Collateralisation bubbles when investors disagree about risk

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Abstract

Survey respondents strongly disagree about return risks and, increasingly, macroeconomic uncertainty. This may have contributed to higher asset prices through increased use of collateralisation, which allows risk-neutral investors to realise perceived gains from trade. Investors with lower risk perceptions buy collateralised loans, whose downside-risk they perceive as small. Investors with higher risk perceptions buy upside-risk through asset purchase and collateralised loan issuance, raising prices. More complex collateralised contracts, like CDOs, can increase prices further. In contrast, with disagreement about mean payoffs, price bubbles arise without collateralisation, which may discipline prices as pessimists demand higher returns on risky loans.

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1 Introduction

From the mid-1990s to the beginning of the Great Recession, the world economy has seen an unprecedented wave of financial innovation, partly in the form of new collateralised debt products.

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At the same time, prices of collateral assets, such as real estate, but also stocks, experienced an unprecedented increase. This paper links these two phenomena to a third, less documented one: disagreement among investors about economic risk. Particularly, we document how retail investors strongly disagreed about riskyness of stock market investments at the beginning of the 2000s, and show that the period of low macro-volatility since the mid-1980s, later called the "Great Moderation", coincided with a strong increase in disagreement about macro-risks in the US more generally. We show how these heterogeneous risk perceptions, when combined with financial innovation in the form of collateralised debt products, can create large asset price bubbles. In the absence of collateralisation, risk-neutral investors trade assets at their common fundamental value even if they disagree about payoff risk. The introduction of simple collateralised loans increases asset prices above this common fundamental value by unleashing perceived gains from trade: while investors with a concentrated posterior distribution of payoffs are less afraid of the downside risk embodied in collateralised debt, those who perceive higher payoff risk value more highly the upside risk of leveraged asset purchases. In equilibrium, this raises the price of collateral assets. This is in contrast to disagreement about mean payoffs, the focus of many previous studies, which, with short-selling constraints, does not require collateralisation to raise prices of assets above their mean valuation. Collateralised loan issuance allows optimists to increase their funds, and thus to drive up prices, even when debt is riskless. Because buyers are pessimistic, optimists are deterred by low prices when debt is risky, which disciplines leverage and asset prices if optimism is mainly about downside risk (Simsek (2013)). In contrast, with disagreement about risk rather than average payoffs, there is no disciplining effect: investors with dispersed posteriors issue risky debt to pass downside risk to those with lower risk perceptions, realising pure perceived gains from trade. Moreover, while an increase in pessimism about mean payoffs lowers prices of risky debt and assets, any rise in disagreement about risk raises asset prices further as it increases the perceived value of the risks that investors have chosen to buy. Finally, we also show how further financial innovation in the form of more complex collateralised debt products can strongly affect prices. We show that this is, perhaps surprisingly, not the case with collateralised debt obligations (CDOs), whose introduction does not affect prices relative to trade in collateralised loans. Allowing agents to use CDOs to collateralise more junior "secondary" CDO contracts, however, strongly raises prices. This is because, by buying and issuing a pair of CDO contracts, investors can buy any sub-set of the payoff distribution. This drives up asset prices to the expectation of its payoffs evaluated at the maximum of all individual probability density functions. For example, with two types of investors that have disjoint payoff distributions, asset prices rise to twice their fundamental value with trade in secondary CDOs.

The literature on the consequences of heteregenous investor beliefs has focused on disagreement about mean payoffs.[1] Thus, Miller (1977)'s seminal article shows how asset prices rise when investors disagree about future mean payoffs and the absence of short-selling makes the marginal investor become more optimistic. Harrison and Kreps (1978) analyse a dynamic version of the model and show that even in periods where optimists are not in the market, pessimists are happy to pay a speculative premium above their fundamental asset valuation in anticipation of rising prices when optimists return to the market in the future, a phenomenon coined "rational bubble" by Scheinkman and Xiong (2003). Geanakoplos (2001) and Geanakoplos (2010) introduce leverage into this framework, whereby investors can issue debt collateralised by the assets they want to buy in order to increase the amount they can invest. This allows optimists to increase their asset purchases and thus makes the marginal investor more optimistic, increasing prices further. At the same time it leads to fluctuations in asset prices in response to changes in investor balance sheets. Simsek (2013) uses a similar model with two groups of investors to show how leveraged investment dampens the effect of belief disagreements on prices when optimism is concentrated on the downside, i.e. when optimists have relatively positive views on the distribution of relatively bad realisations of shocks. If optimists are particularly positive about the upside potential of the asset, however, leveraged investments amplify the effect of belief disagreements. Both Simsek (2013) and Fostel and Geanakoplos (2012) also consider the effect on asset prices of introducing pure derivative contracts, that are contingent on asset payoffs but collateralised by cash. They show how these contracts typically lower asset prices, as they are payoff-equivalent to contracts collateralised by the asset and thus similar to an increase in the supply of collateral assets.

Interestingly, Miller (1977)'s original article associates higher payoff risk with stronger disagreement about mean payoffs. Neither his article, nor the literature that it preceeds, however, has analysed disagreement in beliefs about risk per se.[2] In fact, in Miller (1977)'s original setup where risk-neutral investors simply buy or sell the asset, disagreement about payoff risk around common mean payoffs does not affect prices, which equal their common discounted expected payoff. The introduction of collateralised debt, however, enables investors to realise the perceived gains from trade that arise from heterogeneous risk perceptions by splitting asset payoffs non-linearly. Investors that perceive payoffs to be volatile are, in Simsek (2013)'s words, upside optimists and downside pessimists. They are thus happy to sell the downside risk by issuing collateralised loans, and retain the upside risk they value highly. Investors who believe payoffs to be concentrated are, in contrast, downside optimists and upside pessimists and happily buy collateralised loans whose downside risk they perceive as small. Assets payoffs are thus valued

[1] See (Xiong, 2013) for a survey.

[2] In an early reaction to (Miller, 1977), (Jarrow, 1980) has pointed out the importance of the variance-covariance structure of asset returns for the effect of short-selling constraints on asset prices. His focus is very different to the one in this paper, however.

by optimists throughout the distribution, driving up equilibrium asset prices.

It is important to realise how this complementary relationship between collateralisation and disagreement about risk is different to that arising from disagreement about mean payoffs. In Geanakoplos (2001), with two discrete payoff realisations, optimists find it optimal to issue riskless collateralised loans in order to increase their available funds. Market clearing then implies a more optimistic marginal investor, and an increase in asset prices from a level that, absent short-selling, exceeds the mean of investor valuations even in the absence of leverage. When collateralised loans are risky, the common assumption of first-order stochastic dominance implies a trade-off for optimists, as in Simsek (2013): issuing collateralised debt raises funds for investments, but means selling downside risk at unfavourable prices to pessimists. This is why collateralised contracts can discipline asset prices whenever optimism is about downside risks. Moreover, an increase in disagreement in the form of increased pessimism acts to lower the price of risky loans and assets.
With disagreement about payoff dispersion, we show how the effect of collateralisation is fundamentally different. First, asset prices equal a common fundamental in the absence of collateralised contracts. In other words, a departure of prices from their fundamental requires financial innovation, e.g. in the form of collateralised debt. Second, collateralisation allows investors to realise perceived gains from trade only when debt is risky, by channeling upside and downside risk to those that value them more highly. This implies, fourth, that there is no trade-off, and no disciplining effect of collateralisation: issuing risky collateralised debt realises pure perceived gains from trade. Finally, an increase in disagreement makes collateralised loans and leveraged assets more valuable to those that hold them, and thus always raises asset prices.

Apart from simple collateralised debt, we also look at more complex collateralised loan products, such as CDOs. These make a unit payment whenever the asset payoff is above a certain value. We show that, with two investors that disagree about payoff risk, trade in primary CDOs is exactly equivalent to trade in collateralised loans: the equilibrium asset price is the same in both environments. This is because investors with more concentrated posteriors, who perceive realisations in the lower tail of the distribution as less likely, have a taste for senior CDOs whose riskyness they see as low. But buying all senior CDOs up to a threshold payoff splits payments in exactly the same way as buying a collateralised loan with a face value equal to that threshold. This result changes radically when we allow agents to also trade secondary CDOs (collateralised by other CDOs). Buying a primary CDO and selling a more junior one allows investors to trade claims to any sub-set of the payoff distribution. This increases the price of assets to equal the expectation of payoffs taken over the maximum of individual density functions. We show how this can drive up the asset price to twice its fundamental value in the two-type case.

Why do we think are these theoretical results important? The strong complementarity between heterogeneous risk perceptions and financial innovation in the determination of asset prices provides, in our view, a rationale for the strong boom in prices of collateral assets since the mid-1990s. Moreover, we think that changes on both sides of this complementary relationships could have contributed to that boom. First, technological and regulatory changes encouraged financial innovation. And second, we believe that perceptions of risk became more diverse during the 1990s and 2000s. One reason for this is, in our view, the "Great Moderation" in macroeconomic volatility since the mid-1980s. Recognised as an empirical fact by both academics and investment analysts since the mid-1990s (Kim and Nelson (1999), McConnell and Perez-Quiros (1997), Perez-Quiros and McConnell (2000)), there was, and is so far, no consensus about its origin in, for example, "good luck", "good policy", technological innovation, etc., and therefore about its persistence. To the extent that investment risk is linked to macroeconomic volatility, the strong change in the volatility environment may therefore have led to an increase in belief dispersion about risk going forward. We provide two pieces of evidence for this argument from US surveys: First, supplemenary questions to the Michigan Survey of Consumer Sentiments administered between 2000 and 2005 (summarised in Amromin and Sharpe (2008)) show how retail investors disagree strongly about medium and long-term dispersion of stock returns. To analyse a longer time horizon covering the Great Moderation period, we also show how, since 1980, near-term GDP forecasts from the Survey of Professional Forecasters show an increasing disagreement between forecasters about the dispersion of GDP growth, while disagreement about mean growth has fallen. We take this evidence as motivation to construct a simple example of a dynamic equilibrium in a scenario with learning that tries to capture the main features of the Great Moderation in the US. As a subset of investors adjusts their posterior estimate of volatility more quickly to the Great Moderation than the rest, increasing divergence of posteriors raises asset prices between 5 and 40 percent.

2 Heterogeneity in risk perceptions: evidence from US survey data

Past studies of investor disagreement have focused on heterogeneity in expected, or mean returns. This section, in contrast, shows evidence from US surveys that documents the extent to which investors, or forecasters, disagree about risk, or the dispersion of outcomes around their expections. For this we use two data sources: first, supplementary questions to the Michigan Survey of Consumer Attitudes that, between 2001 and 2005, ask stock market investors for the stock market returns they expect on average and the uncertainty around them in the medium and long-run. And second, a longer history of GDP forecasts elicited in the Survey of Professional Forecasters (SPF) that contains a fully specified histogramme of near-term GDP growth. Relative to other investor surveys, the Michigan survey has the advantage that, for an important asset class - US stocks - it asks actual investors not only for their mean expectations but also for the uncertainty around them.[3] The SPF, in contrast, asks for GDP growth, which is interesting as one of the main macroeconomic determinants of investment returns, if not a perfect predictor. Its advantage is that it contains a long history of histogrammes with a finer support than that of the Michigan survey. It therefore allows us to look at the evolution of disagreement over time during the period of the Great Moderation, where realised macro-volatility was significantly lower than in the preceding post-war period. This is interesting, as part of the disagreemeent about return volatility might arise from differential beliefs about the persistence of this period of low macro-volatility.

2.1 Disagreement about US stock market returns 2001-2005

The Michigan Survey of Consumer Attitudes is a monthly round of about 500 phone interviews among US Households carried out by the Survey Research Center at the University of Michigan. Apart from monthly questions about general sentiment indicators and macroeconomics conditions, the survey has asked those respondents with at least 5000 $ invested in the stock market (between 35 and 40 percent of respondents) a set of supplementary questions about expected stock market returns in 22 interview rounds between September 2000 and October 2005. Specifically, respondents were asked for the “annual rate of return that you would expect a broadly diversified portfolio of U.S. stocks to earn, on average” over the next 10 to 20 years. Importantly, the survey also asks “what do you think the chance is that the average return over the next 10 to 20 years will be within two percentage points of your guess”. Amromin and Sharpe (2008) provide summary statistics for the responses, which we repeat in table 1.[4] The first row of table 1 shows that expected annual returns, averaged across respondents and surveys, equal 9 percent, which coincides almost exactly with the average 10 year annual returns on the S&P total returns index in the period before the last survey in 2005. Disagreement about future mean returns, however, is strong, with 10 percent of respondents expecting an average return of or below 5, and another 10 percent expecting above 16 percent. The perceived riskyness of stock investments,

[3]See Greenwood and Shleifer (2013) for a description of surveys about return expectations in the US. The only other survey that has, to our knowledge, data on return dispersion is the Graham-Harvey Chief Financial Officer survey.

[4]The authors eliminate incomplete responses, those deemed by the interviewer to have a low level of understanding or a poor attitude toward the survey, and those that answered ”50 percent” to all probability questions.

however, also varies strongly across investors: while 10 percent of respondents believe realised returns to fall within 2 percentage points of their expection with a probability of at least 80 percent, another 10 percent expect returns to fall outside this range with at least 80 percent probability. Using a normality assumption to transform these assessments into standard deviations, the 90/10 percentile ratio of standard deviations equals 5.1, compared to 3.2 for expected returns.

Return expectations in the Michigan Survey 2000-2005

	N	Mean	10th pct	25 pct	Median	75th pct	90th pct
Expected return R_e	3,046	10.4	5	7	10	12	16
Prob $\|RR_e\| < 2pp$	3,015	43.3	20	25	50	50	80
Implied σ_{10-20} (in percent)	2,854	4.56	1.56	1.73	2.96	2.96	7.88

The table reports summary statistics of the supplementary questions in the Michigan Survey of Consumer Sentiments, covering 22 surveys in the years 2000 to 2005, taken from Amromin and Sharpe (2008). The first row reports the distribution of investors' answer to the question about the "annual rate of return that you would expect a broadly diversified portfolio of U.S. stocks to earn, on average". The second the probability "that the average return over the next 10 to 20 years will be within two percentage points of your guess", and the third the corresponding standard deviation assuming normally distributed beliefs about stock market returns.

2.2 Disagreement about US Macro Risk 1980-2010

The Michigan Survey data has two disadvantages: first, the uncertainty measure is based on only 1 response about the probability of returns to fall within a constant range around their mean. And second, the supplementary question only starts in 2001, and thus does not allow to look at the evolution of disagreement over a longer time horizon. In order to look at a longer history of risk perceptions, this section focuses on macro-economic risk using data from the Survey of Professional Forecasters (SPF). The SPF is a quarterly survey that asks forecasters to indicate, among other measures, their probability distribution for GDP growth in the current calendar year.[5] Specifically, forecasters report the probability that short-term growth falls in any of 6 brackets.[6] This allows us to study the evolution of disagreement between forecasters about short-term US growth prospects. Particularly, using a normal approximations of the distributions, as

[5]Since 1992, the survey also asks for the same distribution for the following year. We don't use this measure because of the short history.

[6]The brackets have changed slightly in 1990.

in Giordani and Söderlind (2003) we can look at the distribution across forecasters of forecaster-specific means μ_{it} and standard deviations σ_{it} for every quarter since 1980 (when the survey changed from nominal to real GDP projections). Based on this cross-sectional distribution, we look at two measures of disagreement about the mean and dispersion of output growth across forecasters: first, the standard deviations of $\widehat{\mu_{it}}, \widehat{\sigma_{it}}$ defined as

$$\mu_{it} = \widehat{\mu_{it}} + \mu_t$$
$$\sigma_{it} = \widehat{\sigma_{it}}\sigma_t \tag{1}$$

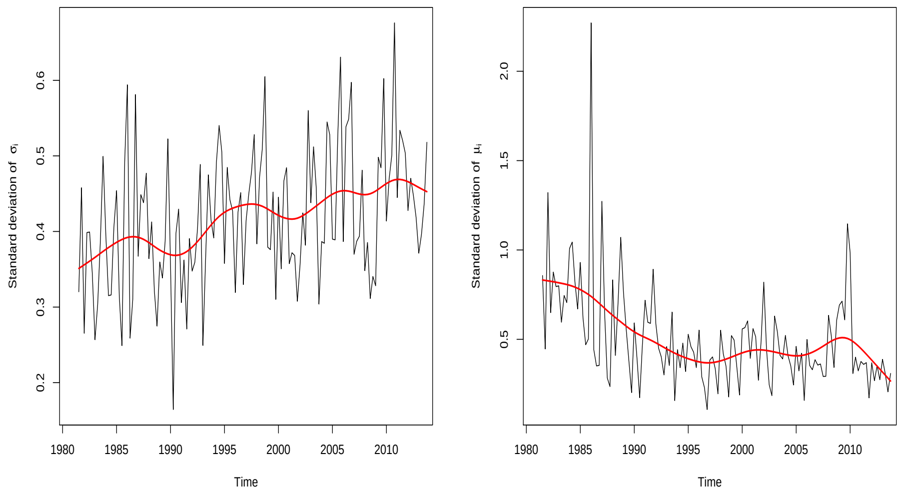
where for the positive variable σ_{it} we use the normalised standard deviation to prevent it from falling to zero mechanically as the mean of σ_{it} falls. The second disagreement measure is based on the integral of absolute differences of any two forecaster-specific normal densities, averaged across forecasters.

$$d = \frac{1}{N_t^2} \sum_i \sum_j \int |f_i(g_y) - f_j(g_y)| dg_y \tag{2}$$

where N_t is the time-varying number of forcasters in the sample. This measure equals zero for any two identical distributions and is bounded above by 2 (for two disjoint distributions). We can calculate the contribution of the heterogeneity in standard deviations to this average disagreement using the formula in (2) with the mean of the two normal distributions held constant ($\mu_{it} = \mu_{jt}$), and define the remaining difference with overall disagreement as the contribution of heterogeneous means.
Figure 1 shows how the dispersion of means and standard deviations of short-term growth forecasts has evolved over time in the survey. In the early 1980s, the standard deviations of means (in the left panel) was about twice that of standard deviations (in the right panel). But while mean forecasts converged - with their standard deviation falling to less than half their initial value before rising abruptly at the beginning of the recent 'great recession' - the dispersion of forecast standard deviations has increased strongly, amid noticeable cyclical swings. Figure 2 shows the contributions to the overall disagreement measure d of heterogeneity in forecaster-specific means (in the left panel) and standard deviations (in the right panel) (where we only use the first quarter of every year to keep the forecast horizon constant and equal to the remainder of the current year). While overall disagreement (not shown) does not follow any trend over the sample, the (smoothed) contribution of heterogeneous standard deviations increases by about 1/3 until the beginning of the recent recession. The contribution of mean growth dispersion, of about the same magnitude at the beginning of the sample, falls by about 1/3 until the recession.

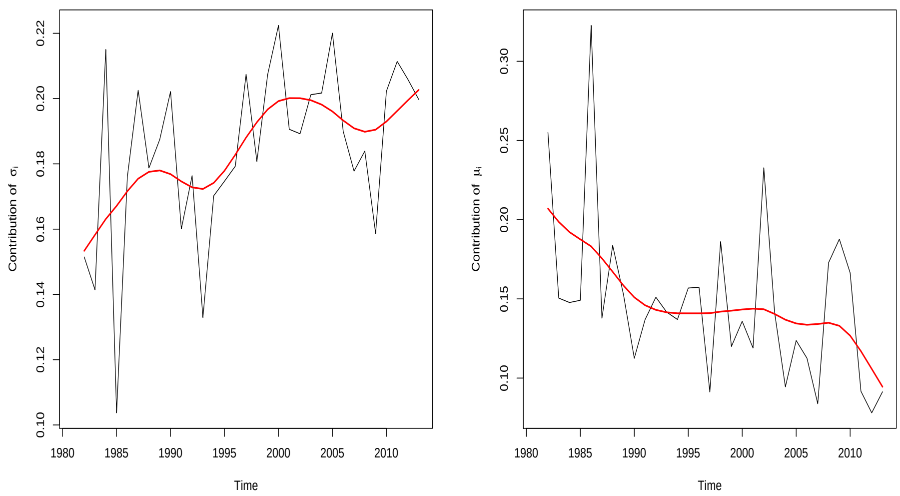
Both the evidence from the Michigan Survey and the SPF thus suggest that there is strong disagreement about the riskyness of stock market returns among US investors and about macroeconomic risk among professional forecasters. The evidence from the SPF moreover suggests that

Figure 1



The figure plots the standard deviations of $\widehat{\mu_{it}}$ and $\widehat{\sigma_{it}}$ as defined in equation (1). The red line shows the trend from an hp-filter with smoothing parameter 1600.

Figure 2



The figure plots the contribution of heterogeneous means and standard deviations to overal disagreement d about current year GDP growth in the SPF as defined in equation (2). The red line shows the trend from an hp-filter with smoothing parameter 25 (to adjust for the annual frequency, see Ravn and Uhlig (2002).

the contribution of heterogeneous perceptions of growth-dispersion has risen strongly since the early 1980s, while disagreement about mean growth has become less important.

3 Asset prices in a continuum economy with disagreement about payoff risk

This section considers an economy with a continuum of investor types who differ in their perception of the riskyness of a single asset. It shows how the introduction of collateralised debt contracts creates an asset price bubble, which we define as a positive deviation of the asset prices from its expected discounted payoff, identical across investors. Moreover, we show how an increase in belief dispersion raises asset prices further.

3.1 The General Environment

We study an economy that exists for two periods $t \in \{0, 1\}$. There is a continuum of agents of unit-mass indexed by i with $i \in I = [0, 1]$. The distribution of agents on I is defined by measure $m : \mathbb{I} \longrightarrow [0, 1]$, where $\mathbb{I}$ is the Borel-algebra of I and m has no mass-points. Denote as g the density function induced by m, and by G the cumulative density function of g with $G(0) = 0$, $G(1) = 1$.

In period 0, agents of type i receive an endowment n_i of the unique perishable consumption good and $\overline{a}_i$ units of a risky asset (a "tree") that pays a stochastic amount $s \in S = [s_{min}, s_{max}]$, $s_{min} > 0$ in period 1. All agents are assumed to be risk-neutral, so they maximise the present discounted sum of expected consumption in both periods i.e. $U_i = c_i + \frac{1}{R}E_i(c_i')$, where E_i is the mathematical expectation of agent i, c_i (resp. c_i') denotes consumption in period 0 (resp. 1) and $\frac{1}{R} < 1$ is the discount factor.

We assume that types differ in their beliefs about the distribution of random payoffs s, summarised by distribution functions $f_i : S \longrightarrow R^+$. We assume that all agents expect payoffs to be the same on average, but that any type $i : i > j$ believes them to be less tightly distributed than type j. In other words, f_j second-order stochastically dominates f_i whenever $i > j$, or formally:

$$\mathbf{A1} : E_i(s) = E_j(s) \equiv E_s,\ f_j \succ^2 f_i \Leftrightarrow j < i$$

where $\succ^2$ denotes second-order stochastic dominance. Thus i is an index of belief dispersion.

3.2 Asset markets

Agents trade in 2 asset markets: In $t = 0$, agent i purchases $a_i - \overline{a}_i$ units of the physical asset in exchange of $p(a_i - \overline{a}_i)$ units of the consumption good. In addition, agents can borrow by pledging part of their future income. However, agents cannot commit to future payments, and therefore have to collateralise their borrowing. We assume that agents only trade the simplest form of these contracts, namely a debt contract. Debt contracts are characterised by a fixed promised face value. The absence of commitment means that agents transfer to their creditor the face value of the loan or the payoff of the assets that serve as collateral, whatever is smaller. We normalise contracts to be secured by 1 unit of the asset as collateral.[7] Thus collateralised loan contracts have unit-payoffs equal to $\min\{s, \overline{s}\}$, where $\overline{s}$ is the promised face value. In $t = 0$, agents trade these contracts at competitive price $q(\bar{s})$. In the following we assume that this price function is Borel measurable.

3.3 Type i's problem for given $\overline{s}$

Take $\overline{s}$ as given. The budget constraints of agent i in $t = 0$ and $t = 1$ respectively are:

$$c_i + pa_i + qb_i \leq n_i + p\overline{a}_i, \tag{3}$$

$$c_i^{'} \leq a_i s + min\{s, \bar{s}\}b_i \tag{4}$$

where a_i and b_i represent agent i's total holdings of risky assets, including the initial endowment, and of collateralised loans respectively. Given that borrowing is subject to a collateral constraint, agent i's positions of collateralized loans sold must satisfy the following condition:

$$b_i \geq -a_{i.} \tag{5}$$

Each unit of collateralized loans sold must be secured by at least one unit of the risky asset that agent i possesses and can be used as collateral. Agent i maximizes his expected utility subject to the budget and collateral constraint. Formally, her optimization problem is:

$$\max_{c_i, c_i', a_i \geq 0, b_i > -a_i} c_i + \frac{1}{R} E(c_i') \tag{6}$$

subject to

(3), (4) and (5)

[7] Note that one unit of a bond with face value 1 collateralised by x units of the asset is payoff-equivalent to x units of a bond of face value 1/x collateralised by one unit of the asset.

3.4 Optimal Behaviour at given prices

3.4.1 Profits

Collateralized Loan Contracts: Buyers of collateralised loans with face value $\overline{s}$ pay a sum q to their counterparty today, for a promise whose expected value is $E_i[\min\{s, \overline{s}\}]$. For a quantity of loans b_i, expected discounted profits are

$$\Pi_i^l = b_i[\frac{E_i[min\{s, \overline{s}\}]}{R} - q]$$

Leveraged Risky Assets: Other than buying assets outright using consumption goods as payment, agents can engage in leveraged asset purchases by using the assets as collateral for collateralised loans. Then, for a given $\overline{s}$, the expected profits from buying a_i units of risky assets partly financed through a collateralised loan of equal size are

$$\Pi_i^a \quad = \quad [\frac{E_i(s) - E_i(\min(s, \bar{s}))}{R} - (p - q)]a_i \tag{7}$$

Figure 3 illustrates how gross unit-profits in period 1 change as a function of the asset payoff s. The definition of profits implies that returns on collateralised loans are convex in s, while those on leveraged asset purchases are concave in s. Given the second order stochastic dominance relationship of beliefs, this immediately implies Proposition 1.

Proposition 1 *Profits and risk perceptions*
Tight-prior agents (low i) have higher expected profits from investing in collateralised loans than those with dispersed priors (high i). The inverse is true for profits from leveraged asset purchases:

$$i > j \Rightarrow \Pi_i^l \leq \Pi_j^l \ \forall \ \overline{s} \in (s_{\min}, s_{\max}), \forall p, q, R$$

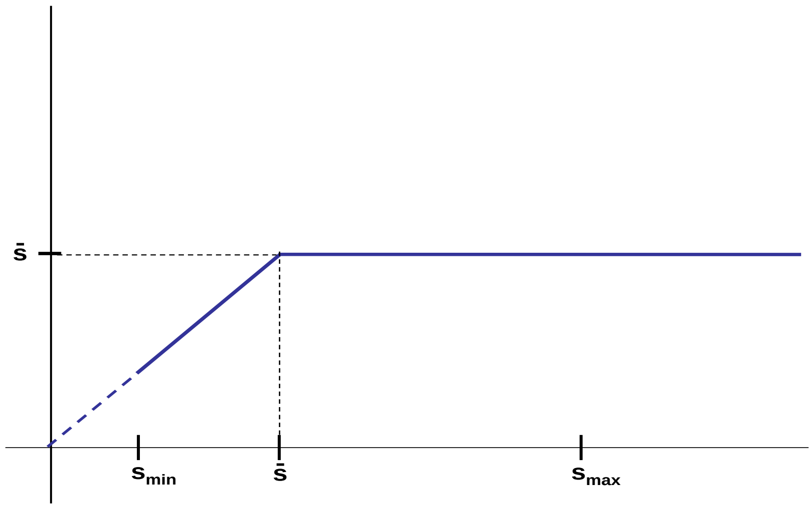
$$i > j \Rightarrow \Pi_i^a \geq \Pi_j^a \ \forall \ \overline{s} \in (s_{\min}, s_{\max}), \forall p, q, R$$

3.5 General Equilibrium

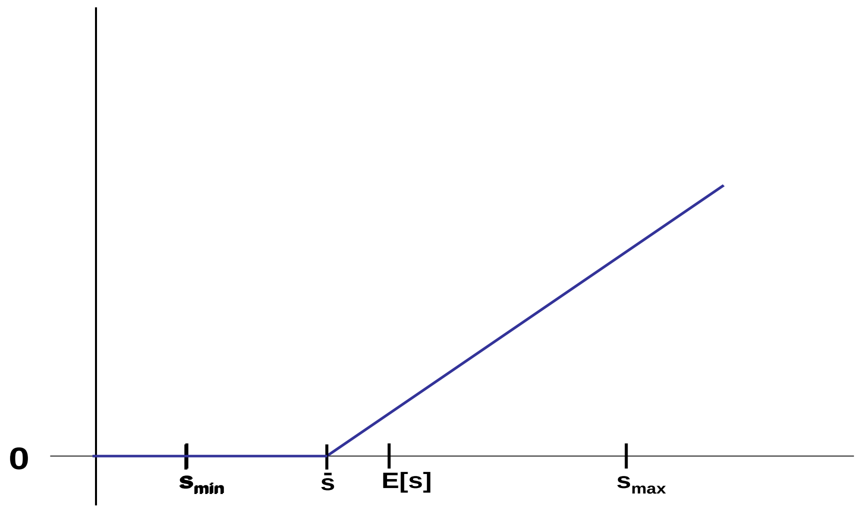
In this section we look at the equilibrium of an economy with exogenous face value $\overline{s} \leq E_s$. For this, we normalise the asset supply to 1 and assume $\overline{a}_i = 1 \ \forall i$. Also, for simplicity, we assume constant consumption endowments $n_i = n \ \forall i$.

Figure 3

Unit-Profits from collateralised loans



Unit-Profits from leveraged asset purchases



The figure plots the profits from collateralised loans (upper panel) that are concave in s, and those from leverage asset purchases (lower panel), which are convex in s.

3.5.1 Equilibrium Definition

A general equilibrium given $\overline{s}$ is a set of prices $(p,\, q)$ and allocations $(c_i, c_i', a_i, b_i)_{i\in[0,1]}$, such that agent $i \in [0,1]$ behaves optimally given p, q and $\overline{s}$, the demand for assets equals the fixed supply,

$$\int_{i\in I} a_i = 1 \tag{8}$$

and the collateralised loan market clears,

$$\int_{i\in I} b_i = 0.$$

3.5.2 Uniqueness of equilibrium and asset price bubbles

Note that for any $p < \underline{p} \doteq \frac{E_0}{R}$ all agents would like to buy risky assets, which cannot be an equilibrium. Similarly, for any given $q > \overline{q} \doteq \frac{E_0(min\{s,\overline{s}\})}{R}$, no agent is willing to buy collateralised loans, but all agents who hold assets make a strict profit by using them as collateral for the issuance of collateralised loans. Again, this cannot be an equilibrium.

Proposition 2 *Uniqueness of a bubble equilibrium*
There is a unique equilibrium with trade in assets of given riskyness $\overline{s}$. This equilibrium has the following properties:

- $p:\ \frac{E_s - E_1[min(s,\overline{s})]}{R} + q \doteq \overline{p} \geq p > \underline{p} \doteq \frac{Es}{R}$
- $q < \overline{q} \doteq \frac{E_0(min\{s,\overline{s}\})}{R}$
- *There are cutoff values $i_q < i_p$ such that all agents with $i > i_p$ invest their whole endowment $n + p$ in leveraged asset purchases, while all agents with $i < i_q$ invest their whole endowment in collateralised loans. Agents with $i : i_q \leq i \leq i_p$ sell their asset endowment and consume the proceeds together with their endowment of consumption goods.*

Proof:
Take any $q < \overline{q}$ and define

$$i_q :\ q = \frac{E_{i=i_q}(min\{s,\overline{s}\})}{R} \tag{9}$$

Take any $p \geq \underline{p}$ and define

$$i_p :\ R^a_{i_p} \equiv \frac{E_s - E_{i=i_p}(min\{s,\overline{s}\})}{p - q} = R. \tag{10}$$

where $R_i^a = \frac{E_s - E_i(min\{s,\bar{s}\})}{p-q}$ is the gross return agent i expects from a unit of own funds invested in leveraged assets. Note that for $p = \frac{E_s}{R}$, $i_q = i_p$ and for $p > \frac{E_s}{R}$, $i_q < i_p$.

Note that for any $p \geq \underline{p}$, all agents weakly prefer to sell their assets and consume the proceeds over holding them outright (i.e. without leverage). Since $E_i[min\{s,\bar{s}\}]$ is decreasing in i, agents with $i > i_p$ ($i < i_q$) expect to make strictly positive profits from leveraged asset (collateralised loan) purchases. So all agents with $i > i_p$ ($i < i_q$) sell their assets and invest the proceeds, together with their consumption endowments, in leveraged assets (collateralised loans). Moreover, since for $p = \frac{E_s}{R}$ $i_p = i_q$, and for $p > \frac{E_s}{R}$ any agent with $i : i_q < i < i_p$ strictly prefers selling her assets and consuming, it has to be that $i > i_p$ agents buy all assets, while $i < i_q$ agents buy all collateralised loans, both of which have supply equal to 1. The market clearing condition for leveraged assets thus becomes

$$\int_{i_p}^{1} a_i g(i) di = \int_{i_p}^{1} \frac{n+p}{p-q} g(i) di = \bar{a} = 1 \tag{11}$$

$$\Rightarrow \qquad (n+p)(1-G(i_p)) = (p-q). \tag{12}$$

where the first equality substitutes for a_i from the budget constraint for $i > i_p$ agents with $c_i = 0$, $\bar{a} = 1$ and $b_i = -a_i$. Note that this immediately puts an upper bound $\bar{p}(q)$ on the asset price at the level where even agents of type $i = 1$, whose beliefs are most dispersed and who thus expect to make the highest profit from leveraged asset purchases, do not want to buy assets

$$\bar{p}(q) \leq \frac{E_s - E_1[min(s,\bar{s})]}{R} + q. \tag{13}$$

The market clearing condition for collateralised loans can be written as

$$\int_{0}^{i_q} b_i g(i) di = \int_{0}^{i_q} \frac{n+p}{q} g(i) di = 1 \tag{14}$$

$$\Rightarrow \qquad (n+p)G(i_q) = q. \tag{15}$$

where the first equality substitutes for b_i from the budget constraint for $i < i_q$ agents with $a_i = 0$, $\bar{a} = 1$ and $c_i = 0$. Note that, since i_q is decreasing in q, so are $G(i_q)$ and the left-hand side of (15), which thus provides a unique mapping from the asset price p into a market clearing price $q^\star$, thus defining $i_q^\star$. From (15) and (12) we get

$$p = n\left(\frac{1}{G(i_p) - G(i_q)} - 1\right). \tag{16}$$

Clearly, this equation has no finite solution for $i_q = i_p$. Hence $i_q < i_p$ in equilibrium and thus $p > \frac{E_s}{R}$. Note that, without loss of generality, we have assumed a tie-breaking rule for agents with $i = i_q$ and $i = i_p$ both of mass zero.

Interpretation

Proposition 2 shows how the convexity of payoffs due to leverage allows to exploit perceived gains from trade arising from heterogenous beliefs about payoff dispersion. Investors who perceive risk to be high (low) expect to make strictly positive profits and invest all their funds in leveraged assets (collateralised loans). Market clearing for consumption goods requires that there be a "middle" interval (i_q, i_p) of agents who consume in the first period. For this to be the case, asset prices must exceed their fundamental value $\underline{p}$. In other words, collateralised debt causes a bubble in asset prices.

Figure 4 illustrates the equilibrium. Types $i \leq i_q$ invest the value of their whole endowment (equal to $n+p$) in collateralised loans, with a total demand equal to $b = G_1 \frac{n+p}{q}$ for $G_1 = \int_0^{i_q} dG(i)$. Similarly, the demand for consumption goods, by agents with $i : i_q \leq i < i_p$ equals $c = G_2(n+p)$ for $G_2 = \int_{i_q}^{i_p} dG(i)$. And finally, total asset demand, by agents with $i > i_p$ equals $a = G_3 \frac{n+p}{p-q}$ for $G_3 = \int_{i_p}^{1} dG(i)$.

3.5.3 Comparative statics: increased belief dispersion and asset prices

This section looks at the effect of increasing belief dispersion on asset prices. For this I define an increase in belief dispersion as a perturbation of the distribution of agents $dG(i)$ that reallocates mass from the middle interval $[i_q, i_p]$ to both extremes $[0, i_q], [i_p, 1]$. In other words, we look at a pair of exogenous small changes $dG_1, dG_3 > 0, dG_2 = -(dG_1 + dG_3) < 0$. The following proposition shows how market-clearing prices rise in response to this marginal increase in belief dispersion.

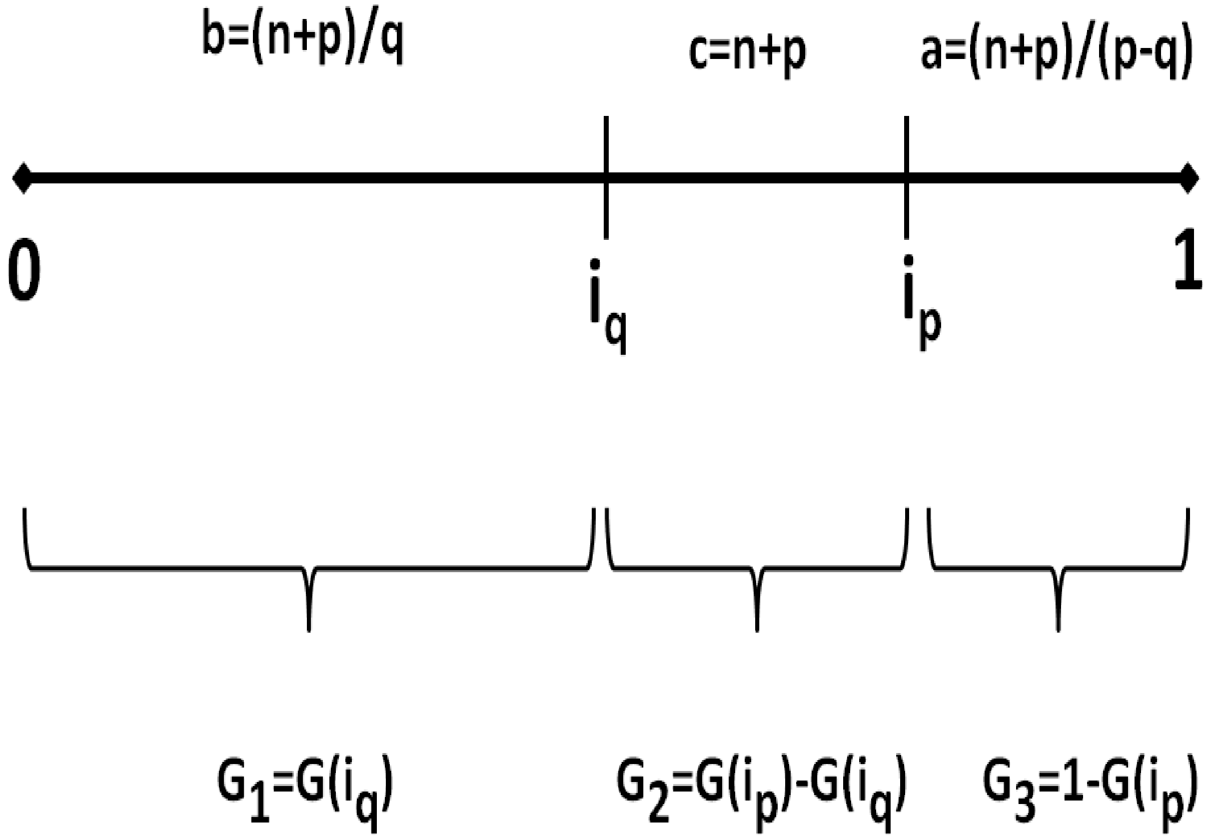
Proposition 3 *Increased belief dispersion raises prices*
A small increse in belief dispersion $dG_1, dG_3 > 0$ *raises prices of both collateralised loans and assets.*

Proof:
Note that at given prices p, q, the change in excess demand for bonds and assets due to the increase in belief dispersion equals individual unit demands multiplied by the change in the mass of agents in the extreme intervals, respectively, $d\widetilde{a} = \frac{n+p}{q} dG_1$ and $d\widetilde{b} = \frac{n+p}{p-q} dG_3$. We are thus looking for a pair of price changes dp, dq that offsets this change to maintain asset market clearing

$$\begin{aligned} db &= -\frac{n+p}{p-q} dG_3 = \frac{n+p}{q} [\frac{\delta G_1}{\delta q} - G_1 \frac{1}{q}] dq \ + \ \frac{1}{q} [\frac{\delta G_1}{\delta p}(n+p) + G_1] dp < 0 \\ da &= -\frac{n+p}{q} dG_1 = \frac{n+p}{p-q} [\frac{\delta G_3}{\delta q} + G_3 \frac{1}{p-q}] dq + \frac{1}{p-q} [\frac{\delta G_3}{\delta p}(n+p) + G_3(1 - \frac{n+p}{p-q})] dp < 0 \quad (17) \end{aligned}$$

Note that, from the definition of G_1, G_3 as well as i_p and i_q in (10) and (9), we have $\frac{\delta G_1}{\delta q} = g(i_q) \frac{\delta i_q}{\delta q} < 0$, $\frac{\delta G_3}{\delta q} = -g(i_p) \frac{\delta i_p}{\delta p} < 0$, $\frac{\delta G_1}{\delta p} = 0$ and $\frac{\delta G_3}{\delta q} = -\frac{\delta G_3}{\delta p} > 0$. Use this, and the

Figure 4



The figure plots the three intervals on [0,1] that correspond to: 1. investors with low belief dispersion ($i \leq i_q$), who have mass G_1 and prefer to buy collateralised loans to consuming or buying assets; 2. investors with medium belief dispersion ($i_q < i \leq i_p$), who have mass G_2 and prefer to consume, rather than invest; and 3. investors with high belief dispersion ($i > i_p$), who have mass G_3 and prefer to buy leveraged assets.

market-clearing conditions $G_1\frac{n+p}{q} = G_3\frac{n+p}{p-q} = 1$,to simplify (17)

$$\begin{aligned} db &= = [\frac{\delta G_1}{G_1 \delta q} - \frac{1}{q}]dq \; + \; \frac{1}{q}G_1 dp \\ da &= [\frac{\delta G_3}{G3\delta q} + \frac{1}{p-q}]dq + [\frac{\delta G_3}{G_3\delta p} + \frac{G_3 - 1}{p-q}]dp \end{aligned}$$

Denoting the vector of price and quantity changes as $\overline{dp}$ and $\overline{dx}$ respectively, and writing

$$\overline{dx} \;=\; A\overline{dp} \Rightarrow \overline{dp} = A^{-1}\overline{dx}$$

we can sign the row i-column j elements of A as $A_{11} < 0$, $A_{12} > 0$, $A_{22} < 0$, $A_{21} > 0$. In other words, the "own-price effects" on asset demand are negative, while the "cross-price effects" (the off-diagonal elements of A) are positive, implying cofactor matrix of A C_A with only negative entries. To conclude the proof, we thus have to show that D_A, the determinant of A, is positive.

$$\begin{aligned} D_A &= [\frac{\delta G_1}{G_1\delta q} - \frac{1}{q}][\frac{\delta G_3}{G_3\delta p} + \frac{G_3-1}{p-q}] \;-\; [\frac{\delta G_3}{G3\delta q} + \frac{1}{p-q}]\frac{1}{q}G_1 \\ &= \frac{1 - G_3 - G_1}{q(p-q)} + \frac{1}{q}\frac{g_3}{G_3}\frac{di_p}{dp}(1 - G_1) + \frac{g_1}{G_1}\frac{di_q}{dq}(-\frac{g_3}{G_3}\frac{di_p}{dp} - \frac{1 - G_3}{p-q}) > 0 \qquad (18) \end{aligned}$$

Interpretation

The proof of proposition 3 shows how we need a rise in both prices to "undo" a rise in excess demand that results from an exogenous increase in belief dispersion. The challenge is three-fold: first, a change in prices changes both the unit demands as well as the size of the intervals G_1 and G_3; second, the unit demands comprise the asset endowment, leading to a positive effect of a rise in asset prices on both quantities; and finally, the cross-price effect of a rise in the price of collateralised loans $dq > 0$ on asset demand is positive, as it makes it cheaper to raise outside funds. The proof exploits market-clearing, and the fact that an equal increase in dp and dq leaves leaveraged asset demand (excluding the endowment effect) unchanged to show that the own price effects dominate the endowent and cross-price effects.

3.5.4 Endogenous choice of $\overline{s}$

So far, we have taken $\overline{s}$, the face value of the loan, as exogenous. This section looks at the optimal choice of $\overline{s}$ subject to an upper bound: $s \leq \overline{s}^{max}$. In other words, we assume that there are some non-modelled features of the economy that put an upper bound to the riskyness of collateralised loans.

The net benefit of a marginal change $d\overline{s}$ to an investor in leveraged asset equals the additional returns on outside funds that increase when selling collateralised loans at a higher price, equal to

$R_i^a \frac{\delta q(\bar{s})}{\delta \bar{s}}$, minus the increase in expected payments on the loan, equal to 1 minus the probability of default $(1 - F_i(\bar{s}))$.

$$\frac{d\Pi_i^a}{d\bar{s}} = \frac{n+p}{p-q}[R_i^a \frac{\delta q(\bar{s})}{\delta \bar{s}} - (1 - F_i(\bar{s}))] \tag{19}$$

Conversely, the net beneft to an agent j from increasing the $\bar{s}$ of the collateralised loan she purchases equals the expected rise in payments $(1 - F_j(\bar{s}))$ minus the loss in profits due to a reduced quantity of loans she can afford at the higher price, equal to $\frac{E_j[min(s,\bar{s})]}{q} \frac{\delta q(\bar{s})}{\delta \bar{s}}$

$$\frac{d\Pi_j^l}{d\bar{s}} = \frac{n+p}{q}[(1 - F_j(\bar{s})) - \frac{E_j[min(s,\bar{s})]}{q} \frac{\delta q(\bar{s})}{\delta \bar{s}}] \tag{20}$$

In order to characterise the equilibrium with endogenous leverage choice we make the following additional assumption:

A2: $\bar{s}^{max} \leq s^\star = min_i min_j (s : F_i(s) = F_j(s) \quad j, i \in [0,1]) > s_{min}$

Note that second-order stochastic dominance implies single-crossing of "adjacent" distributions F_i. The assumption ensures that the maximum leverage is smaller than the minimum of all single-crossing points. For example, if we were to restrict our attention to beliefs that are symmetric around E_s, then we would have $s^\star = E_s$ and $\bar{s}^{max} \leq E_s$, which is equivalent to assuming that the bankruptcy probability cannot exceed fifty percent. The following proposition shows that under this assumption there cannot be an equilibrium with trade in collateralised loans of face value below $\bar{s}^{max}$.

Proposition 4 *Degenerate choice of* $\bar{s}$
Under assumption A_2*, only one collateralised loan contract with* $\bar{s} = \bar{s}^{max}$ *is traded in equilibrium.*

Proof
The choice of $\bar{s}$ depends crucially on the slope of the equilibrium price function $q(\bar{s})$. Note that, for any $\bar{s}$ traded in equilibrium, it has to be that profits expected by the marginal buyer $i_q(\bar{s})$ are weakly decreasing from any $d\bar{s} > 0$. Thus

$$\frac{d\Pi_{i_q}(d\bar{s})}{d\bar{s}}^+ \leq 0 \Rightarrow \frac{\delta q(\bar{s})}{\delta \bar{s}}^+ \geq \frac{(1 - F_{i_q}(\bar{s}))}{R} \tag{21}$$

where $\frac{d\Pi_{i_q}(\bar{s})}{d\bar{s}}^+$ denotes the right-hand-side derivative of profits with respect to $\bar{s}$. We can substitute this into (19), to get

$$\frac{d\Pi_i^a}{\bar{s}}^+ \geq \frac{n+p}{p-q}[\frac{R_i^a}{R}(1 - F_{i_q}(\bar{s})) - (1 - F_i(\bar{s}))] \geq \frac{n+p}{p-q}[(1 - F_{i_q}(\bar{s})) - (1 - F_{i_p}(\bar{s}))] > 0 \tag{22}$$

where the second-to-last inequality follows from $R_i^a \geq R \forall i \geq i_p$, and $(1-F_i(s)) \leq (1-F_{i_p}(s) \forall i \geq i_p$, and the last inequality follows from $(1 - F_{i_q}(\overline{s})) - (1 - F_{i_p}(\overline{s})) > 0 \forall \overline{s} < s^{\star}$. So agents only want to issue loans with maximum leverage $\overline{s}^{max}$.

Interpretation: The marginal buyer of a loan with face value $\overline{s}$ has to weakly prefer that face value to a slightly higher one. This puts a lower bound on the slope of the price function $q(\overline{s})$ at that point. Moreover, for $\overline{s} < \overline{s}^{\star}$, higher i implies a higher bankruptcy-probability, so the additional payment an issuer expects to make on collateralised loans from a small rise $d\overline{s} > 0$ falls with belief dispersion i. Since issuers of loans have higher i than buyers, this implies that at $\overline{s} < \overline{s}^{\star}$ issuers of collateralised loans gain more from a rise in prices $\frac{\delta q(\overline{s})}{\delta \overline{s}} > 0$ than they loose from higher expected payments. So they choose the maximum face value and leverage, equal to $\overline{s}^{max}$. Note that the assumption of an upper bound for the face value $\overline{s}$ is crucial here. Without it, issuers of collateralised loans would potentially choose different $\overline{s}$ and we would face a complicated assignment problem of issuers and buyers of collateralised loans to face values. Geerolf (2014) solves this assignment problem under one particular kind of disagreement about mean payoffs, namely point expectations. With disagreement about risk, this problem becomes, to our knowledge, untractable. This is why the next section looks at a simplified environment with two types that allows to analyse endogenous leverage, and more complex collateralised debt contracts.

4 Leveraged asset trade in a two-type economy

This section looks at an economy with two types $i \in \{0, 1\}$. The environment is similar to Simsek (2013) who, however, assumes that beliefs of one group first-order stochastically dominate those of the other. He shows that when optimists have to partly finance asset purchases through collateralised loans, the resulting leverage amplifies the effect of "upside optimism" about high payoff realisations (which increases expected returns of leveraged assets) but disciplines "downside optimism" about low payoffs (since profits are 0 independently of the exact realisation of s below $\overline{s}$). When agents differ in the perceived dispersion of payoffs, however, investors with more dispersed beliefs can be seen as "upside optimists" and "downside pessimists", in Simsek (2013)'s language. The result that leveraged investments amplify upside optimism but dampen downside optimism then implies that disagreement about second moments of payoffs necessarily leads to scope for asset trade and a rise in prices above their common fundamental value. Moreover, we show how an increase in belief-divergence further inflates this price bubble as long as investors are sufficiently cash rich.

We assume that the distribution g has two mass points at $i \in \{0, 1\}$, corresponding to two groups of agents, each of unit mass, whose beliefs satisfy $f_0 \succ^2 f_1$, as before. We call type 0 agents, with less dispersed beliefs about payoffs, "more confident", and type 1 agents, who believe payoffs to be more dispersed, "less confident". Since as before $\Pi_1^l <= \Pi_0^l, \Pi_1^a > \Pi_0^a \ \forall \ \overline{s} \in (s_{\min}, s_{\max}), \forall p, q, R$, more confident agents are the natural buyers of collateralised loans, and less confident agents the natural investors in leveraged assets. In other words, if there is trade in collateralised loans in equilibrium $-b_1 = b_0 > 0$. Agents are endowed with 1 unit of the asset and n_1, n_0 units of the consumption good respectively. So the value of their period 1 endowment in consumption goods is $n_i + p \ i = 0, 1$. We assume that more confident type 0 agents are cash-rich

$$A3 : n_0 \geq \frac{E_s}{R}.$$

This assumption is similar to Simsek (2013)'s assumption *A1* of cash-richness. Under assumption $A3$ any asset price below the fundamental value $\frac{E_s}{R}$ would lead to excess asset demand for loans, so $p \geq \frac{E_s}{R}$. Moreover, given $A3$, the total value of type 0 agents' endowment equals $n_0 + p \geq 2\frac{E_s}{R} \geq 2max_{\overline{s}}q(\overline{s})$. So type 0 agents can afford to buy all collateralised loans at their maximum expected value. In turn, this implies that 0 types bid up the price of any collateralised loan issued by type 1 agents to their expected discounted value, where they are indifferent between investing and consuming, implying a bond price function

$$q(\overline{s}) = \frac{E_0[\min\{s, \overline{s}\}]}{R} \tag{23}$$

4.1 Type 1's problem and the choice of $\overline{s}$

In contrast to the exogenous $\overline{s}$ in the previous section, less confident type 1 agents now choose both current consumption, which through the budget constraint determines their investments, and the level of leverage $\overline{s}$ given p and the price function $q(\overline{s})$.

$$\begin{aligned} \max_{c_1, \overline{s}} \quad U_1 &= c_1 + \frac{(n_1 + p - c_1)}{R} \frac{[E_s - E_1(\min\{s, \overline{s}\})]}{p - \frac{E_0\{\min\{s, \overline{s}\}\}}{R}} \\ &= c_1 + \frac{(n_1 + p - c_1)}{R} R_1^a \end{aligned} \tag{24}$$

where again $R_1^a \doteq \frac{[E_s - E_1(\min\{s, \overline{s}\})]}{p - \frac{E_0\{\min\{s, \overline{s}\}\}}{R}}$ is the leveraged gross return of the asset using a loan with riskiness $\overline{s}$. The first order condition for $\overline{s}$ can be written as

$$\frac{(n_1 + p)}{Rp - E_0\{min\{s, \overline{s}\}\}}[(1 - F_1(\overline{s})) - \frac{R_1^a}{R}(1 - F_0(\overline{s}))] = 0 \tag{25}$$

Proposition 4: Interior choice of $\overline{s}$.

Suppose that p is such that $\frac{E_s}{R} = \underline{p} < p < \overline{p} \doteq \frac{E_s + E_0(min(s, \overline{s})) - E_1(min\{s, \overline{s}\})}{R}$ holds for some

$\overline{s} \in (s_{min}, s_{max})$, *such that agent 1 expects to make profits for some* $\overline{s}$ *when she buys assets at* p *that exceeds the fundamental value. Then* $R_1^a(p, \overline{s})$ *has an interior maximum at some* $\overline{s}^\star \in (s_{min}, s_{max})$.

Proof: Note that $R_1^a(p, s_{max}) = 0$. Also, if $p > \frac{E_s}{R}$, $R_1^a(s_{min}) = \underline{R_1^a} < R$. But if at some $\overline{s}'$ $p < \frac{E_s + E_0(min\{s,\overline{s}'\}) - E_1(min(s,\overline{s}'))}{R}$, then $R_1^a(\overline{s}') > R$. The statement then follows from continuity of R_1^a.

4.2 Equilibrium Characterisation

Definition: A general equilibrium is an endogenous face value of collateralised loans $\overline{s}$, a set of prices $(p,\ q)$ and allocations $(c_i, c_i', a_i, b_i)_{i\in\{0,1\}}$, such that (23) holds, agent 1 solves the optimization problem (24), the demand for assets equals the fixed supply,

$$a_0 + a_1 = 2$$

and the collateralized loan maket clears,

$$b_1(\overline{s}) + b_0(\overline{s}) = 0 \ \forall \overline{s}.$$

The following proposition shows that equilibrium is defined by two conditions: first, the optimal choice of leverage $\overline{s}$; and second, the market clearing for leveraged assets, which defines the price such that type 1 agents either exhaust all their wealth buying assets, or are indifferent between investing and consuming. Intuitively, as agent 1 wealth rises, their increasing demand for assets bids up the price until it reaches indifference level $\overline{p}$.

Proposition 5: Existence and uniqueness of equilibrium.
Denote as $n_1^{\max}(\overline{s}) = n_1 + 2\frac{E_0[\min\{s,\overline{s}\}]}{R}$ *the resources available to type 1 agents for net purchases of assets when they issue collateralised loans backed by the whole asset endowment of the economy.* p *and* $\overline{s}$ *are given by the unique solution of the following equations*

$$\mathbb{C} \equiv [E_s - E_1(\min\{s,\overline{s}\})](1 - F_0) - (1 - F_1)(Rp - E_0[min\{s,\overline{s}\}]) = 0 \quad (26)$$

$$p = max\{\overline{p}, p^\star\} \quad (27)$$

$$p^\star = n_1^{\max}(\overline{s}) \quad (28)$$

where the lefthand side of (28) *are the net purchases of assets and the right-hand side equals the available resources, both weighed by the mass of type* 1 *agents.*

Proof: Equation (26) is simply the optimality condition for leverage choice. To understand equations (27) and (28), note that for any $p < \frac{E_s}{R}$ all agents would like to buy risky assets, which cannot be an equilibrium. Equivalently, for any $p > \overline{p} \doteq \frac{E_s + E_0(min(s,\overline{s})) - E_1(min\{s,\overline{s}\})}{R}$ both type 0 and type 1 agents would like to sell their risky assets, again contradicting equilibrium. Agent 1 optimality implies that they invest all resources in leveraged assets when $\frac{E_s}{R} \leq p < \overline{p}$, but are indifferent between buying leveraged assets and consuming at $p = \overline{p}$. Thus, for $\overline{s}(\overline{p})$ the value of $\overline{s}$ that solves (26) when $p = \overline{p}$, if $n_1^{\max}(\overline{s}(\overline{p})) \geq \overline{p}$, type 1's endowment is large enough to buy type 0's assets at the maximum price $\overline{p}$ that ensures her participation. There is thus an equilibrium price $\overline{p}$ at which type 1 agents are happy to consume in period 0 any resources that remain after purchasing all of type 0's assets.

If for some price $p : \frac{E_s}{R} \leq p < \overline{p}$ $n_1^{\max}(\overline{s}(p)) < p$, type 1 agents cannot buy all assets at that price but expect to make strictly positive profits $R_1^a > R$, so invest all their resources to buy type 0's assets, implying market clearing condition (28).

Finally, to prove uniqueness, since (28) is trivially strictly upward-sloping, it suffices to show that (26) is downward-sloping. This follows by differentiating (26) totally

$$\frac{dp}{d\overline{s}} = -\frac{\frac{d\mathbb{C}}{d\overline{s}}}{\frac{d\mathbb{C}}{d\overline{p}}} \tag{29}$$

Weak concavity of $R_1^a(\overline{s})$ at the optimum choice of $\overline{s}$ implies that the numerator is weakly negative. Since $\frac{\mathbb{C}}{d\overline{p}} < 0, \forall p, \overline{s}$ the result follows.

Interpretation: Similarly to the continuum economy, with heterogeneity in perceived risks across two types, unique equilibrium prices of risky assets are necessarily above their common fundamental valuation. So with heterogeneous risk perceptions, collateralised contracts lead to a bubble in asset prices also in the two type economy. Moreover, it is easy to see from (28) that a rise in resources of type 1 agents (weakly) increases prices. Unlike the continuum economy, however, there is now a unique endogenous choice for leverage $\overline{s}$.

4.3 Comparative statics

This section looks at the effect of "belief-divergence", in the sense of a further mean-preserving contraction to f_0, or equivalently a dispersion of f_1. For this, we concentrate on economies where funds of less confident type 1 agents are large either because their endowments are high, or because they can raise enough funds from issuing collateralised loans.

$$A4 : n_1 \geq \underline{n_1} = \frac{E_s - E_0(min\{s, s^\star\}) - E_1(min\{s, s^\star\})}{R}.$$

$A4$ implies that asset prices are at their upper bound, as the following corollary states

Lemma 1 *$A4$ implies $p = \overline{p}$.*

Proof
Under A4 we have

$$\begin{aligned} n_1^{\max}(s^\star) &\geq \frac{E_s - E_0(min\{s,s^\star\}) - E_1(min\{s,s^\star\}) + 2E_0[\min\{s,s^\star\}]}{R} \\ &\geq \frac{E_s + E_0[\min\{s,\overline{s}\}] - E_1(min\{s,s^\star\})}{R} = \overline{p} \end{aligned} \tag{30}$$

which implies that type 1 agents have ressources larger than the value of assets evaluated at any $p \leq \overline{p}$. So equilibrium requires type 1 agents to be indifferent between consuming and investing in leveraged assets, implying an equilibrium price equal to $\overline{p}$.

Corollary 1 *For any symmetric distributions f_1, f_0, $\underline{n_1} < 0$. So A4 trivially holds.*

Note that for any symmetric distribution $E_s = s^\star = \frac{1}{2}(s_{max} + s_{min})$. So

$$\begin{aligned} \underline{n_1} = \frac{E_s - E_0(min\{s,s^\star\}) - E_1(min\{s,s^\star\})}{R} \\ \leq \frac{E_s - 2s^\star(1 - F_0(s^\star))}{R} \\ = \frac{E_s - 2E_s\frac{1}{2}}{R} = 0 \end{aligned} \tag{31}$$

were the last line follows from $s^\star = E_s$ and $1 - F_0(s^\star) = 1 - F_1(s^\star) = \frac{1}{2}$ due to symmetry.

To look at belief divergence we assume that the distribution function f_i is parameterised by a variable v such that

1. f_i is continuous in v for all s, $i = 0, 1$
2. $E_{i,v}(s) = E_s, \forall v$, $i = 0, 1$
3. $f_0(v)$ second order dominates $f_0(v')$ whenever $v > v'$
4. $F_0(v, s) - F_0(v', s)$ is downward sloping in s whenever $v > v'$ and crosses the zero line once at $s^\star$.

In the following we define 'belief-divergence' as small changes in the beliefs of more and less confident types f_0, f_1 through a pair of small changes in their corresponding values of v $dv_0 \geq 0, dv_1 \leq 0$ with at least one strict inequality, corresponding to a mean-preserving contraction to f_0 and a mean-preserving spread to f_1.

From the pricing equation for bonds (23), it is immediately clear that $dv_0 > 0$ increases the valuation of collateralised loans by type 0 agents, and thus their price.

Lemma 2 *A fall in risk perceived by type* 0 *increases prices of collateralised loans* $\frac{\delta q}{\delta v_0} > 0$

Also, under assumption $A4$ we can identify the effect of belief-divergence on asset prices

Proposition 5 *Belief-divergence increases asset prices*
Under A4, $\frac{\delta p}{\delta v_0} > 0$ *and* $\frac{\delta p}{\delta v_1} < 0$.

Proof: $A4$ implies $p = \overline{p} = \frac{E_s + E_0(min(s,\overline{s})) - E_1(min\{s,\overline{s}\})}{R}$, and $F_0(\overline{s}) = F_1(\overline{s})$ from (26), so $\overline{s} = s^\star$ from single-crossing. Since $s^\star$ does not change in response to dv_i, neither does $\overline{s}$. But $\overline{p}$ rises with a mean preserving spread $dv_0 \geq 0, dv_1 \leq 0$ as $\frac{\delta E_0[min\{s,s^\star\}]}{\delta v_0} \geq 0$ and $\frac{\delta E_0[min\{s,s^\star\}]}{\delta v_1} \leq 0$.

Discussion

A mean-preserving contraction in f_0 is equivalent to lenders updating their beliefs to a lower level of risk. This increases the expected payoff from a collateralised loan of given riskyness, and thus increases the amount they are willing to pay for collateralised loans. For investors, this always increases expected profits at a given asset price and level of loan riskyness, and thus raises $\overline{p}$. Under $A4$, less confident type 1 agents have enough resources to drive up the equilibrium price to $p = \overline{p}$, so increased attractiveness of leveraged investments immediately raises asset prices. When $A4$ does not hold, a mean preserving spread in beliefs has an ambiguous effect on marginal profits and thus the optimal value of riskyness $\overline{s}$. Specifically, while a rise in v_0 increases the return at any given riskyness, in an asymmetric equilibrium it can increase or decrease $1 - F_0$, the marginal effect of a change in $\overline{s}$ on profits at given returns. Under $A4$ this effect is absent as $p = \overline{p}$.

5 Collateralised Debt Obligations[8]

This section analyses the equilibrium of the two-type economy when agents can trade a more generalised set of collateralised contracts. Particularly, we look at trade in Collateralised Debt Obligations (CDOs). CDOs are debt-like instruments, whose payments are, typically, derived on the basis of an underlying pool consisting of a large number of credit contracts (unsecured credit to households, mortgage contracts, etc). Specifically, the issuer of a CDO sells the cash-flow of the credit-pool in 'tranches' that correspond to ordered percentiles of credit repayments. Thus, a unit-debt obligation collateralised by the xth percentile of credit payments is a promise to pay 1 dollar to the buyer if at least x percent of the creditors pay, and 0 otherwise. Tranches with

[8]The results in this section have benefited from a conversation with Julian Kolm.

low x are called 'senior', while those with high (highest) x are 'junior' ('equity') tranches. Note that as before, in the absence of a cash technology, all claims are ultimately collateralised by the asset, which is in fixed supply. This explains the difference of the results in this section with respect to those in, for example, Fostel and Geanakoplos (2012) or section 6 in Simsek (2013), where agents can also use a cash technology to collateralise claims.

In our framework, for $s \in [s_{min}, s_{max}] = [0, 1]$, $f_i(s)$ is the probability that exactly a fraction s of the underlying credit pool pays off. A CDO is a payment $\Psi(x)$ that equals 1 unit of consumption when $s \geq x$ and 0 otherwise. We call x the 'seniority' of the CDO (although it is strictly its 'juniority'). Its expected payoff to agent i equals $E_i[\Psi(x)] = 1 - F_i(x)$. From the single-crossing property of second-order stochastic dominance, we know that there is a value $s^\star$ such that $E_1[\Psi(x)] \geq (<) E_0[\Psi(x)]$ when $x \geq (<) s^\star$. Thus, less confident agents, with more volatile beliefs, are the natural buyers of the junior and equity tranches. We denote CDO prices as $Q(x)$. Note that when the asset is tranched fully into CDOs, the total payout for a realisation s equal $\int_0^s 1 \times dx = s$. So CDO payments exhaust the total payoffs.

5.1 Trade in primary CDOs

Consider the two type environment from the previous section, and assume, for now, that agents can only issue CDOs backed by the original asset payoffs. So $B_i(x) \geq -a_i$. This implies a budget constraint

$$c_i = n_i + p - pa_i - \int_0^1 Q(x) B_i(x) dx \tag{32}$$

$$c_i' = a_i s + \int_0^s B_i(x) dx \tag{33}$$

where now $B_i(x)$ denote i's purchases of CDOs of seniority x. Type i's problem is thus

$$\max_{c_i, c_i', a_i \geq 0, B_i(x) \geq -a_i} c_i + \frac{1}{R} E(c_i') \tag{34}$$

subject to

(32) and (33)

Definition: A general equilibrium is a set of prices of the asset p and of CDOs with seniority x $Q(x)$, and an allocation $(c_i, c_i', a_i, B_i(x))_{i \in \{0,1\}}$, such that given prices $(c_i, c_i', a_i, B_i(x))_{i \in \{0,1\}}$ solve agent i's problem (35), the demand for assets equals the fixed supply,

$$a_0 + a_1 = 2$$

and the market for primary CDOs clears,

$$B_1(x) + B_0(x) = 0 \ \forall.$$

Note that buying all CDOs with seniority $x : x \in X$ for a set $X \subset [0,1]$ is equivalent to buying the asset and selling all CDOs in the complementary set $X^{-1} = [0,1]\backslash X$. We will normalise contracts such that type 1 agents buy the asset and sell CDOs to type 0 agents. The latter will want to buy all CDOs with $Q(x) < (1 - F_0(x))$. Under assumption $A3$, they can afford this because $n_0 + p \geq 2\frac{E_s}{R} = 2\int_0^1 (1 - F_0(s))ds \geq 2\int_{x:Q(x)<(1-F(0))} 1 \times Q(x)dx$. Thus, market clearing requires $Q(x) \geq (1 - F_0(x))$. In other words, type 0 agents drive up the price of all CDOs to at least their reservation value. Since $1 - F_0(x) > 1 - F_1(x)\ \forall x < s^\star$, more confident type 0 agents value CDOs with seniority below the single-crossing point $x < s^\star$ strictly more than the less confident type 1 agents. Therefore, type 1 agents do not buy any CDOs with $x < s^\star$ at $Q(x) \geq (1 - F_0(x))$, as this implies an expected loss. Note, however, that at price $Q(x) = (1 - F_0(x))$ type 1 agents expect to make a strict profit from keeping CDOs with seniority $x > s^\star$, which they value more than type 0. Note also that for any realisation of the asset payoff $s \in [0,1]$ the second period payments on CDOs with seniority $x > s^\star$ in the second period equal exactly the payments on a collateralised loan with face value $s^\star$, since $\int_0^{min\{s,s^\star\}} 1 \times dx = \min\{s, s^\star\}$. This implies, if assumptions $A3$ and $A4$ are satisfied, that the asset price with primary CDO trade equals that with trade in collateralised loans.

Proposition 6 *Under assumptions* $A3$ *and* $A4$*, the equilibrium asset price* p *in the economy with trade in CDOs equals that in an economy with trade in collateralised loans.*

Proof
According to lemma 1 the asset price with trade in collateralised loans equals $\overline{p}$. At this price, according to (26) the optimal choice of face value for loans $\overline{s}$ is equal to the single crossing point of the CDFs $s^\star$.
With trade in CDOs, $Q(x) = (1 - F_0(x))$ for $x < s^\star$ as type 0 agents are cash-rich. When type 1 agents buy assets, they thus find it optimal to issue all CDOs with seniority $x > s^\star$ to type 0 agents. This implies type 1's return on funds invested in the asset and partly financed through CDO issuance equals

$$R_1^{aCDO} = \frac{E_s - \int_0^{s^\star}(\int_s^1 f_1(x)dx)ds}{p - \frac{\int_0^{s^\star}(1-F_0)(x)dx}{R}} = \frac{E_s - E_1[min\{s, s^\star\}}{p - \frac{E_0[min\{s,s^\star\}]}{R}} \tag{35}$$

where the last line follows since $\int_s^1 f_1(x)dx = 1 - F_1(s)$ and

$$\int_0^{s^\star} (1 - F_i(x))dx = s^\star - [xF_i(x)|_0^{s^\star} - \int_0^{s^\star} xf_i(x)dx]$$
$$= (1 - F_i(s^\star))s^\star + \int_0^{s^\star} xf_i(x)dx = E_i[min\{s, s^\star\}] \tag{36}$$

Equation (36) states that both the proceeds from CDO issuance in the first period and the expected payments by type 1 agents in the second period are equal to those from the issuance of collateralised loans in the previous section. The indifference condition $R_1^{aCDO} = R$ thus implies a reservation price of $\overline{p} = \frac{E_s + E_0(min(s,\overline{s})) - E_1(min\{s,\overline{s}\})}{R}$, exactly equal to that with collateralised loan trade. Since the payments in period 1 are exactly the same as with collateralised loan trade, type 1 agents can afford to buy all assets at this price.

Discussion

Proposition 6 shows how 'horizontal' tranching of payoffs through CDOs does not allow agents to exploit more gains from trade than with trade in collateralised loans. This is because the decision by type 1 agents to issue a CDO of seniority x has exactly the same marginal cost (in terms of additional payments in the second period) and benefits (in terms of funds raised today) as a marginal increase in the face value of a loan at $\overline{s} = x$. Costs are proportional to $1 - F_1(x)$ while benefits are proportional to $1 - F_0(x)$. So single-crossing implies that the additional flexibility that CDOs allow is not used in equilibrium.

5.2 Trade in primary and secondary CDOs

A buyer of a CDO tranche of seniority x can easily use the cash-flow from this asset to collateralise an additional asset that pays 1 dollar whenever $s \geq x' \geq x$. We call such an asset equivalently a 'secondary CDO' of seniority x'. Note that unlike so-called 'synthetic' CDOs, which are pure derivative contracts, secondary CDOs are not sold 'naked', but collateralised by more senior primary or secondary CDOs. Denoting by $\widehat{B}_i(x)$ the amount of secondary CDOs of seniority x held by agent 1, this implies a 'downward collateral constraint'

$$\widehat{B}_i(x) \geq -\int_0^x (B_i(s) + \widehat{B}_i(s))ds, \ \forall x \in [0,1] \tag{37}$$

and budget constraints

$$c_i = n_i + p - pa_i - \int_0^1 Q(x)(B_i(x) + \widehat{B_i}(x))dx \tag{38}$$

$$c_i' = a_i s + \int_0^s (B_i(x) + \widehat{B_i}(x))dx \tag{39}$$

which exploit the arbitrage condition that primary and secondary CDOs of seniority x must have the same price $Q(x)$, as they represent the same claims.

Trade in secondary CDOs allows agents to trade claims to any range of payoff in S. To see how this increases expected profits, consider the equilibrium prices with primary CDO trade of the previous section and a type 0 agent who has purchased a primary CDO of seniority $x < s^\star$ from type 1 agents. At the price $Q(x) = 1 - F_0(x)$, she is exactly indifferent between consuming or buying the CDO. Introducing secondary CDOs, however, allows her to sell of the tail risk $1 - F_0(x'), x' > s^\star$ by issuing a new CDO collateralised by the original primary CDO. Type 1 agents, who expect to make 0 profits on their asset portfolio, are happy to buy this secondary CDO at price $Q(x') = 1 - F_1(x')$. Expected profits for type 0 agents from this pair of trades are strictly positive as $1 - F_0(x) - (1 - F_0(x')) - (Q(x) - Q(x')) = F_0(x') - F_1(x') > 0$.

More formally, with trade in primary and secondary CDOs, type i's problem becomes

$$\max_{c_i, c_i', a_i \geq 0, B_i(x) \geq -a_i, \widehat{B_i}} c_i + \frac{1}{R} E(c_i') \tag{40}$$
$$\text{subject to}$$
$$(38), (39) \text{ and } (37)$$

Definition: A general equilibrium is a price of the asset p and of CDOs with seniority x $Q(x)$, and an allocation $(c_i, c_i', a_i, B_i(x), \widehat{B_i}(x))_{i \in \{0,1\}}$, such that given prices $(c_i, c_i', a_i, B_i(x), , \widehat{B_i}(x))_{i \in \{0,1\}}$ solve agent i's problem (40), the demand for assets equals the fixed supply,

$$a_0 + a_1 = 2$$

and the market for both primary and secondary CDOs clears,

$$B_1(x) + B_0(x) = 0 \;\forall x$$
$$\widehat{B_1}(x) + \widehat{B_0}(x) = 0 \;\forall x$$

Note, again, how we have imposed the arbitrage condition that, for a given seniority x, the prices of primary CDOs need to equal those of secondary CDOs, in our definition of equilibrium. It is easy to see how trade in secondary CDOs completes the asset market, as the joint purchase and sale of CDOs with seniority x and $x + dx$ respectively is equivalent to the purchase of an asset that is contingent on state x as $dx \longrightarrow 0$. Proposition 7 shows how, even in the presence of collateral constraints, this implies asset prices equal to those that would pertain with unconstrained trade of Arrow-Debreu securities. For this to be true, we have to replace assumption $A4$ by an alternative condition that ensures type 1 agents have enough resources to buy the assets that they value highly at their own valuation.

$$A4': \; n_1 \geq E_s - \int_{X_0} s f_0(s) ds \tag{41}$$

where X_0 is defined as the set of states that type 0 agents perceive as more likely $X_0 = \{x \in [0, 1] : f_0(x) > f_1(x)\}$, and X_1 is its complement $X_1 = [0, 1] \backslash X_0$.

Proposition 7 *There is an equilibrium with trade in secondary CDOs where the asset price equals*

$$p^{SCDO} = \frac{\int_0^1 s \;\; max_i\{f_i(s)\}}{R} ds \tag{42}$$

Proof[9] Note that, as in the previous section with trade in primary CDOs only, portfolios are not uniquely defined in equilibrium. So, again, we normalise portfolios such that agents of type 1 buy all the assets. Suppose that they also buy all secondary CDOs with seniority $x \in X_1$ and issue primary CDOs (backed by their asset holdings) and secondary CDOs (backed by more senior secondary CDOs) for any $x \in X_0$. Suppose also that type 0 agents buy all primary CDOs with seniority $x \in X_0$ and sell secondary CDOs (backed by any more senior CDOs they hold) with seniority $x \in X_1$. This leaves type 1 agents with claims from their asset holdings and CDO purchase equal to $2s + \int_0^s B_1(x) + \widehat{B_1(x)}dx = 2s$ for all $s \in X_1$, and 0 otherwise. This is because for any $s \in X_1$ type 1 agents buy secondary CDOs that exactly equal their previous issuance of more senior CDOs, leaving a 0 net claim from CDOs, such that total net claims equal those from initial asset holdings. For $s \in X_0$, in contrast, claims from primary CDO issuance are equal to $-2s$ and claims from secondary CDOs bought and sold cancel, leaving a zero claim to payoffs overall. Similarly, net claims by type 0 agents due to their purchase and sale of CDOs equal $\int_0^s B_1(x) + \widehat{B_1(x)}dx = 2s$ for all $s \in X_0$, and 0 otherwise.
Suppose $Q(1) = 0$, $\frac{\delta Q}{\delta s} = -\frac{max_i\{f_i(s)\}}{R}$, and that the asset is priced by arbitrage $p = p^{SCDO}$. To show that this is an equilibrium, note that all agents expect to make 0 profits from their trading strategies. Moreover, any deviation from agent 1's strategy by buying and selling CDOs of seniority $x \in X_0$ and $x' > x$ respectively, implies a loss as expected profits equal $E_1[s \in [x, x']] - [Q(x) - Q(x')] = \int_x^{x'} f_1(x)dx + \int_x^{x'} \frac{\delta Q(x)}{\delta x}dx = \int_x^{x'} (f_1(x) - max\{f_0(x), f_1(x)\})dx < 0$ (and equivalently for type 0 agents). Finally, agents are indifferent between consuming and buying the asset at p^{SCDO}, partly financed through appropriate issuance of CDOs. So trading strategies are optimal. To show that they are also affordable, write the period 1 budget as endowment plus net claims sold minus net claims purchased

$$\begin{aligned} n_0 + p^{SCDO} - \frac{2\int_{X_0} sf_0(s)ds}{R} &\geq \frac{E_s + \int_{X_1} sf_1(s)ds - \int_{X_0} sf_0(s)ds}{R} \geq 0 \\ n_1 + \frac{2\int_{X_0} sf_0(s)ds}{R} - p^{SCDO} &= n_1 + \frac{\int_{X_0} sf_0(s)ds - \int_{X_1} sf_1(s)ds}{R} \geq 0 \end{aligned} \tag{43}$$

where the last inequality follows from assumption $A4'$.

[9] The proof considers the case where the mass of $x : f_1(x) = f_0(x)$ is zero, such that agents almost surely disagree on the pdf. This is without loss of generality, as it is trivial to account for 'regions of agreement'.

Discussion

Proposition 7 shows how trade in secondary CDOs allows agents to exploit all perceived gains from trade, as they can separately trade claims to any subset of S. When the economy is sufficiently cash-rich, and given the collateral constraints that restrict the asset supply, this drives up the price of a portfolio of primary and secondary CDOs that has a unit net payoff in state s to the maximum valuation across agents. The following corollary shows how this equilibrium drives up the price of debt in line with that of assets.

Corollary 2 *There is an equilibrium with trade in secondary CDOs where the price of CDO contracts equals*

$$Q^{SCDO}(x) = \frac{\int_x^1 \ max_i\{f_i(s)\}}{R} ds \tag{44}$$

The next proposition compares the asset price with trade in secondary CDOs to that with collateralised trade or primary CDO trade.

Proposition 8 *When f_1, f_0 are symmetric and have no mass points, the asset price bubble $p^{SCDO} - \frac{E_s}{R}$ with trade in secondary CDOs is at least twice as large as that with trade in collateralised loans or primary CDOs.*

Proof

$$\begin{aligned}
p^{SCDO} * R &= \int_0^1 s \ max\{f_i(s)\}ds = E_s + \int_0^1 s(max\{f_i(s)\} - f_1(s))ds \\
&= E_s + \int_0^{\frac{1}{2}} (s + (1-s))(max\{f_i(s)\} - f_1(s))ds \\
&\geq E_s + 2[\int_0^{\frac{1}{2}} s(f_0 - f_1))ds + \int_{\frac{1}{2}}^1 \frac{1}{2}(f_0 - f_1)ds \\
&= E_s + 2[E_0[min\{\frac{1}{2}, s\} - E_1[min\{\frac{1}{2}, s\}] \\
&\Rightarrow p^{SCDO} - \frac{Es}{R} \geq 2(\overline{p} - \frac{Es}{R})
\end{aligned} \tag{45}$$

where the second line follows from symmetry, the third follows since $1 - s \geq s$ for $s \leq \frac{1}{2}$ and $fmax\{f_i(s)\} \geq f_0(s)$ as well as the fact that $\int_{\frac{1}{2}}^1 c(f_0 - f_1) = 0$ for any constant c when f_0, f_1 are symmetric around $\frac{1}{2}$.

Discussion

Proposition 8 shows how trade in secondary CDOs allows agents to at least double the bubble in asset prices, as they can exploit differences in beliefs more efficiently. Particularly, while both collateralised loans and primary CDOs only allow to trade on differences in agents' CDFs, secondary CDOs allow trade on divergent beliefs regarding the PDF. An appendix shows how, at the price of a significantly more cumbersome proof, the condition of symmetry can be relaxed in favour of the more general assumption that $s^\star \geq \frac{1}{2}$.

Proposition 9 *Whenever f_1 and f_0 are disjoint, the equilibrium price of the asset and that of the most senior CDO equal twice their expected discounted payoff $p^{SCDO} = 2\frac{E_s}{R}$, $Q^{SCDO}(0) = 2$.*

Proof

$$\begin{aligned} p^{SCDO} * R &= \int_0^1 s \; max\{f_i(s)\} \\ &= \int_0^1 s \; f_0(s)ds + \int_0^1 s \; f_1(s)ds \\ &= 2E_s \end{aligned} \tag{46}$$

where the second line follows from the fact that f_1, f_0 are disjoint. The proof for the price of the most senior CDO is equivalent.

Discussion

With disjoint distributions f_0, f_1, type 1 perceives no cost from selling claims to agent 0, as for any x in $X_0 = \{x : f_0(x) > 0\}$ she expects to make 0 payments in the second period. Proposition 9 shows how this, intuitively, drives up asset prices to twice their fundamental value.

6 An example of a dynamic economy with learning about the Great Moderation

The main results of this paper are derived in a static environment. It is evident, however, that some of the issues at the heart of the analysis have an important dynamic dimension. Thus, for all long-lived assets, future price movements are an important determinant of both the prices investors are ready to pay today, and of the return risk they face over and above fluctuations in payoffs. Also, the wealth distribution across agents of different beliefs - shown to be an important determinant of asset prices in the static environment - should be expected to vary over time and thus lead to fluctuations in prices. And finally, disagreement about the distribution of payoffs and returns, which we took as given in our analysis, can be expected to evolve over time in response to the the realisations of uncertainty. The latter point seems particularly relevant, as the analysis of this paper is partly motivated by the experience of the Great Moderation with its strong fall in macroeconomic volatility. As the evidence from the SPF showed, this period coincided with rising disagreement among US forecasters about the dispersion of near term GDP growth around its expected value. One interpretation of this evidence is that forecasters adjusted to the new volatility environment in heterogeneous ways, in line with the disagreement about the origins, and therefore persistence, of the period of low macro-volatility.

We leave an in-depth dynamic analysis for future research. To illustrate some of the issues involved, however, we present in this section a simple example of a dynamic version of the two-type model where belief heterogeneity arises because some agents adjust their risk perception more quickly than others in reaction to an observed fall in macro-volatility. Specifically, we look at a version of the two-type economy where time is infinite $t = 0, 1, 2, ...$ and physical assets pay a random amount of the consumption good s_t that is independent across periods. Agents of both types are infinitely-lived and receive consumption endowment $n_i, \forall t$. Agents maximise the present discounted value of consumption through decisions on consumption c_{it} and on purchases of assets and collateralised loans, a_{it+1}, b_{it+1} respectively, every period. As before, they trade physical assets, whose quantity we normalise to 1, and collateralised loans whose face value for next period $\overline{s}_{t+1}$ is agreed on in t.

Suppose for now that agents specialise their investment in only 1 asset class, with i and j agents buying leveraged assets and collateralised loans, respectively. The budget constraint for leveraged investors is then

$$a_{it+1}(p_t - q_t) + c_{it} \leq n_i + max\{a_{it}(p_t + s_t - \overline{s}), 0\}) \tag{47}$$

The budget of agents that purchase collateralised loans is characterised by

$$b_{jt+1}q_t + c_{it} \leq n_i + b_{jt}(min\{p_t + s_t, \overline{s}\}) \tag{48}$$

Again we assume that type j buyers of collateralised loans make 0 surplus, so

$$q_t = \frac{E_{jt}[min\{p_{t+1} + s_{t+1}, \overline{s}]}{R}. \tag{49}$$

Agents invest in leveraged assets if

$$R_i^a = \frac{E_{it}[p_{t+1} + s_{t+1} - min\{p_{t+1} + s_{t+1}, \overline{s}\}]}{p_t - \frac{E_j[min\{p_{t+1}+s_{t+1},\overline{s}\}]}{R}} \tag{50}$$

$$= \frac{E_{it}[p_{t+1}] + E_s - E_{it}[min\{p_{t+1} + s_{t+1}, \overline{s}\}]}{p_t - \frac{E_j[min\{p_{t+1}+s_{t+1},\overline{s}\}]}{R}} > R \tag{51}$$

where expectations are now allowed to vary both across type and time as agents learn from empirical evidence. In Broer and Kero (2013), we analyse how risk-averse homogeneous investors price assets under different Bayesian and ad-hoc learning rules in a model with time-varying volatiltiy but without leverage. Here, we use an anticipated-utility approach with a particularly simple learning rule to illustrate the asset price dynamics that can result from introducing leverage in a model where, absent leverage, prices would equal the constant common valuation of dividends by risk-neutral investors. Again, we concentrate on payoffs of a generic asset as the source of randomness in the environment, which agents believe to be distributed according to the distribution $f_{i,t+1}$ next period. Motivated by the fact that the Great Moderation was established as an empirical fact by the mid-1990s, we analyse a scenario at the beginning of which agents perfectly observe a change in the payoff distribution from h^{pre} to h^{post}, but differ in the way they adjust their beliefs about the future. Specifically, agents adjust last period's belief $f_{i,t}$ by a constant fraction g_i towards the current observed distribution of payoffs h_t

$$f_{i,t+1} = (1 - g_i)f_{i,t} + g_i h_t \tag{52}$$

In line with the characterisation of type 0 agents as more and type 1 agents as less confident in the static environment, we assume $g_0 > g_1$, such that type 0 adjust believe quicker in the new environment of low volatility. Finally, we adopt an anticipated utility approach to behaviour as in Cogley and Sargent (2008), whereby agents update their beliefs in line with the simple rule in (52), but do not anticipate to learn further in the future.

Equilibrium definition A competitive equilibrium consists of

1. Sequences of prices and quantities as functions of the state of the economy $(s_t, \overline{s}_t, a_{it}, a_{jt})$ s.t.
2. Agents optimise given belief f_{it}
3. Markets for consumption and assets clear

Discussion

Note how we specify an equilibrium where agents agree on the price but disagree on the distribution of exogenous shocks. Their beliefs change over time, but, in line with our anticipated utility approach, agents use constant beliefs to calculate expectations for the future.

The following proposition shows the equilibrium with trade in collateralised loans when both h^{pre} and h^{post} are continuous and symmetric, implying symmetric distributions $f_{0,t}, f_{1,t}$. For this, we assume that assets are initially endowed to type 0 agents.[10]

[10]This is convenient as it implies that the present discounted value of tomorrow's asset holdings $\frac{\overline{p}}{R}$ nets

Proposition 10 *Stationary Dynamic Equilibrium*
If $n_1 \geq 2\frac{E_s - E_1[min\{s,s^\star\}]}{R}$*, there is an equilibrium with the following properties*

- $p_t = \overline{p} = \frac{E_s + E_0(min(s,\overline{s})) - E_1(min\{s,\overline{s}\})}{R-1} \ \forall \ t \geq 0$
- $q_t = \frac{E_0(min(s,\overline{s})) + \overline{p}}{R} \ \forall \ t \geq 0$
- $\overline{s} = s^\star + \overline{p} \ \forall \ t \geq 0$

where again $s^\star$ *is the single crossing point of type* 1 *and* 0*'s CDFs.*

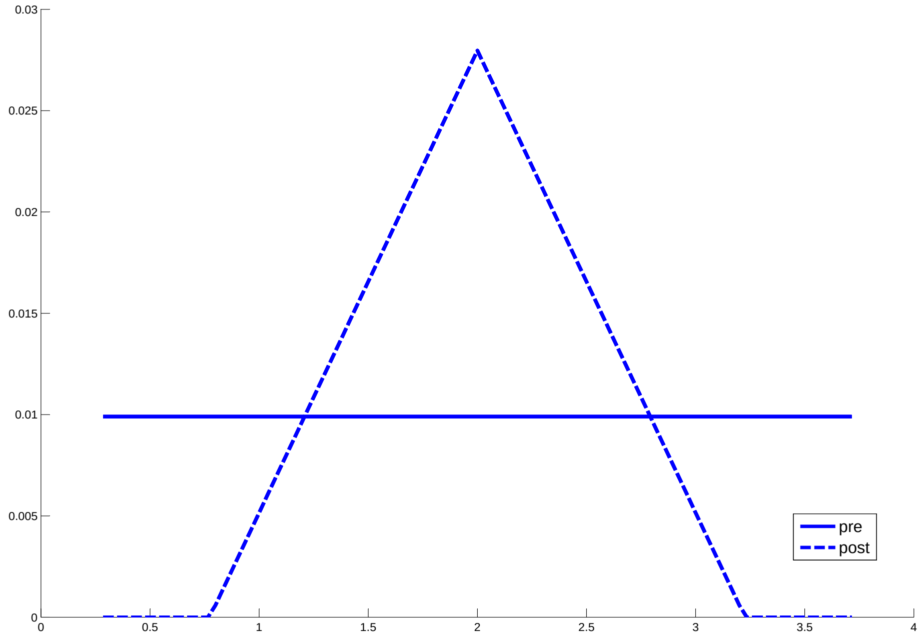
Proof:
To verify that the postulated investment rules and price process form an equilibrium, we must show that type 1's choices are optimal and affordable given the price process, and that markets clear.
Note that at the postulated price, type 1 agents are exactly indifferent between consuming and investing as $E_1[p_{t+1}] = E_0[p_{t+1}] = p_t = \overline{p}$ implies $R_1^a = R$. Moreover, $E_1[min\{p_{t+1} + s_{t+1}, \overline{s}\}] < E_0[min\{p_{t+1} + s_{t+1}, \overline{s}\}]$, so $R_1^a > R_0^a$ and agent 0 strictly prefers to consume or invest in collateralised loans, rather than invest in leveraged assets.
That $\overline{s} = s^\star + \overline{p}$ is an optimal choice for loan riskyness follows from $R_1^a = R$ and the first order condition (25) that applies unchanged as prices are constant over time.
Finally, the assumption that $n_1 \geq 2\frac{E_s - E_1[min\{s,s^\star\}]}{R}$ implies that less confident type 1 agents have large enough consumption endowments to buy all the assets at the price $\overline{p}$. Note also that payments from type 1 agents to type 0 agents to buy the assets are greater than the value of collateralised loans in the economy, so type 0 agents can afford to buy all collateralised loans.

6.1 A calibrated example

To illustrate the quantitative effect of leverage on asset prices in this simple example economy, we normalise expected payoffs to 2 and the standard deviation during the pre-Great Moderation period to 1 by choosing h^{pre} to have a uniform distribution on $[0.285, 3.715]$. In line with the fall in the standard deviation of both US GDP and consumption growth during the Great Moderation to half their previous values, we choose h^{post} to be a triangular distribution with standard

out in type 0 agents' budget, who sell 2 units of the asset but by 2 units of collateralised loans. Alternative assumptions that are consistent with type 0 agents being able to afford all collateralised loans, however, would imply the same equilibrium prices.

Figure 5: Payoff distribution before and during the Great Moderation



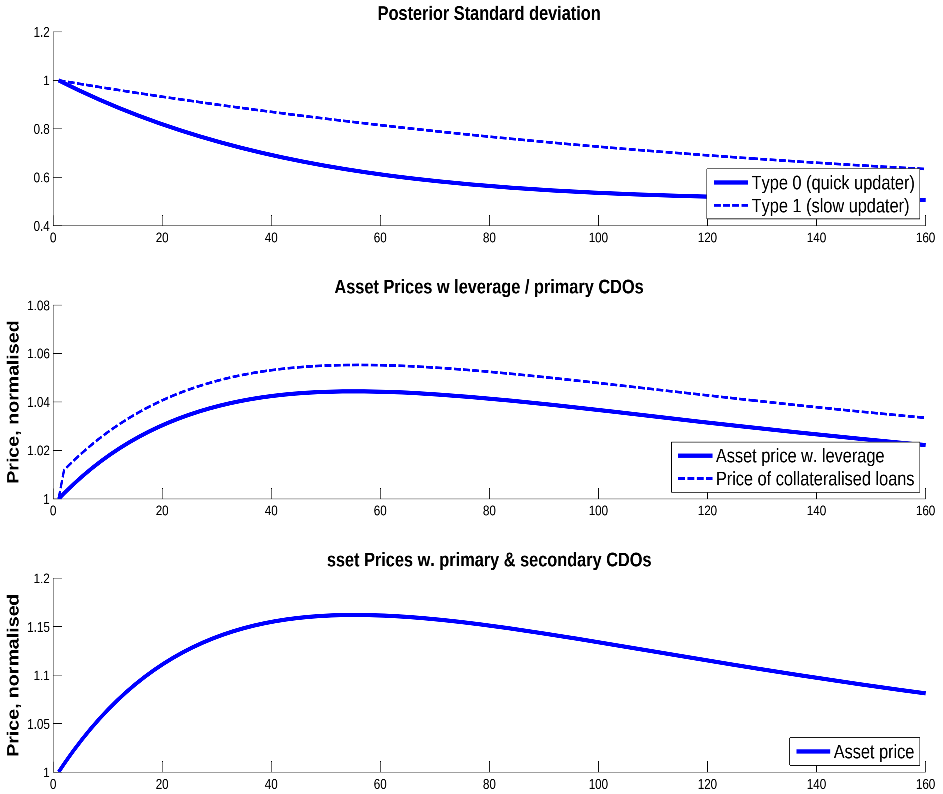
The figure plots the distribution of payoffs before (solid line) and during the Great Moderation (dashed lined).

deviation $\frac{1}{2}$. Figure 5 illustrates the two distributions.[11] Finally, we look at two calibrations of the learning parameters. First, we analyse a scenario of "temporary disagreement", where $g_1 = 1\%$ and $g_0 = 3\%$, chosen to have asset prices peak after 55 periods, in line with the US experience where the Great Moderation started in the second half of the 1980s and prices peaked towards the end of the 1990s. A second calibration is based on the evidence from the SPF, where disagreement about risk continued to rise throughout the Great Moderation period. In this second calibration we thus set $g_1 = 0$, implying that type 1 agents continue to have 0 confidence in the fall in macro-volatility. We set quarterly interest rates to 1 percent (which affects the level of asset prices, but not the normalised levels presented below).

Figure 6 presents the results for the first scenario. The top panel shows how, after the onset of the Great Moderation in $t = 0$, the standard deviations of the posterior payoff distribution first diverge before slowly re-converging. Mean asset prices with trade in collateralised loans (or

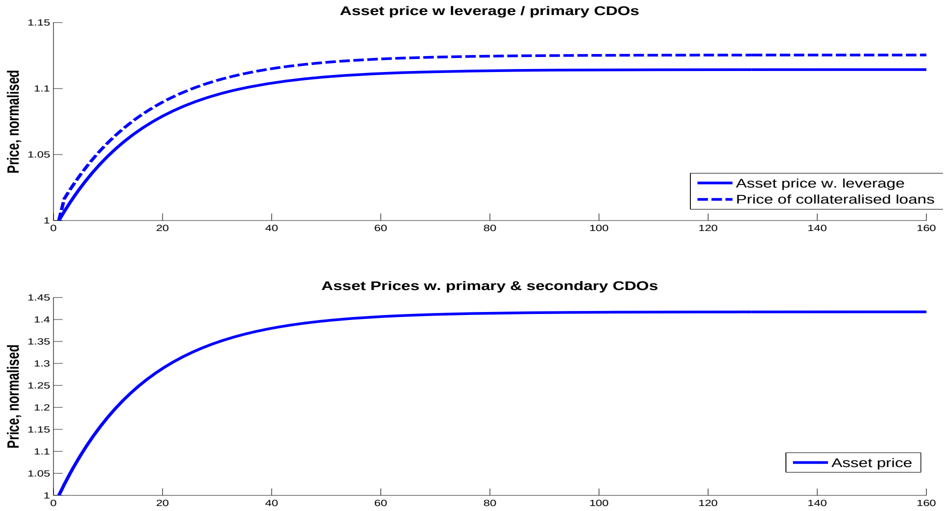
[11] We opt for a triangular distribution for h^{post}, rather than a uniform distribution, because, for infinitesimally small disagreement, the latter implies that $f_{0,t}(s) > f_{1,t}(s)$ for all $s \in (0.285, 3.715)$, potentially leading to extreme swings in asset holdings for small changes in beliefs with trade in secondary CDOs.

Figure 6: Dynamic Example, Temporary Disagreement



The figure plots the standard deviations of payoffs for both types (top panel), the prices of collateralised loans and assets (central panel), and the asset price with trade in primary and secondary CDOs (bottom panel).

Figure 7: Dynamic Example, Permanent Disagreement



The figure plots the prices of collateralised loans and assets (central panel), and the asset price with trade in primary and secondary CDOs (bottom panel).

primary CDOs), in the central panel, follow a hump-shaped pattern and peak at about 5 percent above their initial value. The price of collateralised loans follows closely that of the collateral asset. The bottom panel of figure 5 shows how, with trade in primary and secondary CDOs, the hump-shape of asset prices is more pronounced and the magnitude of the asset price boom, which peaks at about 15 percent, larger.
Figure 7 presents the results for the second scenario, where type 1 agents continue to believe in the high volatility of the pre-Great Moderation era. As type 0 agents' beliefs slowly converge to the new observed distribution, disagreement, as well as asset prices, increase monotonically. With trade in collateralised debt, prices of assets and collateralised loans rise by 11 and 13 percent, respectively. But with trade in primary and secondary CDOs, the rise in asset prices is increased strongly, to more than 40 percent.

We see these quantitative results as illustrative. In our view they show how, potentially, the financial innovation of the 1990s and 2000s may have contributed to a significant rise in asset prices, as the introduction of secondary CDOs raises the asset price bubble to between 3 and 4 times its level in the scenario with trade in collateralised loans only. We would like to stress, however, that the highly stylised nature of this example makes it, more than anything, an illustration of promising avenues for future research.

7 Conclusion

This paper has looked at the role of collateralised asset trade in economies where investors disagree about risk, rather than mean payoffs considered in the literature. The analysis was motivated by the fact that US surveys on expectations of stock returns and GDP growth showed strong, and in the case of GDP growth rising disagreement about the dispersion of outcomes around their mean value. We presented a simple static model of investor disagreement, where in the absence of collateralisation, risk-neutral investors trade assets at their common fundamental value even if they disagree about payoff risk. The introduction of simple collateralised loans increased asset prices above this common fundamental value by unleashing perceived gains from trade. This was because investors with a concentrated posterior distribution of payoffs are less afraid of the downside risk embodied in collateralised debt, while those who perceive higher payoff risk value more highly the upside risk of leveraged asset purchases. In equilibrium, this raised the price of collateral assets. We also showed how further financial innovation in the form of more complex collateralised debt products can strongly affect prices. This was, perhaps surprisingly, not the case with primary CDOs, whose introduction did not affect prices relative to trade in collateralised loans. Allowing agents to use CDOs to collateralise more junior "secondary" CDO contracts, however, strongly raised prices. Finally we illustrated this mechanism in a highly stylised quantitative example with learning at different speeds about an abrupt fall in volatility, as in the Great Moderation of macro-volatility. The rise in asset prices due to the resulting disagreement about risk was modest with trade in collateralised loans, but substantial with trade in primary and secondary CDOs. Again, this result underlined the main message of this paper, that disagreement about risk and collateralised contracts of different degrees of sophistication are strongly complementary in their effects on asset prices. We believe that, given the coincidence of strongly heterogeneous risk-perceptions and financial innovation in the 1990s and early 2000s, this could partly explain the strong boom in the prices of collateral assets and collateralised debt during that period. Moreover, we think that the results could be interesting for policy-makers, as they suggest that regulation of risk-taking and leverage could be important even in the absence of traditional forms of "exuberance" about mean returns, because investors who believe risk to be relatively high have strong incentives to increase leverage to capture the upside risk of investing.

We hope that our analysis opens some avenues for further research. Particularly interesting seems a less stylised dynamic analysis, with stochastic variation in prices that opens scope for much richer forms of disagreement than that about payoff dispersion considered here. Also, one should consider further financial innovation through pure derivative contracts that are contingent on asset payoffs but collateralised by cash, as in Simsek (2013) and Fostel and Geanakoplos (2012) where their introduction lowers asset prices. Finally, it would be interesting to analyse the reasons behind disagreement about the distributions of payoffs, taken as given in this paper, and how perceived risk varies with investor characteristics.

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8 Appendix A: Asset price bubbles with trade in synthetic CDOs

This proposition generalises the conditions in proposition 8 in the main text.

Proposition 11 *When f_0 has no mass points and $s^\star \geq \frac{1}{2}$, the asset price bubble $p^{SCDO} - \frac{E_s}{R}$ with trade in secondary CDOs is at least twice as large as that with trade in collateralised loans or primary CDOs.*

Proof

$$
\begin{aligned}
p^{SCDO} * R &= \int_0^1 s\ max\{f_i(s)\}ds = E_s + \int_0^1 s(max\{f_i(s)\} - f_1(s))ds \\
&= E_s + \int_0^{s^\star} s(max\{f_i(s)\} - f_1(s))ds + \int_{s^\star}^1 s(max\{f_i(s) - f_1(s), 0\})ds \\
&\geq E_s + \int_0^{s^\star} s(f_0(s) - f_1(s))ds + s^\star \int_{s^\star}^1 max\{f_0(s) - f_1(s), 0\}ds \\
&\geq E_s + [\int_0^{s^\star} s(f_0(s) - f_1(s))ds + \int_{s^\star}^1 s^\star(f_0(s) - f_1(s))ds] + s^\star min_{\widehat{f_0},\widehat{f_1}}\{\int_{s^\star}^1 max\{\widehat{f_0}(s) - \widehat{f_1}(s), 0\}\}ds \\
&\geq E_s + \Delta E[min\{s, s^\star\} + s^\star \frac{\Delta E[min\{s, s^\star\}}{1 - s^\star} \\
\Rightarrow &\ p^{SCDO} - \frac{Es}{R} \geq 2(\overline{p} - \frac{Es}{R})\ \ \forall s^\star \geq \frac{1}{2}
\end{aligned}
$$

The fourth line follows by adding $\int_{s^\star}^1 c(f_0(s) - f_1(s))ds = 0$, which holds for any constant c, and because $\Gamma(f_0, f_1) \geq min_{\widehat{f_0},\widehat{f_1}}\Gamma(\widehat{f_0}, \widehat{f_1})$ for any function Γ and $\widehat{f_0}, \widehat{f_1}$ fulfilling the assumptions of equal expected payoffs $\int_{s^\star}^1 s(\widehat{f_0}(s) - \widehat{f_1}(s))ds = -\int_0^{s^\star} s(f_0(s) - f_1(s))ds+$ and given mass above the single-crossing point $\int_{s^\star}^1 \widehat{f_i}(s)ds = 1 - \int_0^{s^\star} f_i(s), i = 0, 1$. The minimisation problem can be written as

$$
\begin{aligned}
& min_{\widehat{f_0},\widehat{f_1}:[s^\star,1]\longrightarrow R+} \int_{s^\star}^1 (max\{\widehat{f_0}(s) - \widehat{f_1}(s), 0\})ds \\
& s.t. \\
& \int_{s^\star}^1 s(max\{\widehat{f_1}(s) - \widehat{f_0}(s), 0\})ds - \int_{s^\star}^1 s(max\{\widehat{f_0}(s) - \widehat{f_1}(s), 0\})ds = \int_0^{s^\star} s(f_0(s) - f_1(s))ds > 0 \qquad (53) \\
& \int_{s^\star}^1 \widehat{f_i}(s)ds = 1 - \int_0^{s^\star} f_i(s)ds, i = 0, 1 \qquad (54)
\end{aligned}
$$

Along the first and second constraint, disagreement can be reduced by concentrating $s : max\{\widehat{f_1}(s) - \widehat{f_0}(s), 0\} > 0$ at $s = 1$ and $s : max\{\widehat{f_0}(s) - \widehat{f_1}(s), 0\} > 0$ at $s = s^\star$. To see that this is the minimum disagreement consider a small reallocation $d\widehat{f_1}$ $(d\widehat{f_0})$ from 1 $(s^\star)$ to some other value $s \in (s^\star, 1)$. In order not to violate constraint (53), this requires an increase in $\int_{s^\star}^1 s(max\{\widehat{f_1}(s) - \widehat{f_0}(s), 0\})ds = \int_{s^\star}^1 \mathbf{1}(max\{\widehat{f_1}(s) - \widehat{f_0}(s), 0\})ds$ or a decrease in $\int_{s^\star}^1 s(max\{\widehat{f_0}(s) - \widehat{f_1}(s), 0\})ds = \int_{s^\star}^1 \mathbf{s}^\star(max\{\widehat{f_0}(s) - \widehat{f_1}(s), 0\})ds$. Since disagreement is concentrated at the extremes of the interval $[s^\star, 1]$, these changes necessitate an increase in disagreement, which increases the objective.

The resulting equal mass points of $\widehat{f}_0$ at $s^\star$ and of $\widehat{f}_1$ at 1 imply, after substituting the constraint of equal expected payoffs, $\widehat{f}_0(s^\star) = \widehat{f}_1(1) = \frac{\int_0^{s^\star\ 1} s(f_0(s)-f_1(s))ds}{1-s^\star} = \frac{\Delta E[min\{s,s^\star\}]}{1-s^\star}$. Together with $\int_0^{s^\star} s(f_0(s) - f_1(s))ds + \int_{s^\star}^1 s^\star(f_0(s) - f_1(s))ds = \Delta E[min\{s, s^\star\}]$ this implies the fifth line.