series of cross-sectional distribution represented by (f_t) on the time series $(\langle w^C, g_t \rangle)$.

In particular, we define v^C to be the longrun response function of (f_t) on $(\langle w^C, g_t \rangle)$, which we may interpret as summarizing the longrun effect of (f_t) on the longrun movement of (g_t) in the direction of w^C.

Possible Number of Cointegrating Relations

Clearly, there are at most r-number of linearly independent distributional cointegrating relationships, $r \leq \min(p, q)$, between (f_t) and (g_t).

Otherwise we would have a cointegrating vector c of the form $c = (-a_1, \ldots, -a_p, 0, \ldots, 0)'$ or $c = (0, \ldots, 0, b_1, \ldots, b_q)'$, which implies that there is a linear combination of $v_1, \ldots, v_p$ or $w_1, \ldots, w_q$ whose inner product with (f_t) or (g_t) becomes stationary.

This contradicts the assumption that $v_1, \ldots, v_p$ and $w_1, \ldots, w_q$ are linearly independent functions that span F_N and G_N, respectively.

Distributional Cointegration

The distributional cointegration does not presume any distributional regression relationship like $g_t = \mu + Af_t + e_t$. However, for two time series of densities (f_t) and (g_t) that are given by the above distributional regression model, we may easily deduce that

Lemma Let (f_t) and (g_t) be given by the distributional regression model $g_t = \mu + Af_t + e_t$ with some stationary (e_t). Then for any $w \in G_N$, we have $A^*w \notin F_S$ and the distributional cointegration

$$\langle w, g_t \rangle = \pi + \langle v, f_t \rangle + u_t$$

holds with $v = P_N A^* w$.

Longrun Response to Cross-sectional Moments

If (f_t) and (g_t) are given by the distributional regression $g_t = \mu + Af_t + e_t$, then we have

$$G_C = G_N \quad \text{and} \quad r = q \leq p,$$

In this case, there exists a distributional cointegrating function (v^C, w^C) with

$$w^C = Q_N \iota_\kappa$$

Then it follows that

$$\langle w^C, g_t \rangle = \langle Q_N \iota_\kappa, g_t \rangle = \langle \iota_\kappa, Q_N g_t \rangle = \langle \iota_\kappa, g_t^N \rangle,$$

where $g_t^N = Q_N g_t$ is the nonstationary component of (g_t).bigskip Therefore, we may interpret the corresponding v^C as the longrun response function of (f_t) to the κ-th cross-sectional moment of (g_t^N), or the κ-th longrun cross-sectional moment of (g_t).

Test for Distributional Cointegration

Assume that we find p and q, the numbers of unit roots in (f_t) and (g_t), and obtain consistent estimates (v_i^T) of (v_i) and (w_j^T) of (w_j), $i = 1, \ldots, p$ and $j = 1, \ldots, q$, which span the nonstationary subspaces F_N and G_N of (f_t) and (g_t).

To test for distributional cointegration, we let (z_t^T) be defined as

$$z_t^T = \big(\langle v_1^T, x_t\rangle, \ldots, \langle v_p^T, x_t\rangle, \langle w_1^T, y_t\rangle, \ldots, \langle w_q^T, y_t\rangle\big)'$$

Clearly, the test τ_n^T to determine the number of distributional unit roots may be used to test for the number of unit roots in (z_t), $z_t = \big(\langle v_1, x_t\rangle, \ldots, \langle v_p, x_t\rangle, \langle w_1, y_t\rangle, \ldots, \langle w_q, y_t\rangle\big)'$.

The maximum number of unit roots for (z_t) is of course given by $p + q$ (no distributional cointegration in (f_t) and (g_t)).

n-number of unit roots for (z_t) implies r-number of cointegrating relationships with $r = (p + q) - n$.

Empirical Illustrations

Income-Consumption Dynamics

Interactive Income-Consumption Dynamics

As an application of our model and methodology, we analyze the interactions between the income and consumption dynamics.

For our analysis, we apply our theory developed thus far with (f_t) and (g_t) representing the time series of household income and household consumption distributions.

Data

The cross-sectional observations of household income and consumptions are obtained at monthly frequency from Consumer Expenditure Survey (CES), collected for Bureau of Labor Statistics, US Census Bureau.

CES consists of two surveys - Quarterly Interview Survey and Diary Survey, that provide information on buying habits, expenditures, income, and consumer unit (families and single consumers) characteristics. CES is the only Federal survey to provide information on complete range of consumer expenditures and incomes.

The data set provides 400 time series observations from October 1979 to February 2013, with cross-sectional observations for each month ranging from 1,537 to 5,406.

During this sample period, each household is included in the survey at most five times, and therefore CES provides a pseudo panel data.

More on Data

In order to construct monthly household income and consumption, we follow the definitions in Krueger and Perri (2006), and aggregate the monthly values provided in Universal Classification Code (UCC) level for each month and year.

We then deflated the nominal income and consumption values by monthly CPI provided by BLS for all urban households with using a base year which varies among 1982, 1983 and 1984.

The survey uses topcodes which may change annually and be applied at a different starting point. We drop all top-coded values.

As in Krueger and Perri (2006), we correct expenditure on food, impute services from vehicle and primary residence, and exclude observations with possible measurement error or inconsistency problem.

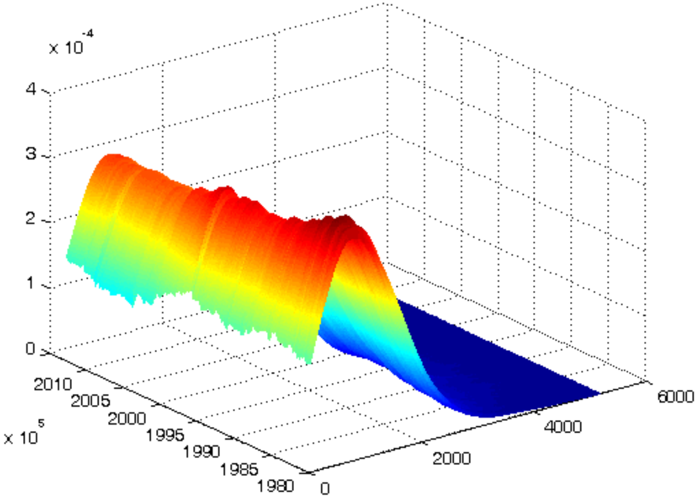
Interactive Dynamics of Income and Consumption

If

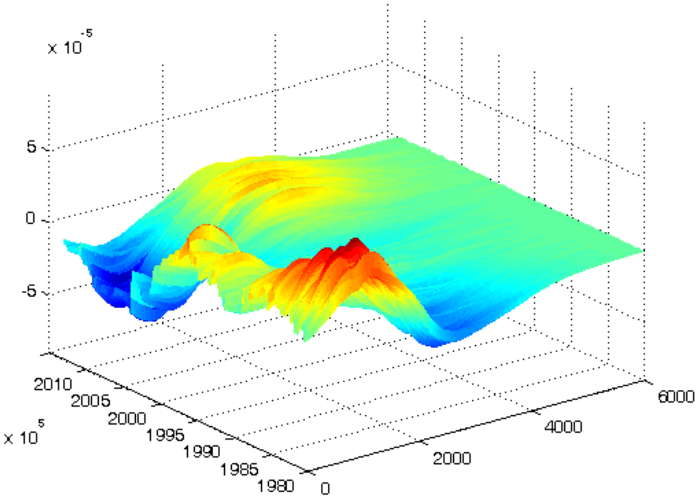
- the time series of income distributions has p unit roots
- the time series of consumption distributions has q unit roots
- there are r cointegrating relationships between them

Then, there are $(p+q)-r$ unit roots in their time series combined together.

Densities of Household Incomes



Demeaned Densities of Household Incomes



Unit Root Dimension - Incomes

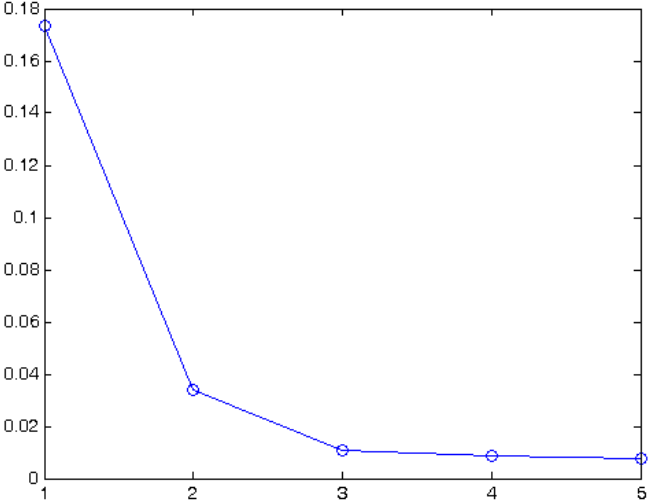
To determine the unit root dimension n in the time series of cross-sectional distributions of household incomes, use the test $\hat{\tau}_n^T$ to test $\mathrm{H}_0 : \dim(H_N) = n$ against $\mathrm{H}_1 : \dim(H_N) \leq n - 1$ with $n = 1, \ldots, 5$.

M	1	2	3	4	5
$\hat{\tau}_n^T$	0.1734	0.0338	0.0106	0.0088	0.0076

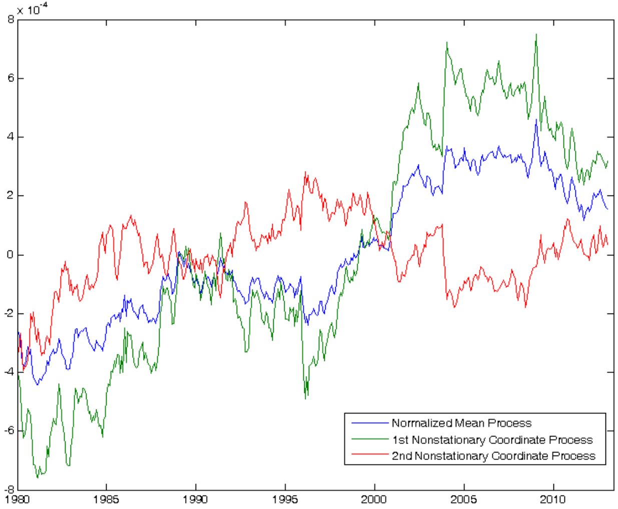
Our test, strongly and unambiguously, rejects H_0 against H_1 successively for $n = 5, 4, 3$. Clearly, however, the test cannot reject H_0 in favor of H_1 for $n = 2$.

We conclude that there exists two-dimensional unit root, and set $\hat{n}_T = 2$.

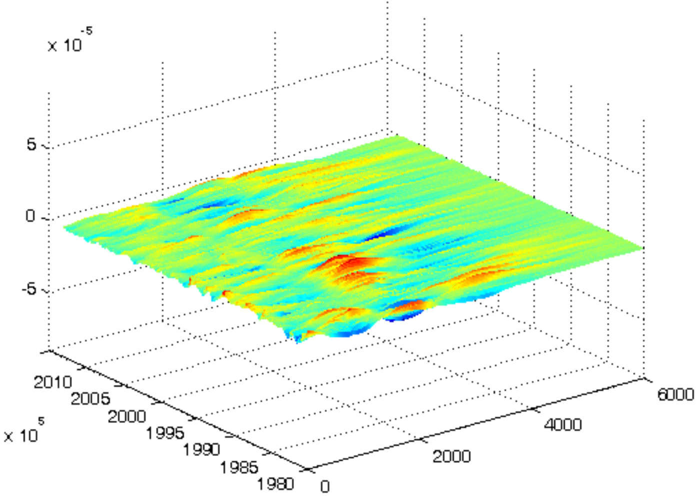
Scree Plot of Eigenvalues - Incomes

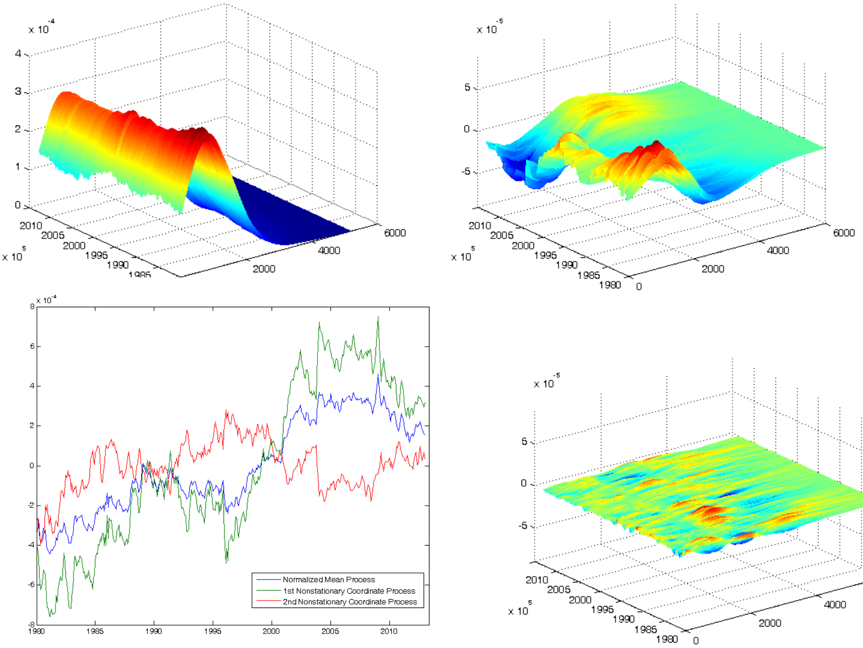


Integrated Coordinate Processes - Incomes



Stationary Components - Incomes





x 10^-4
x 10^5
x 10^-5
Normalized Mean Process
1st Nonstationary Coordinate Process
2nd Nonstationary Coordinate Process

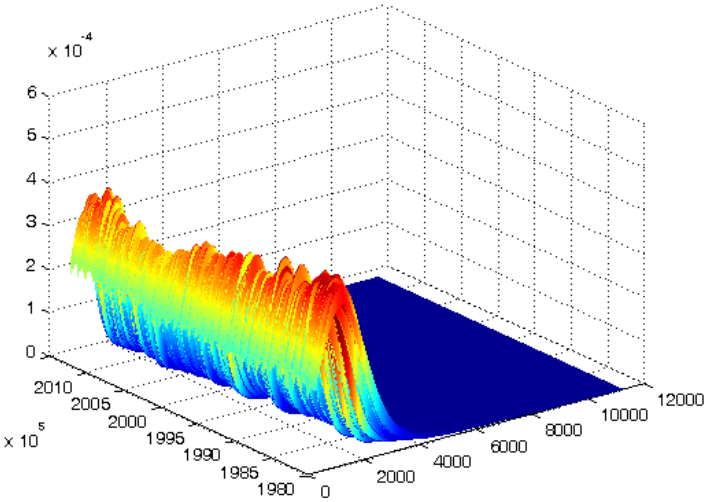
UR Proportions in Moments - Incomes

Compute the unit root portion estimates $\hat{\pi}_\kappa^T$ for the cross-sectional distributions of household incomes with $\hat{n}_T = 2$ for the first four moments.

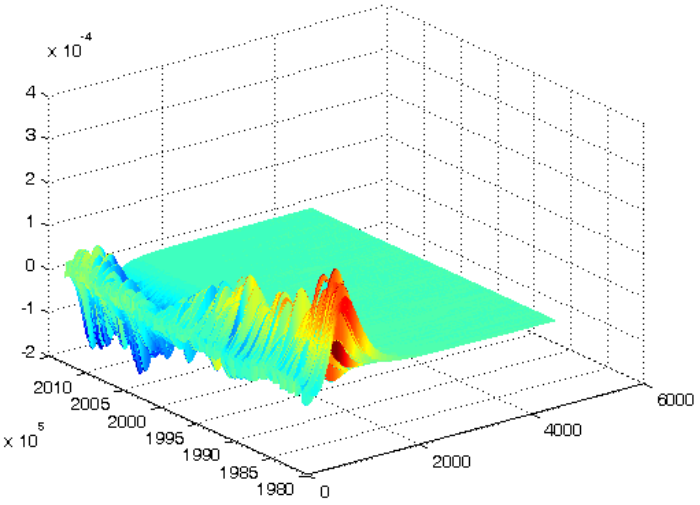
$\hat{\pi}_1^T$	$\hat{\pi}_2^T$	$\hat{\pi}_3^T$	$\hat{\pi}_4^T$
0.5734	0.3943	0.2755	0.2011

The unit root proportions for the first four moments of the cross-sectional household income distributions are all substantially large. In particular, the unit root proportions for the first two moments are quite substantial.
Nonstationarity in the cross-sectional household income distributions would certainly make their volatilities more persistent.

Densities of Household Consumptions



Demeaned Densities of Household Consumptions



Unit Root Dimension - Consumptions

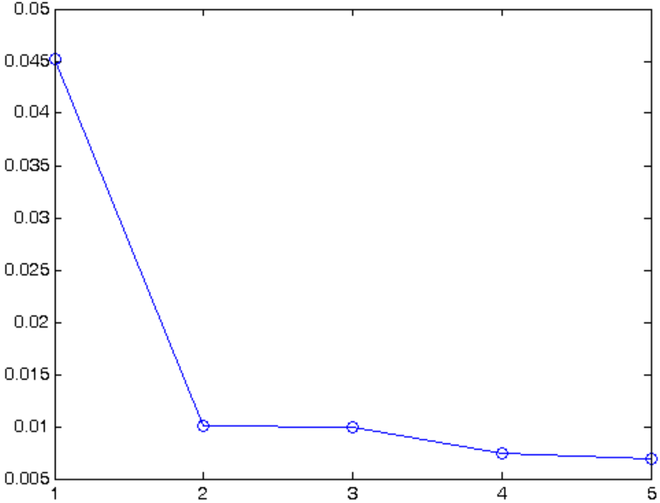
To test for existence of unit root in time series of cross-sectional distributions of household consumptions, use the statistic $\hat{\tau}_n^T$ to test $\mathsf{H}_0 : \dim(H_N) = n$ against $\mathsf{H}_1 : \dim(H_N) \leq n - 1$ with $n = 1, \ldots, 5$.

M	1	2	3	4	5
$\hat{\tau}_n^T$	0.0452	0.0100	0.0099	0.0075	0.0069

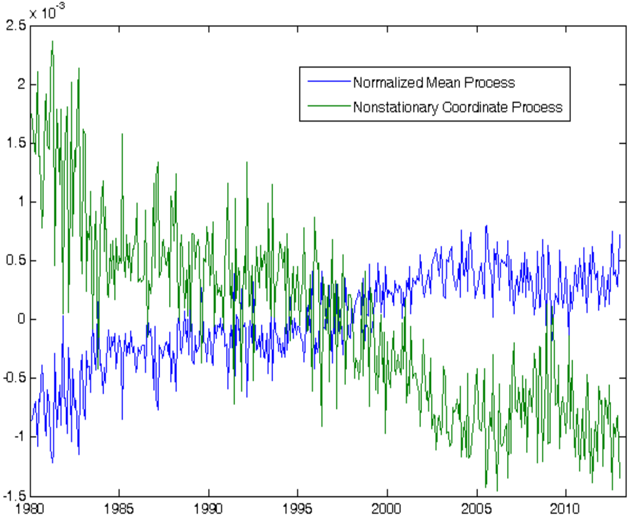
Our test successively rejects the null against the alternative for $n = 5, 4, 3, 2$.

However, at 5% level, the test cannot reject H_0 in favor of H_1 for $n = 1$. Our test result implies $\hat{n}_T = 1$.

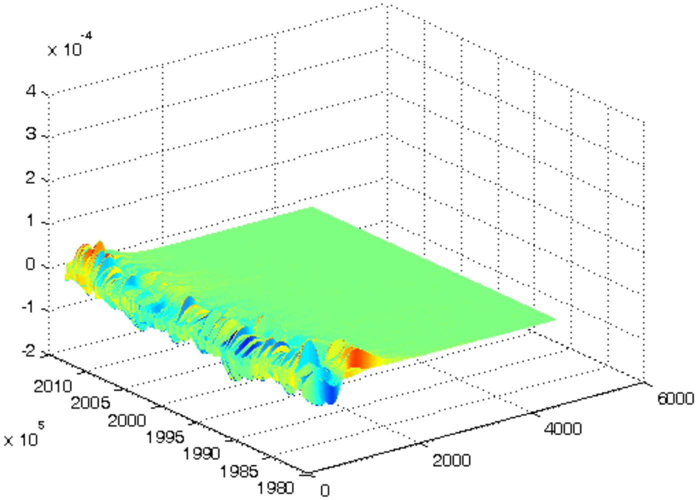
Scree Plot of Eigenvalues - Consumptions

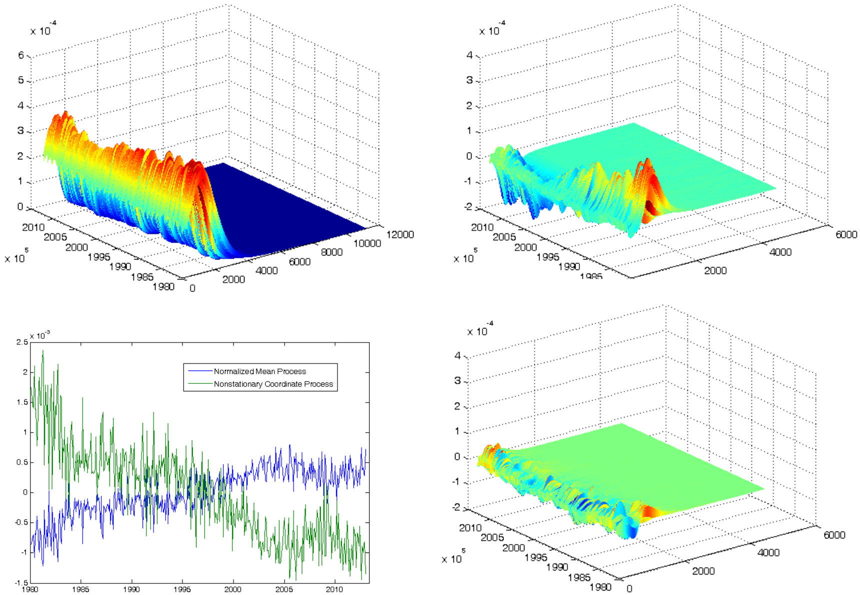


Integrated Coordinate Processes - Consumptions



Stationary Components - Consumptions




x 10-4
x 10 5
2010
2005
2000
1995
1990
1985
1980
0
2000
4000
6000
8000
10000
12000
x 10-3
Normalized Mean Process
Nonstationary Coordinate Process
1980
1985
1990
1995
2000
2005
2010

UR Proportions in Moments - Consumptions

Compute the estimates $\hat{\pi}_\kappa^T$ of the unit root proportions π_κ for the first four moments of the cross-sectional distributions of household consumption, with $\hat{n}_T = 1$.

$\hat{\pi}_1^T$	$\hat{\pi}_2^T$	$\hat{\pi}_3^T$	$\hat{\pi}_4^T$
0.5598	0.4483	0.3595	0.3169

The unit root proportions are also substantial for all of the first four moments.

Distributional Cointegration

$H_N(f)$ and $H_N(g)$ are estimated to be 2- and 1-dimensional and there are 2- and 1-unit roots in (f_t) and (g_t), denoting income and consumption distributions.

Therefore, v_1, v_2 and w_1 span $H_N(f)$ and $H_N(g)$, such that $\langle v_1, f_t \rangle$, $\langle v_2, f_t \rangle$ and $\langle w_1, g_t \rangle$ are unit root processes.

If 3-dimensional process (z_t)

$$z_t = \big(\langle v_1, f_t \rangle, \langle v_2, f_t \rangle, \langle w_1, g_t \rangle \big)'$$

is cointegrated with the cointegrating vector

$$c = (\alpha_1, \alpha_2, \beta_1)',$$

then the distributional cointegration holds with the cointegrating functions of (f_t) and (g_t) given by

$$v^C = \alpha_1 v_1 + \alpha_2 v_2 \quad \text{and} \quad w^C = \beta_1 w_1.$$

Test for Distributional Cointegration

We may use τ_n^T also in this case to find the number of unit roots in (z_t), containing all unit root process from the time series of income and consumption distributions by testing $\mathrm{H}_0 : (p+q) - r = n$ against $\mathrm{H}_1 : (p+q) - r \leq n-1$.

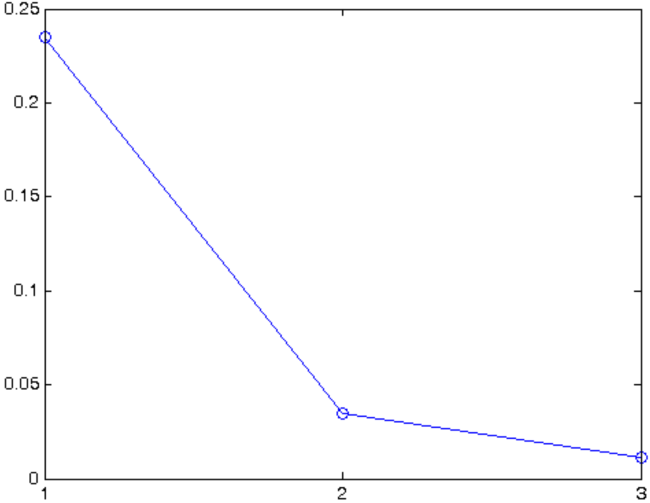
Given $p = 2$ and $q = 1$, we may have up to three unit roots in the time series of income and consumption distributions together. Therefore, we consider only $n = 1, 2$ and 3.

n	1	2	3
$\hat{\tau}_n^T$	0.2347	0.0350	0.0113

Our test rejects H_0 against H_1 for $n = 3$. However, the test cannot reject H_0 in favor of H_1 for $n = 2$, giving $(p+q) - r = 2$.

This implies $r = 1$, i.e., the presence of a single cointegrating relationship between income and consumption distributions.

Scree Plot - Distributional Cointegration Test



Cointegrating Function

Let v_1 and v_2 be orthonormal functions that span the nonstationary subspace F_N of the time series (f_t) of income distributions, and let w be the normalized function generating the nonstationary subspace G_N of the time series (g_t) of consumption distribution.

We find one cointegrating relation between income and consumption distributions, and therefore, there exists constants a_1, a_2 and b such that

$$b\langle w, g_t\rangle = \delta + a_1\langle v_1, f_t\rangle + a_2\langle v_2, f_t\rangle + u_t$$

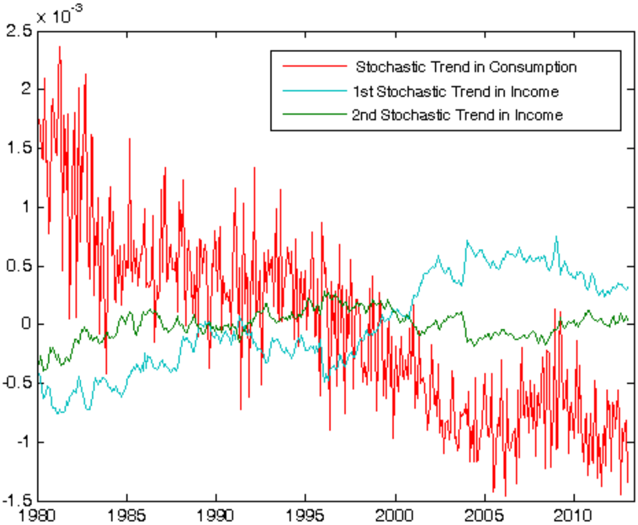
with some constant function δ and general stationary process (u_t) with mean zero.

In this case, we have

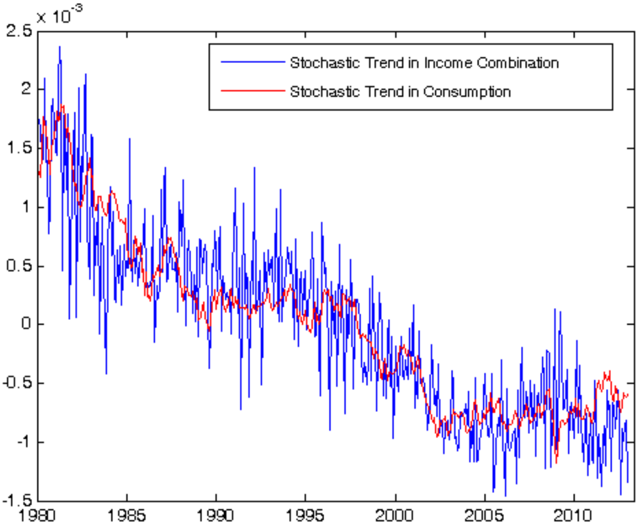
$$v_C = a_1 v_1 + a_2 v_2 \quad \text{and} \quad w_C = bw,$$

where (v_C, w_C) is the cointegrating function of (f_t) and (g_t).

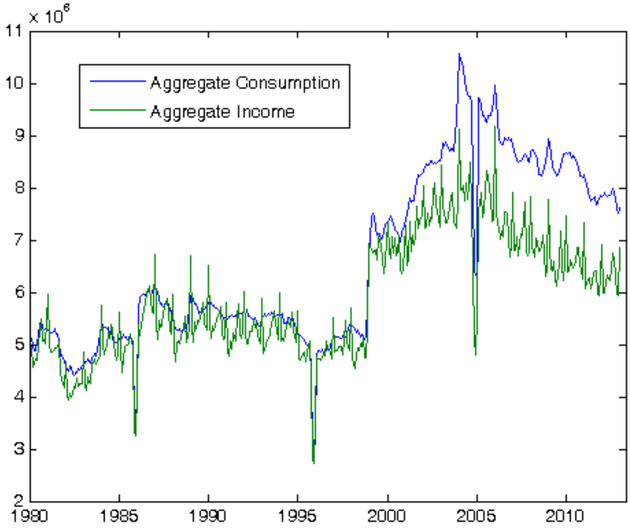
Stochastic Trends in Income and Consumption



Common Trends in Income and Consumption



Aggregate Income and Aggregate Consumptions



Longrun Response of Income to Consumption

We may readily obtain estimates of v_C and w_C, which we define as

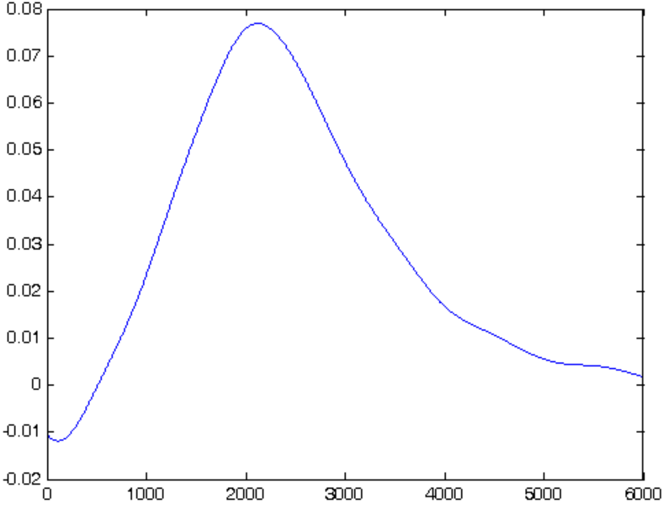
$$v_C^T = a_1^T v_1^T + a_2^T v_2^T \quad \text{and} \quad w_C = b^T w^T,$$

from our estimates v_1^T, v_2^T and w^T of v_1, v_2 and w, and a_1^T, a_2^T and b^T of a_1, a_2 and b.

The estimates v_1^T, v_2^T and w^T are obtained from our testing procedure for distributional unit roots, and the estimates a_1^T, a_2^T and b^T from our testing procedure for distributional cointegration, respectively in and between household income and consumption distributions.

The estimated longrun response function of income distribution to consumption distribution is given by v_C^T.

Longrun Response Function



Empirical Findings

The longrun trend in consumption is most affected by the income group with monthly earnings slightly over $2,000. Roughly, all households with monthly earnings between $1,000 and $4,000 seem to play important roles in determining the persistent stochastic trend in consumption. As the level of monthly earning decreases below $1,000, the longrun component of household's income has very little impact on the longrun consumption.

The longrun component of household's income for the rich also does not have any major effect on the longrun consumption, though the magnitude of their effect decreases at a slower rate as their income increases than the rate it decreases as the income decreases for the poor.

Note

The income response to consumption is estimated to be negative for the household with monthly earnings less than approximately $500, which we believe to be just an evidence of insignificant response.

Observations for households with monthly earnings below approximately $500 are scarce and irregular, so we do not expect to have any reliable results over very low income levels.