Constraints on Matching Markets Based on Moral Concerns

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Abstract

In many markets, bans on monetary transfers are enforced driven by concerns that otherwise wealthier people may have better access to resources. This study discusses whether transfer bans adequately reflect such inequality concerns. We consider an assignment problem involving agents with heterogeneous wealth endowments and preferences with positive income effects. To address concerns of unequal access, we introduce discrimination-freeness as a constraint. Discrimination-freeness requires that the allocation of objects is independent of the wealth endowments of the individuals. We show that for large wealth inequalities, transfers are necessary for efficiency but the Pareto-efficient frontier of discrimination-free social choice functions can be reached without transfers. Furthermore, a market designer who must not use monetary transfers faces the same restrictions for the implementation of an object allocation as a designer who is bound by discrimination-freeness. For small wealth inequalities, the results are different. Depending on the characteristics of the objects, efficiency can be achieved without using transfers or money is needed to reach the Pareto-efficient frontier of discrimination-free social choice functions. Also, discrimination-free allocations can exist that are not implementable without transfers. If money can be used outside a market designer's control, additional restrictions beyond mere transfer-freeness might be necessary to address concerns of discrimination.

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1 Introduction

Why worry that we are moving toward a society in which everything is up for sale? … One [reason] is about inequality … Where all good things are bought and sold, having money makes all the difference in the world.

Michael Sandel in "What Money Can't Buy" (Sandel, 2012, p. 8)

Monetary transfers are prohibited or undesirable in various markets, such as the ban of selling organs or the provision of free education in many countries. From a classical utilitarian perspective, banning transfers seems counterintuitive, as price mechanisms are known to promote the efficiency of resource allocation. The aversion towards market prices for certain goods appears to stem from ethical concerns regarding inequality. *"From the egalitarian's angle of vision, what underlies noxious markets … is a prior and unjust distribution of resources, … the fairness of the underlying distribution of wealth and income is extremely relevant to our assessment of markets"* (Satz, 2010, p. 5).[1] When wealth inequality in a society is high, differences in the willingness to pay for a good can be the consequence of wealth inequality rather than differences in the benefits from consuming the good (where the concept of "benefits" will be defined precisely later on). Consequently, classical market mechanisms may reinforce existing disparities by allocating resources mainly based on wealth.

In this work, we study in what way a ban on monetary transfers reflects these concerns of unequal access. Our research makes a two-fold contribution. First, we introduce discrimination-freeness as a fairness criterion to address a desire for wealth-independent access to goods.[2] While the standard assumption of quasilinear preferences fails to capture these concerns adequately, we assume positive income effects and explicitly incorporate wealth inequality. Second, we study the impact of wealth inequality on efficiency properties of social choice functions (SCFs) and compare the implications of requiring discrimination-freeness to an exogenous ban on monetary transfers.

Our research findings indicate that if wealth inequalities are large, SCFs without trans-

[1] See also, e.g., Kahneman, Knetsch and Thaler (1986), Frey and Pommerehne (1993), and Roth (2007). Ambuehl, Niederle and Roth (2015) empirically show that the assessment of markets depends on the financial perspective of market participants.

[2] While our focus is on inequality concerns, we acknowledge that there are other arguments against the use of money.

fers are inefficient. However, requiring discrimination-freeness implies the same restrictions on implementing an object allocations a ban on monetary transfers does. For minor wealth inequalities, the results can substantially differ – depending on the characteristics of the objects considered. If wealth inequalities are small, a SCF without transfers is either efficient or discrimination-free Pareto-improvements exist. Furthermore, for certain spaces of object values, prices can be used to incentivize truthful reporting without violating discrimination-freeness. Our analysis contributes to the understanding of the link between banning transfers for assigning goods and ensuring wealth-independent access to them.

Notably, the significance of a desire for discrimination-free access varies across different markets. While it appears to have less relevance in markets for conventional consumer goods such as fashion or cars, it becomes critical in markets that affect life opportunities, such as education and health. The Universal Declaration of Human Rights (1948), for instance, upholds both the "right to education" and the "right to health," emphasizing the critical importance of ensuring discrimination-free access to these fundamental aspects of human well-being.[3]

Our concrete setting is as follows: a market designer assigns a set of indivisible objects and monetary transfers to a group of agents. The agents have additive separable preferences over objects and money with decreasing marginal utility in money.[4] Each agent is characterized by a type $(\theta, e) \in \Theta \times E$ where θ is a vector of object utilities and e his initial wealth endowment. A SCF assigns objects and monetary transfers based on the type profile.

We define a SCF as discrimination-free if its allocation of objects does not depend on wealth realizations. Discrimination-freeness complements well-established inequality and fairness constraints by capturing the impact of wealth heterogeneity on market design. Notably, our assumption of positive income effects is crucial to consider discrimination as a valid concern, as otherwise, the willingness to pay would be unaffected by wealth endowments.

In our analysis, we investigate whether efficiency or the efficient frontier of discrimination-

[3]See, e.g., articles 25 and 26. By General Comment No. 14 (2000): "Health facilities, goods, and services have to be accessible to everyone without discrimination [...]".

[4]Additive separability is not necessary for the analysis, but is useful to separate between a wealth-independent marginal object utility and the marginal utility of money. This eases notation and interpretation. In Section 6.2 we discuss how to extend the ideas to general preference spaces.

free SCFs can be reached without the use of monetary transfers and whether implementable and discrimination-free SCFs are implementable without monetary transfers as well. The possibility and impossibility results we derive crucially depend on the degree of wealth inequality, measured by the lower and upper limits of endowments of the wealth space E, as well as on characteristics of the object utility space Θ.

For large wealth inequalities, we find that efficiency of a SCF cannot be achieved without using monetary transfers. Even if a SCF allocates the objects such that object utilities are maximized, it is inefficient. However, SCFs at the efficient frontier of transfer-free SCFs are at the efficient frontier of discrimination-free SCFs as well. It implies that any trading incentives that may exist after assigning the objects cannot be resolved without violating discrimination-freeness. Furthermore, we show that for any implementable and discrimination-free SCF, there exists an implementable and transfer-free SCF that induces the same allocation of objects. Therefore, for large wealth inequalities, a market designer constrained from using monetary transfers has the same means to allocate the objects as a market designer constrained by discrimination-freeness has. The main driver of these results is that the willingness to pay increases in wealth.

For low wealth inequalities, the results can be substantially different – depending on the characteristics of the object utility space Θ. First, if Θ is such that all objects are sufficiently different, a transfer-free SCF that maximizes the sum of object utilities is efficient. The reason is that positive income effects imply that an agent's willingness to pay for an object is lower than his willingness to accept for giving up the object. As a result, small wealth inequalities do not create incentives to trade. If Θ is such that not only the objects are different enough but all agents have sufficiently similar object utilities, a transfer-free SCF can be efficient even if it is only ordinal efficient (i.e., the agents do not want to exchange objects). It implies that if types are private information, transfer-free SCFs can be efficient even if implementability is required. Therefore, for positive income effects, maximizing object utilities is neither necessary nor sufficient to achieve efficiency - which is in contrast to the standard case of quasilinear preferences. Second, if the object utility space is such that an implementable SCF without transfers is *not* efficient for small wealth inequalities (i.e., objects can be very similar or the variation in object utilities is high), transfer-free SCFs are even not at the efficient frontier of discrimination-free SCFs. More specifically, there exist trading incentives that are not driven by wealth. Third, a market designer who can use monetary transfers might be able to use prices to implement an object allocation without violating discrimination-freeness. For this, it is sufficient that the closure of the object utility space Θ is not convex.

For illustration of the last result, consider an example where only one object is assigned, and receiving the object requires payment of a price. For such a mechanisms to be implementable and discrimination-free, for any realization of object utilities an agent's willingness to pay the price must be independent of her wealth endowment. A necessary condition for this is that wealth inequality is sufficiently small. Furthermore, the structure of the object utility space is crucial. For instance, discrimination-freeness and implementability can be achieved if the object utility space has only two elements such that the agent either has a high or a low utility for the object. If then wealth inequality is low enough for the high object utility the agent is willing to pay the price for any wealth realization and for the low object utility she never pays the price. In this scenario, a market designer constrained by discrimination-freeness may have a wider range of tools for the object allocation than a designer constrained by the absence of money.

Finally, we discuss that banning monetary transfers may not be sufficient to ensure discrimination-freeness once monetary transfers can be used outside the assignment procedure. To formalize such settings, we use the concept of bribing. Schummer (2000) defines bribes as one agent paying another to misreport preferences in a way that benefits both parties. He shows that for quasilinear preferences, an object assignment is bribe-proof (i.e., no incentives for such bribes exist) if and only if an agent's assignment does not depend on the preferences of other agents. Extending this result to our framework of nonlinear preferences creates a significant constraint on SCFs: if discrimination-freeness is required to hold even when bribes are possible, an agent's allocation of objects must not depend on the preferences of others. Bribes can be interpreted more broadly as the use of money outside a centralized mechanism to influence one's access to a good. For example, one might choose to attend a private school rather than a public school. Alternatively, wealth may allow one to buy a house in a neighborhood near a desirable school, thereby increasing one's chances of gaining admission to that school.

Overall, our results show that prohibiting transfers can be interpreted as a response to large wealth inequalities when unequal access is a concern and the mechanism cannot mitigate potential wealth inequalities. The goal of ensuring wealth-independent access is not the only reason for wanting to prohibit transfers, but our work highlights that it alone can explain the prohibition of transfers. However, we also find that for smaller wealth inequalities, it may not be necessary to ban transfers to achieve discrimination-freeness. Translating our findings into real-world applications, countries with a more equitable distribution of wealth may have greater flexibility in allowing transfers than countries with substantial wealth inequalities. Furthermore, the kind of goods is also relevant for the assessment of the market. The more valuable the objects are, the more relevant the impact of positive income effects is.

We organize the paper as follows. In the rest of the introduction, we discuss the related work. In Section 2 we describe the basic model. We introduce discrimination-freeness in Section 3 and derive implications for market design compared to banning money in Sections 4 and 5. In Section 6 we consider discrimination-concerns that can arise through the use of money outside the mechanism and discuss specific model assumptions. We conclude with Section 7. Proofs are in the appendix A.

1.1 Related Work

Repugnance on markets. For some markets, society appears to be repugnant towards using monetary transfers or prices to allocate goods. For instance, Kahneman et al. (1986), Frey and Pommerehne (1993), and Roth (2007) deal with the desire of third parties to restrict monetary transfers. Ambuehl et al. (2015) show that the assessment of markets depends on the financial perspective of market participants. Inequality appears to be a central concern for market disapproval (e.g., Sandel (2012), Satz (2010)). In our work, we explicitly integrate the concern of unequal access due to wealth inequality into an economic model. There also exist other concerns associated with monetary transfers. Slippery slope arguments point out that the introduction of money may itself lead to unintended consequences like on the market, e.g. organ commercialism (Bruzzone (2010)). Gneezy and Rustichini (2000)) argue that the existence of a monetary fine can induce unexpected behavior. Prices may also come along with unwanted external effects (see, e.g., Jehiel, Moldovanu and Stacchetti (1996) for an example of selling nuclear weapons and Satz (2008) and Rippon (2014) for kidney sales). Ambuehl (2023) examines potential harmful effects of undue inducements but does not find support for this concern.[5]

Inequality, income effects, and price controls. Our research contributes to the literature dealing with price controls as a response to inequality in markets. Dworczak, Kominers and Akbarpour (2020) model inequality as heterogeneity in marginal utilities of money. By studying the trade-offs between allocative efficiency and redistribution they show that inequality may give reason for price regulations. Reuter and Groh (2020) use a similar framework as Dworczak et al. (2020) and show that it may be optimal not to sell to those who are willing to pay the most. Che, Gale and Kim (2013) model inequality via budget constraints to argue that market clearing prices may not be optimal. While in Dworczak et al. (2020) the driving argument for price controls is redistribution opportunities, in Che et al. (2013) it is allocative efficiency. In contrast to the above-mentioned literature, we model inequality as heterogeneity in endowments accompanied by decreasing marginal utilities in money - implying positive income effects. The focus of our work is less on how inequality impacts on optimality of a mechanism but rather on a market designer's toolset if discrimination-freeness is a constraint.

There are several examples in the literature showing that income effects can have a substantial impact on market design results and policy implications. Maskin and Riley (1984) show that for risk-averse bidders, the first-price auction and the second-price auction are neither equivalent nor optimal as it is the case for quasilinear preferences. Baisa (2017) shows that with positive income effects, random allocations may Pareto-dominate the second-price auction. Also, random allocations may Pareto-dominate selling the good for a market clearing price as shown in Huesmann (2017) for decreasing marginal utility of money and in Che et al. (2013) for budget constraints. Garratt and Pycia (2014) demonstrate that under income effects, the efficient bilateral trade problem (Myerson and Satterthwaite (1983)) is solvable for certain settings.

Fairness constraints. There is a wide range of fairness concepts in the market design literature that constrain the distribution of goods. Our formalization of discrimination-freeness is closely linked to the popular concept of *anonymity* requirements imposed on

[5]There is furthermore a large literature dealing with how incentives impact on the moral behavior of individuals (Frey and Oberholzer-Gee, 1997, Gneezy and Rustichini, 2000, Mellström and Johannesson, 2008, Richard, 1970). In contrast, we are interested in how monetary incentives impact who receives what.

SCFs, which typically demand that the outcome is independent of the agents' identities (see, e.g., Thomson (2011)). In contrast, discrimination-freeness solely pertains to the part of the SCF that allocates the objects and requires that it remains unaffected by variations in wealth endowments.

Other commonly discussed fairness criteria mostly pertain to how an agent perceives their allocated bundle in relation to others. For instance, *no envy* implies that no agent prefers another agent's bundle, while *equal treatment of equals* asserts that an agent does not prefer another agent's bundle when both agents have identical preferences (e.g., Bogomolnaia and Moulin (2001)). Notably, our definition of discrimination-freeness implies that two agents are considered as equal (before any tie-breakers are involved) when their object values are the same but wealth may differ. In our view, the requirement of discrimination-freeness does not render other fairness criteria unnecessary. Instead, it is a complementary criterion to existing ones, enriching the understanding of fairness in resource allocation if inequality is a concern.

2 Model

Consider the problem of assigning a set Ω of objects to a set N of n agents. Ω contains k distinct objects and a Null-object 0 that corresponds to staying unassigned. Each object has a capacity $c(\omega)$ with $\sum_{\omega \neq \{0\}} c(\omega) \leq n$ and $c(0) = n$.[6] Each agent receives at most one object and the assignment has to respect capacities.

Payoff environment. Each agent i has preferences about owning an object ω and wealth e described by an additive separable utility function $u_i : \Omega \times \mathbb{R} \to \mathbb{R}$ with

$$u_i(\omega, e) = \theta_i(\omega) + h(e). \tag{2.1}$$

Objects are distinct, for the Null-object we normalize $\theta(0) = 0$. The marginal utility in money is positive (i.e., $h' > 0$) and decreasing (i.e., $h'' < 0$). We do not explicitly assume

[6]This condition rules out settings with irrelevant objects which won't be assigned to any agent because all other objects are preferred. Omitting this assumption would require some case distinctions in the analysis that rather distract from the main point.

a budget constrained but wealth might become negative.[7] Furthermore, $\lim_{e\to\infty} h(e) = \infty$ and $\lim_{e\to\infty} h'(e) = 0$.

Each agent i is endowed with some initial wealth $e_i \in \mathbb{R}$, and the agent's utilities attached to the k objects are described by $\theta_i \in \mathbb{R}^k_+$. From the perspective of the market designer and the other agents, each agent's object utilities and wealth endowment are drawn independently from some distributions with supports $\Theta \subset \mathbb{R}^k_+$ and $E \subset \mathbb{R}$. We call $\Theta \times E$ the type space, Θ the object utility space, and E the wealth space. Furthermore, we denote

$$\underline{e} = \inf E > -\infty \quad \text{and} \quad \overline{e} = \sup E \leq \infty.$$

Agent i's preferences over individual assignments are thus described by a $k+1$-dimensional type $t_i = (\theta_i, e_i) \in \Theta \times E$. An object assignment ω and a monetary transfer m is evaluated according to $\theta_i(\omega) + h(e_i + m)$.

$R(\theta_i)$ is the rank order $r_i : \Omega \to \{1, ..., k+1\}$ of objects for an agent i implied by any $\theta_i \in \Theta$ with $r_i(\omega) < r_i(\omega')$ if and only if $\theta_i(\omega) > \theta_i(\omega')$. Since the Null-object provides the lowest utility, $r_i(0) = k+1$. With $\Theta(r_i)$ we denote the set of all $\theta_i \in \Theta$ with $R(\theta_i) = r_i$.

The assumption of additive separability enables a clear distinction between implications that are driven by object utilities and those that are driven by wealth. This eases the distinction between the drivers of the later results. Note that as long as income effects are positive, the core arguments remain to hold even for fairly general preference spaces (e.g., object utilities may also depend on wealth). For a more detailed discussion see Section 6.2.

Social Choice Functions. $\varphi = (\sigma, m)$ represents a *social choice function* (SCF) that selects for each type profile $(\theta_i, e_i)_{i\in N} \in (\Theta \times E)^n$ an assignment $(\sigma_i, m_i)_{i\in N}$. We refer to σ: $(\Theta \times E)^n \to (\Omega \cup \{0\})^n$ as the *object assignment* and to m: $(\Theta \times E)^n \to \mathbb{R}^n$ as the *transfer rule.* $\sigma_i \in \Omega$ denotes the object assigned to agent i and $m_i \in \mathbb{R}$ the monetary transfer.[8] If types are private information, the SCF represents the corresponding direct mechanism that maps reported types to outcomes. We limit our attention to the set Φ of

[7]The qualitative results do not change when requiring $e > 0$. However, it require some more case distinctions that may distract from the main drivers of the results. Importantly, our results are not driven by budget constraints but by positive income effects. Incorporating budget constraints but assuming a constant marginal utility of money does not yield the same results as our specification of preferences.

[8]σ and m thus denotes either the assignment that maps types to outcomes or the outcomes themselves.

SCFs with $\sum_{i\in I} m_i \leq 0$ (no subsidy).[9] With $\Phi_{TF} \subset \Phi$ we refer to the set of all SCFs that are transfer-free. Therefore, for $\varphi \in \Phi$ it holds that $\varphi \in \Phi_{TF}$ if and only if $\varphi = (\sigma, 0)$.

A SCF φ may use tie-breaking rules, such as priorities or lotteries. Such tie-breakers are determined *before* the mechanism is conducted and are fixed for each agent independently of the realization of types. We thus take the perspective of an interim stage, where the tie-breakers may introduce a non-anonymous aspect to the SCF, even if it is anonymous in an ex-ante stage. This perspective allows us to focus on deterministic outcomes. This approach of considering an interim stage is more suitable for our analysis since we are interested in whether money can increase efficiency, rather than whether ex-ante efficiency gains can be achieved through probabilistic assignments. Evaluating probabilistic assignments under income effects is not straightforward and requires a separate assessment.

Definitions. A SCF $\varphi' = (\sigma', m') \in \Phi$ *Pareto-dominates* a SCF $\varphi = (\sigma, m) \in \Phi$ if for all type realizations $(\theta_i, e_i)_{i\in N} \in (\Theta \times E)^n$ all agents are weakly better off and for at least one realization there is one agent who is strictly better off. $\varphi = (\sigma, m) \in \Phi$ is *(Pareto-)efficient* if there is no $\varphi' = (\sigma', m') \in \Phi$ that Pareto-dominates φ. $\varphi = (\sigma, m) \in \Phi$ is *ordinal-efficient* if there is no object assignment σ' that Pareto-dominates σ (holding m fixed). For any $\Phi' \subset \Phi$, a SCF $\varphi \in \Phi'$ is at the (Pareto-)*efficient frontier* of Φ' if there is no SCFs $\varphi' \in \Phi'$ that Pareto-dominates φ.

A SCF $\varphi = (\sigma, m)$ is *implementable* if there exists a mechanism with a dominant strategy equilibrium such that, for all type profiles, the equilibrium outcome is the outcome of the social choice function.[10] We limit our attention to social choice functions for which truthtelling is a dominant strategy. Truthtelling is a dominant strategy if and only if for any agent i with type $t_i = (\theta_i, e_i) \in \Theta \times E$ and all types $t_{-i} = (\theta_{-i}, e_{-i}) \in (\Theta \times E)^{n-1}$ of the other agents:

$$u_i(\sigma_i(t_i, t_{-i}), e_i + m_i(t_i, t_{-i})) \geq u(\sigma_i(t_i', t_{-i}), e_i + m_i(t_i', t_{-i})) \quad \forall\ t_i' \in \Theta \times E.$$

[9] The qualitative results continue to hold if we require $\sum_{i\in I} m_i \leq F$ with $F \in \mathbb{R}$. $F > 0$ corresponds to a fund size that has to be raised, and $F < 0$ corresponds to a budget for subsidies.

[10] Requiring individually rationality is not relevant for our results.

3 Discrimination-Free Social Choice Functions

We define a social choice function as *discrimination-free* (concerning wealth) if the object assignment does not depend on wealth endowments. Formally,

Definition 1 (Discrimination-Free)**.** *Let* $(\Theta \times E)^n$ *be the agents' type space. A SCF* $\varphi = (\sigma, m)$ *is discrimination-free (concerning wealth) if and only if*

$$\sigma(\theta, e) = \sigma(\theta, e') \quad \forall (\theta, e), (\theta, e') \in (\Theta \times E)^n$$

$\Phi_{DF} \subset \Phi$ *denotes the set of all discrimination-free social choice functions.*

Discrimination-freeness pertains to factors that determine the object allocation and does not a priori impose restrictions on monetary transfers. Positive income effects are crucial for a meaningful definition of discrimination-freeness. Without income effects, preferences over outcomes are wealth-independent, and discrimination-freeness does not impose restrictions on how a SCF depends on preferences. However, with income effects, discrimination with respect to wealth becomes a valid concern because the willingness to pay is not only driven by the object utility but also positively depends on wealth. A discrimination-free SCF may incorporate differences in object utilities but must not consider differences in wealth endowments for the object assignment. The transfer rule is a priori not restricted in whether wealth endowments are taken into account. For instance, if wealth is known to the market designer, she may redistribute incomes. Thereby, our definition of discrimination-freeness reflects inequality concerns concerning access to goods. In contrast, most other fairness considerations in the literature refer to the whole assignment.

Note that discrimination-freeness per se does not induce an aim to equalize utilities and therefore does not necessarily incorporate a desire for redistribution. This is in contrast to a utilitarian market designer who is primarily concerned with classical inequality aversion. Importantly, even if an agent is compensated for giving up a good, discrimination-freeness may restrict this trade if its feasibility depends on wealth endowments.

4 (Constrained) efficiency of discrimination-free SCFs without transfers

A central objective in assigning objects is efficiency such that the agents do not want to trade ex-post. A key advantage of using money in assigning indivisible resources is its ability to trade off differences in object utilities, making it a primary tool for realizing Pareto improvements. In this section, we discuss under which conditions money is needed for efficiency of SCFs and whether requiring discrimination-freeness instead of banning transfers can offer advantages in efficiently assigning the objects. In other words, can a market designer who must not use mnetary transfers efficiently assign the objects when facing wealth inequality? If efficiency cannot be reached, can she at least reach the Pareto-efficient frontier of discrimination-free SCFs without using transfers? As a starting point for our following analysis, we first discuss trading incentives between two agents if positive income effects exist.

4.1 Trading incentives among two agents

To begin with, consider an agent with type (θ, e). Object ω provides a utility of $\theta(\omega)$. Define $k(\theta(\omega), e)$ as the willingness to pay *(wtp)* for ω, i.e.,

$$\theta(\omega) + h(e - k(\theta(\omega), e)) = h(e). \tag{4.1}$$

$c(\theta(\omega), e)$ is the willingness to accept *(wta)* for to giving up ω, i.e.,

$$\theta(\omega) + h(e) = h(e + c(\theta(\omega), e)). \tag{4.2}$$

Quasilinear preferences, (i.e., $h'' = 0$) imply $k(\theta(\omega), e) = c(\theta(\omega), e) = \theta(\omega)$. In contrast, for positive income effects (i.e., $h'' < 0$), $k(\theta(\omega), e)$ and $c(\theta(\omega), e)$ increase in both object utility and wealth. By definition $k(\theta(\omega), e) = c(\theta(\omega), e - k(\theta(\omega), e)$, which implies

$$k(\theta(\omega), e) < c(\theta(\omega), e). \tag{4.3}$$

Therefore, the money an agent is willing to accept for giving up an object exceeds what he is willing to pay for the object. This gap between the wta and the wtp has consequences for trading incentives between two agents. Even if the buyer is richer and derives a higher utility from consumption, two agents may not want to trade. More specifically, let a seller S own object ω and buyer B own ω'. Both prefer ω to ω', i.e.,

$$\Delta\theta_i = \theta_i(\omega) - \theta_i(\omega') > 0 \quad \text{for} \quad i = S, B.$$

Then, $k(\Delta\theta_B, e_B)$ is the buyer's willingness to pay for owning ω instead of ω'. Likewise, $c(\Delta\theta_S, e_S)$ is the seller's willingness to accept for receiving ω' instead of ω. For positive income effects (i.e., $h'' < 0$), $k(\Delta\theta_B, e_B)$ and $c(\Delta\theta_S, e_S)$ increase in both object utility and wealth. Since for equal types, the *wta* exceeds the *wtp*, $\Delta\theta_B > \Delta\theta_S$ is neither necessary nor sufficient for trade. If $\Delta\theta_B < \Delta\theta_S$, the agents want to trade if B is rich enough. Conversely, even if $\Delta\theta_B > \Delta\theta_S$, the agents may not want to trade if neither the object utilities nor the wealth levels differ much. This is in contrast to quasilinear preferences, for which agents trade if and only if $\Delta\theta_B > \Delta\theta_S$ such that wealth endowments are irrelevant.

The following example illustrates how this gap between the *wtp* and *wta* might be reflected in an example of school choice.

Example 1. *Assume there are two schools, school A and school B. School B has unlimited capacity while School A has limited capacity but is the preferred school for all students. Going to A requires a fee of* 50,000 *USD while there is no fee for going to B. Consider a family who is not willing to pay the fee for going to school A. Now assume that the school places are randomly assigned without the need to pay the fee for school A. The above-mentioned family receives a place at school A. Another family being assigned to B is now offering* 50,000 *USD for the place to that family. Positive income effects can explain why the family may not be willing to give up the place for* 50,000 *USD though it exceeds their willingness to pay for school A. The decrease in marginal utility of money when getting richer implies that paying* 50,000 *USD hurts more than receiving the* 50,000 *USD benefits.*

4.2 (Constrained) efficiency with known types.

We start by considering a market designer who knows the agents' types and must not use monetary transfers for the assignment of objects. Trading incentives among two agents can be caused by differences in object utilities as well as differences wealth. Let φ^c be a

SCF without monetary transfers that assigns the objects by maximizing the sum of object utilities. Formally,

$$\varphi^c = (\sigma^c, 0) \quad \text{with} \quad \sigma^c(\theta) = argmax_\sigma \sum_{i=1}^{n} \theta_i(\sigma_i(\theta)). \tag{4.4}$$

φ^c accounts for any differences in object utilities such that they do not cause trading incentives.

For quasilinear preferences, wealth endowments are irrelevant to efficiency of a SCF because they do not impact preferences. Then, a SCF φ is efficient if and only if $\varphi = \varphi^c$.

With positive income effects, the intuition of the efficiency properties of φ^c is well reflected by a special case of two agents S and B and one object ω. If $\theta_S > \theta_B$, the SCF φ^c assigns the object to agent S. φ^c is efficient if and only if

$$k(\theta_B, e_B) - c(\theta_S, e_S) \leq 0 \quad \text{for all} \quad \theta_S \geq \theta_B \in \Theta \quad \text{and} \quad e_S, e_B \in E. \tag{4.5}$$

Since both $k(\cdot,\cdot)$ and $c(\cdot,\cdot)$ are increasing in their arguments, (4.5) is equivalent to

$$k(\theta, \overline{e}) - c(\theta, \underline{e}) \leq 0 \quad \text{for all} \quad \theta \in \Theta. \tag{4.6}$$

For $\overline{e} = \underline{e}$, the inequality is strict (see 4.3). Then, φ^c is efficient if and only if wealth inequality is small enough, measured by $\overline{e}$. Furthermore, $k(\theta, \overline{e}) - c(\theta, \underline{e})$ is concave in θ and equals zero for $\theta = 0$ as well as for some $\theta^* > 0$. Efficiency of φ^c thus requires, ceteris paribus, that either inequality is small enough or that the object utility is high enough for all possible type realizations. The less valuable the object ω is (compared to not being assigned), the less inequality already leads to trading incentives.

Since φ^c incorporates any differences in object utilities, potential trading incentives are driven by wealth inequality. Whenever the potential buyer is poorer than the seller, there is no incentive to trade. Therefore, trading incentives cannot be resolved without violating discrimination-freeness and φ^c is at the efficient frontier of discrimination-free SCFs.

Proposition 1 generalizes the above intuitions to a setting of many objects and agents. With more than one object, the structure of Θ is relevant through $\inf \Theta$ measuring how similar any two objects can be evaluated that might be assigned.

$$\inf \Theta := \inf_{\theta \in \Theta, \omega, \omega' \in \Omega^*} |\theta(\omega) - \theta(\omega')| \tag{4.7}$$

For $\sum_{\omega \neq \{0\}} c(\omega) = n$ we set $\Omega^* = \Omega \setminus \{0\}$, otherwise $\Omega^* = \Omega$. The reason is that for measuring how similar two objects can be, the Null-object is only relevant if at least one agent needs to stay unassigned.

Proposition 1. *Let* (Θ, E) *be some type space. There exists* $e^c \geq \underline{e}$ *such that* $\varphi^c = (\sigma^c, 0)$ *is efficient if and only if* $\overline{e} \leq e^c$. $e^c = e^c(\inf \Theta, \underline{e})$ *strictly increases in* $\inf \Theta$ *with* $e^c(0, \underline{e}) = \underline{e}$ *and* $\lim_{\inf \Theta \to \infty} e^c(\inf \Theta, \underline{e}) = \infty$. *Furthermore,* φ^c *is at the efficient frontier of* Φ_{DF}.

Once $\varphi^c = (\sigma^c, 0)$ is inefficient, any transfer-free and discrimination-free SCF is inefficient as well. Proposition 1 thus implies that whether monetary transfers are needed for achieving efficiency depends on wealth inequality (measured by $\overline{e}$) as well as on how similar the objects can be evaluated (measured by $\inf \Theta$). However, by maximizing the sum of object utilities, the efficient frontier of discrimination-free SCFs can be reached without using transfers. The characteristics of $e^c(\inf \Theta, \underline{e})$ reflect that if the object utility space Θ is such that two objects can be arbitrarily similar (i.e., $\inf \Theta = 0$), φ^c is efficient if and only if no inequality exists (i.e., $\overline{e} = \underline{e}$). For $\inf \Theta > 0$, however, φ^c can be efficient even if wealth inequality exist (i.e., $\overline{e} > \underline{e}$). This result is implied by the gap in the *wtp* and *wta*, see (4.3). Conversely, for any $\underline{e} > \underline{e}$ and $\overline{e} < \infty$ given, φ^c is efficient once all objects are distinct enough (i.e., $\inf \Theta$ is large enough).

4.3 (Constrained) efficiency with unknown types.

Maximizing the sum of object utilities is attractive but requires information about types. If types are private information, implementability restricts the set of SCFs the market designer can use. Without monetary transfers, implementability requires that an agent's object assignment must only depend on his type through his object ranking and particularly not on his intensity of preferences or wealth endowment.[11] The efficient frontier of money-free SCFs then is the set of ordinal-efficient SCFs. An example for such a mechanism is the Serial Dictatorship in which agents take turns picking an object. Ex-post

[11] To see this, consider a transfer-free SCF $\varphi = (\sigma, 0)$ and θ_i, θ_i' with $R(\theta_i) = R(\theta_i')$. Assume $\varphi(\theta_i, \theta_{-i}) \neq \varphi(\theta_i', \theta_{-i})$ for some $\theta_{-i} \in \Theta^{N-1}$. It implies that $\sigma_i(\theta_i) \neq \sigma_i(\theta_i')$. Since $R(\theta_i) = R(\theta_i')$, agent i then has either an incentive to misreport for θ_i or θ_i'. Therefore, φ is not implementable.

trading incentives then may arise not only due to wealth inequality but also due to differences in the intensities of object utilities. In the following, we discuss the role of the type space (Θ, E) for whether an implementable SCF at the efficient frontier of transfer-free SCFs is efficient or at the efficient frontier of discrimination-free SCFs.

To fix ideas, consider again two agents and one object. An implementable and transfer-free SCF φ^o then assigns the object independent of types and is at the efficient frontier of transfer-free SCFs. Agent S then might receive the object even if agent B values it more (i.e., $\theta_S < \theta_B$). φ^o is efficient if and only if

$$k(\theta_B, e_B) - c(\theta_S, e_S) \leq 0 \quad \text{for all} \quad \theta_S, \theta_B \in \Theta \quad \text{and} \quad e_S, e_B \in E. \tag{4.8}$$

Monotonicity properties make it sufficient to check trading incentives if the buyer has the highest willingness to pay that can be induced by the type space and the seller has lowest. More specifically, for $\overline{e} < \infty$, $\inf \Theta > 0$ and $\sup \Theta < \infty$ let $V = \frac{\sup \Theta}{\inf \Theta}$ be the maximal relative variation the object utilities in Θ can have. φ^o then is efficient if and only if

$$k(V \inf \Theta, \overline{e}) - c(\inf \Theta, \underline{e}) \leq 0 \tag{4.9}$$

Inefficiency of φ^o thus depends on $\underline{e}$, $\overline{e}$, $\inf \Theta$ and V induced by the type space. For large $\overline{e}$ (i.e., wealth inequality is large), φ^o is inefficient. For $\overline{e}$ close to $\underline{e}$ efficiency of φ^o depends on $\inf \Theta$ and V. Notably, keeping any $\overline{e} \leq \underline{e}$ and V fixed, (4.9) holds if $\inf \Theta$ is large enough. The smaller V or the larger $\inf \Theta$, the more wealth inequality can exist without inducing trading incentives. For $V = 1$ (i.e., both agents attach the same utility to the object), φ^o is efficient if and only if $\overline{e} \geq e^c$ (see Proposition 1).

If ^{o} is inefficient, we are furthermore interested in whehter φ^o is at the efficient frontier of discrimination-free SCFs. To answer this, it is relevant whether the trading incentive is caused by wealth inequality or by differences in object utilities. If some trading incentive exists independent of wealth realizations it is solely driven by differences in object utilities. Then, a discrimination-free Pareto-improvement exists. Conversely, if any trading incentive disappears once the buyer is poor and the seller is rich, a discrimination-free Pareto-improvement does not exist and φ^o is at the efficient frontier of discrimination-free SCFs. Formally, φ^o is at the efficient frontier of discrimination-free SCFs if and only if

$$k(V \inf \Theta, \underline{e}) - c(\inf \Theta, \overline{e}) \leq 0. \tag{4.10}$$

Inequalities (4.10) and (4.9) are similar except that $\underline{e}$ and $\overline{e}$ are interchanged. The reason is that here the relevant case is whether potential trading incentives survive if the seller is poor and the buyer is rich.

In the remainder of this section, we transfer the above insights to the general case of many agents and many objects. First, we treat the limit cases for the type space. These are arbitrarily high wealth inequality (i.e., $\overline{e} = \infty$), arbitrarily close objects (i.e., $\inf \Theta = 0$) or arbitrarily distinct objects (i.e., $\sup \Theta = \infty$). All three cases imply inefficiency of transfer-free SCFs while the results for constrained efficiency differ.

Proposition 2. *For $\overline{e} = \infty$, any implementable SCF at the efficient frontier of Θ_{TF} is inefficient but at the efficient frontier of Θ_{DF}. For $\inf \Theta = 0$ or $\sup \Theta = \infty$, any implementable SCF at the efficient frontier of Θ_{TF} is inefficient and, if $\overline{e} < \infty$, not at the efficient frontier of Φ_{DF}*

In the following, assume $\inf \Theta > 0$, $\sup \Theta < \infty$ and $\overline{e} < \infty$. First, we establish that wealth inequality is a key driver for efficiency and constrained efficiency.

Proposition 3. *Let (Θ, E) be such that $\inf \Theta > 0$ and $\sup \Theta < \infty$. Let φ^o be an implementable SCF at the efficient frontier of Φ_{TF}. There exists a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ which is independent of the wealth space E, such that*

- *φ^o is efficient if and only if $\overline{e} \leq e^*$ with $e^* = f(\underline{e})$.*
- *φ^o is at the efficient frontier of Θ_{DF} if and only if $\overline{e} > \hat{e}$ with $\hat{e} = f^{-1}(\underline{e})$.*

Proposition 3 implies that for large wealth inequalities (measured by $\overline{e}$) any SCF at the efficient frontier of transfer-free SCFs is inefficient but at the efficient frontier of discrimination-free SCFs. For low wealth inequalities (i.e., $\overline{e} \approx \underline{e}$), one of these efficiency properties does not continue to hold. $e^* < \underline{e}$ implies that the SCF is inefficient even without wealth inequality (i.e., $\overline{e} = \underline{e}$). However, since $\hat{e} = f^{-1}(\underline{e})$, $e^* < \underline{e}$ implies $\hat{e} > \underline{e}$. Then, for low wealth inequalities, the SCFs is not only inefficient but also not at the efficient frontier of discrimination-free SCFs. Conversely, if $e^* \geq \underline{e}$, efficiency can be reached without the use of transfers if wealth inequality is low enough. Furthermore, it implies that $\hat{e} < \underline{e}$, such that independent of wealth inequality, transfer-free SCFs at the efficient frontier of transfer-free SCFs are at the efficient frontier of discrimination-free SCFs as well.

Corollary 1. *For any type space (Θ, E) and any implementable SCF φ^o at the efficient frontier of Θ_{TF} it holds that $e^* < \underline{e}$ if and only if $\hat{e} > \underline{e}$.*

The actual size of the critical wealth e^* depends on the object utility space Θ, the lower bound of wealth $\underline{e}$ as well as on the concrete SCF φ^o considered. Based on the intuitions derived in our introductory discussion of the case with two agents and one object, we now develop upper and lower bounds for e^* (and for $\hat{e}$) that do not depend on the specific SCF considered but only on key characteristics of the object utility space Θ like how similar objects can be evaluated and how much object utilities can vary.

To measure how much object utilities in Θ can vary, we define V_Θ as the maximal relative variation in object utilities that can realize for the type space Θ. Formally,

$$V_\Theta = \frac{\sup\limits_{\theta \in \Theta, \omega, \omega' \in \Omega^*} |\theta(\omega) - \theta(\omega')|}{\inf\limits_{\theta \in \Theta, \omega, \omega' \in \Omega^*} |\theta'(\omega) - \theta'(\omega')|} = \frac{\sup \Theta}{\inf \Theta} \tag{4.11}$$

Again (see (4.7)), $\Omega^* = \Omega$ if at least one agents stays unassigned and $\Omega^* = \Omega \setminus \{0\}$ if no agent needs stay unassigned. We define v_Θ as the minimal relative variation of two objects in the following way.

$$v_\Theta = \inf_r \inf_{\omega, \omega' \in \Omega^*} \sup_{\theta, \theta' \in \Theta(r)} \frac{|\theta(\omega) - \theta(\omega')|}{|\theta'(\omega) - \theta'(\omega')|}. \tag{4.12}$$

In words, v_Θ is defined such that for any rank order and any pair of objects selected, the relative variation in object utilities with this rank order is at least v_Θ.

Proposition 4. *Let (Θ, E) be some type space with $\inf \Theta > 0$ and $\sup \Theta < \infty$. There exist $\delta(\inf \Theta, V_\Theta, \underline{e})$ and $\rho(\inf \Theta, v_\Theta, \underline{e}) \leq e^c(\inf \Theta, \underline{e})$ such that for all implementable SCFs φ^o at the efficient frontier of Φ_{TF} it holds that*

$$\delta(\inf \Theta, V_\Theta, \underline{e}) \leq e^* \leq \rho(\inf \Theta, v_\Theta, \underline{e}) \tag{4.13}$$

$\delta(\inf \Theta, V_\Theta, \underline{e})$ increases in $\inf \Theta$ and $\underline{e}$ and decreases in V_Θ with $\lim\limits_{\inf \Theta \to \infty} \delta(\inf \Theta, V_\Theta, \underline{e}) = \infty$. $\rho(\inf \Theta, v_\Theta, \underline{e})$ increases in $\inf \Theta$ and $\underline{e}$ and decreases in v_Θ with $\lim\limits_{v_\Theta \to \infty} \rho(\inf \Theta, v_\Theta, \underline{e}) <$

$\underline{e}$ *and* $\lim_{\inf \Theta \to 0} \rho(\inf \Theta, v_\Theta, \underline{e}) \leq \underline{e}$.

Keeping some $\inf \Theta$, v_Θ, *and* V_Θ *fix,* $\hat{e}$ *satisfies* $\rho^{-1}(\underline{e}) \leq \hat{e} \leq \delta^{-1}(\underline{e})$.

By Proposition 4, for $\overline{e} \leq \delta$ any implementable SCF φ^o at the efficient frontier of transfer-free SCFs is efficient. $\lim_{\inf \Theta \to \infty} \delta(\inf \Theta, V_\Theta, \underline{e}) = \infty$ implies that $e^* > \underline{e}$ holds if $\inf \Theta$ is large enough. In words, once all objects are distinct enough, efficiency can be reached without using transfers as long as wealth inequality is small enough. Even more, for any $V_\Theta < \infty$ and $\underline{e} < \overline{e} < \infty$ fixed, any implementable SCF at the efficient frontier of Φ_{TF} is efficient if $\inf \Theta$ is large enough. Conversely, $\overline{e} \geq \rho$ implies that any implementable SCFs without transfers is inefficient. By the characteristics of ρ derived in Proposition 4, this holds if, ceteris paribus, the minimal variation v_Θ is large enough or $\inf \Theta$ is small enough.

The estimates for $\hat{e}$ follow in an analogous way. The more variation the object utilities in Θ have, measured by minimal variation in v_Θ or the more similar object can be, measured by $\inf \Theta$, the higher the lower bound for $\hat{e}$ is. It mirrors the intuition that the more the object utilities can differ among two agents, the less the potential trading incentives depend on wealth, increasing the opportunities for discrimination-free Pareto-improvements.

This section's efficiency results show that for large wealth inequalities, money is needed to achieve efficiency but it is not needed to reach the efficient frontier of discrimination-free SCFs. Therefore, banning transfers cannot only ensure discrimination-freeness but it still allows to reach the efficient frontier of discrimination-free SCFs. For low wealth inequalities, implementable SCFs without transfers can be even efficient. However, if for low wealth inequalities, implementable SCFs without transfers are inefficient, they are even not at the efficient frontier of discrimination-free SCFs. Then, a ban of transfers restricts more than requiring discrimination-freeness would require since discrimination-free Pareto-improvements exist.

Example 2. *As in the Example 1, consider two schools A and B. School B has unlimited capacity, and school A has limited capacity but is preferred by all. School B' utility is normalized to zero. Whether efficiency or the efficient frontier of discrimination-free SCFs can be reached without the use of monetary transfers depends on the type space. In particular, it depends i) on wealth inequality, ii) on how low A can be evaluated, and iii) on how much the valuations for A can vary among the students.*

If the market designer is informed about types, she can assign school A to those with the highest utility for A. However, if students might evaluate school A arbitrarily close to B, even small wealth inequalities induce trading incentives. Any trades that may occur are wealth dependent which implies that any Pareto-improvement violates discrimination-freeness. Conversely, as long as wealth is bounded above and A is sufficiently distinct from B for all students, there won't be any students who want to trade ex-post.

If preferences are private information, the market designer has to rely on implementable mechanisms, e.g., a lottery. For efficiency results, it becomes relevant how the valuations for A can vary. If wealth inequality is low, AND school A is important enough to all, AND the utilities for A cannot differ too much among the students, a lottery of school places is efficient. Then, even if a rich student receives school B and a poor student school A, the students won't trade. The more heterogeneously the agents may evaluate A, the more likely it is that those with high benefits from A may want to buy the school places from those benefiting much less. If A can be so much more valuable for some compared to others, they want to buy even if they are poor and those selling are rich. Such a trade is then independent of wealth. Then, a lottery is inefficient and a discrimination-free Pareto-improvement exists.

5 Implementability of discrimination-free SCFs with money

In the previous section, we took the perspective of a market designer who must not use monetary transfers and discussed how it depends on the type space whether she can efficiently assign the objects or reach the efficient frontier of discrimination-free SCFs.

Now consider a market designer who can use monetary transfers but faces discrimination-freeness as a constraint. If types are private information, prices can facilitate to account for preferences intensities. However, linking the assignment of an object to the payment of a price may lead to discrimination, as the willingness to pay depends on the realized wealth endowments. The central question is now under which conditions transfers can be employed without violating discrimination-freeness to implement an object allocation that is not implementable without using transfers. In other words, does a market designer who has to comply with discrimination-freeness possess a broader toolkit for allocating objects than a market designer who must not use monetary transfers?

To gain insight into the intuition of the problem, consider again the case of only one object. Suppose $\varphi = (\sigma, m)$ is implementable and discrimination-free. Additionally, assume that the object allocation σ cannot be implemented without the use of money. It implies that there is an agent who, when announcing some high object value θ_H, receives the object, while for a low value θ_L, he does not. For implementability, receiving the object needs to be linked to the payment of a price. Discrimination-freeness requires that keeping any realization $\theta \in \Theta$ fixed, whether or not the agent is willing to pay the price is independent of his wealth endowment. The larger wealth inequalities can be, the more difficult it is to satisfy these requirements. Furthermore, if Θ is an interval, it is possible to find a type $(\theta^*, e^*) \in \Theta \times E$ with $\underline{e} < e^* < \overline{e}$ who is indifferent between buying or not. Then, keeping θ^* fixed, the agent wants to buy the object when being rich enough but does not want to buy it when being poor enough. It implies that φ can only account for intensities in object utilities when wealth inequalities are not too large and the closure of Θ is not an interval. A market designer who a priori knows that values can be either high or low but not intermediate thereby may use money for implementation. Intuitively, the lower wealth inequality is, the smaller the gap can be.

The following propositions generalize this intuition.

Proposition 5. *Let $\underline{e} < \overline{e}$. If $\overline{e} = \infty$ or if the closure $\overline{\Theta}$ of Θ is convex, for any implementable $\varphi = (\sigma, m) \in \Phi_{DF}$, $\varphi^0 = (\sigma, 0) \in \Phi_{TF}$ is implementable as well.*

Large wealth inequalities or convex object utility spaces thereby imply that the market designer is restricted as if she must not use monetary transfers since she can implement the same object allocation without transfers as well. Note that it does not mean that she must not use moonetary transfers at all. For instance, if softening the condition of budget-balance and instead allowing to collect money, fees that are independent of the object allocation may be charged to cover costs associated with the objects.

By Proposition 5, non-convexity of $\overline{\Theta}$ is a necessary condition to find implementable and discrimination-free SCFs for which the object allocation is not implementable without monetary transfers. We show that if $\overline{\Theta}$ is a cartesian product, non-convexity implies that there exists some wealth space E such that an implementable SCF $\in \Phi_{DF}$ exists whose object allocation is not implementable without transfers.

Proposition 6. *Let $\overline{\Theta}$ be not convex and assume that $\overline{\Theta} = \overline{\Theta}(\omega_1) \times .... \times \overline{\Theta}(\omega_k)$. For any $\underline{e} > e_0$, there exists $\overline{e} \in (\underline{e}, \infty)$ such that if $E \subset [\underline{e}, \overline{e}]$, some implementable SCF $\varphi = (\sigma, m) \in \Phi_{DF}$ exists such that $\hat{\varphi} = (\sigma, 0) \in \Phi_{TF}$ is not implementable.*

Example 3. *We again turn to the example of two schools A and B. School B has unlimited capacity while school A has limited capacity but is preferred by all over school A. The utility of school B is normalized to zero. The market designer might use monetary transfers but she has to ensure discrimination-freeness. Implementability requires that if a student's assignment depends on school A's utility, being assigned to A comes along with the payment of a price. The price needs to be independent of the student's type. Discrimination-freeness requires that if keeping the utility for A fixed, whether or not the student is willing to pay the price is independent of wealth realizations.*

Now assume that the utility space for A is such that students either value A quite low or quite high, but not at an intermediate level. Then, there exists a price such that, if keeping some wealth level fixed, only those with high utility for A will pay the price but those with low utility for A won't. If inequality is small enough this holds independent of wealth realizations. There must not exist intermediate utilities for A because otherwise for these intermediate utilities, the willingness to pay would depend on their realization of wealth.

6 Discussion

6.1 Using Money Outside the centralized procedure

Numerous real-world applications demonstrate that even if a mechanism to allocate objects is transfer-free, there are other factors that can be influenced by money and, consequently, affect the outcomes. When a market designer lacks the ability to prevent the influence of money on these factors, wealth disparities can impact the allocation of resources, giving again rise to concerns of discrimination.

The concept of bribing is a useful formalization of using money outside the centralized mechanism. Schummer (2000b) defines bribes as an agent pays another agent to misreport preferences, resulting in a mutually beneficial outcome for both compared to a scenario with truthful reporting. Several real-world applications can be interpreted as a form of bribing. For instance, at Emory University, US, the successful students have priority for receiving a place in specific courses. It turned out that not only seats were sold after the assignment but also that students with high priorities misreported their preferences in favor of popular courses to be able to sell them (Chronicle, 2019).

To fix ideas, consider the serial dictatorship mechanism where agents take turns selecting an object. The mechanism is implementable and discrimination-free. Suppose that the first and second agents in line have the same rank order of objects. If now the second agent's willingness to pay for trading the assignments is higher than the first agent's willingness to accept, both agents would benefit if the second agent were to bribe the first agent to misreport preferences. Therefore, even with discrimination-free SCFs, money might be used for bribes which, in turn, it allows to gain advantages for agents with higher wealth. A way to circumvent discrimination concerns arising from bribes is to require that a SCF is not only discrimination-free but also bribe-proof. A SCF is called bribe-proof if no bribery incentives as described above exist (Schummer, 2000b). By Schummer (2000a), in a setting with quasilinear preferences over bundles of objects and transfers, bribe-proofness requires that each agent's outcome is independent of other agents' preferences.[12] The general idea can be applied to our setting with non-linear preferences. If a SCF is required to be bribe-proof, the market designer might be restricted beyond not using transfers, particularly if wealth inequality is high.[13]

Remark 1. *Assume that $\overline{e} = \infty$ and consider some implementable and discrimination-free SCF $\varphi \in \Phi_{DF}$. φ is bribe-proof if and only if $\varphi_i(t_i, t_{-i}) = \varphi_i(t_i, t'_{-i})$ for all $t_i \in T$ and $t_{-i}, t'_{-i} \in T^{n-1}$.*

If a SCF is not bribe-proof, it can put the rich in an advantageous position. However, requiring that an agent's assignment is independent of other agents' types heavily restricts the information about preferences a market designer can use to assign objects. This is at the expense of efficiency. In particular, if there are exactly as many objects as agents, the allocation is essentially constant, i.e., types are irrelevant for the assignment. A simple lottery satisfies this condition. If more objects than agents exist it implies wastefulness: there is a type profile such that an object remains unassigned that is preferred to the assigned object by at least one agent.

If the SCF is non-bossy, i.e. an agent cannot alter another agent's assignment without

[12]In Schummer (2000b) it is generalized to a broader class of quasilinear settings.
[13]In the appendix we provide a formal definition of bribe-proofness and a formal proof of the remark.

changing his own, bribe-proofness is necessary to ensure discrimination-freeness. A SCF that is not non-bossy does not necessarily cause discrimination if bribing incentives exist, even if wealth inequality is large. This is because if misreporting of one agent alters another agent's assignment without changing his own, bribes are essentially costless and thus do not depend on wealth endowments.

The arguments are also transferable to setting with two market sides like patients and donors in the context of kidney donation where bribes among patients and donors could become a concern.

Investing in Priority. Many centralized assignment procedures use priorities as a substitute for preferences of a second side of the market. In our setting, they could be an ex-ante criterion to break ties. In school choice problems, for instance, each school ranks the students according to priority criteria. A priority criteria frequently used is the distance of a student's home to the school. Since those being able to afford high house prices have the choice where to live, students can gain priority at a preferred school by the means of money. Black (1999), for instance, showed that house prices are positively correlated with the quality of the school in the neighborhood. In the context of kidney transplants, organs are typically distributed to patients on a waiting list based on priority measures. Steve Jobs reportedly obtained his liver transplantation because he was advised to raise his chances by subscribing to waiting lists in other states than his home state California.[14] This approach required to be wealthy enough to be able to quickly move to any location, e.g., by private plane.

Investing in priorities can be interpreted as a special case of bribes among two sides of a market. As long as an assignment procedure depends on parameters that can be influenced via costly investments, wealth influences the assignment and discrimination concerns become relevant.

Co-existing Private Markets. In addition to objects assigned via a centralized mechanism, there might exist further objects that are distributed through a private market. Once on the private market objects are distributed via prices, wealth impacts on the access to the objects which in turn gives rise to discrimination concerns. A classical example are private schools that charge admission fees. Interpreting the admission decision of a

[14] See, e.g., CNN (2009).

private school as its preference report illustrates that a co-existing private market is a special case of bribes. As long as a private market co-exists and charges prices or fees which are paid dependent on the wealth endowment, discrimination concerns are not fully addressed.

6.2 Model Assumptions

In the following, we discuss some assumptions on the model and illustrate how the basic model presented might be extended to address several settings relevant for applications.

Preference space. We have introduced several assumptions regarding the space of preferences, with the most restrictive one being additive separability. Additive separability allows us to isolate the impact of wealth on the marginal utility of money, making it easier to point out the drivers of our results. However, in the light of some applications, wealth may also influence the benefits derived from object consumption. For instance, the poor might benefit from a good school more than the rich because the rich might easier be able to substitute shortcomings.

The core intuition and results of our analysis are driven by the assumption that the willingness to pay increases with income. Softening the other assumptions does not alter the character of our analysis. As long as the willingness to pay increases with income, the presence of higher wealth inequality can lead to greater trading incentives if objects are allocated without the use of monetary transfers. This, in turn, makes it more challenging to realize these trades without violating discrimination-freeness. It also continues to hold that with small wealth inequalities, a SCF without transfers can be efficient since for positive income effects the willingness to accept exceeds the willingness to pay.

In our basic model, for small wealth inequalities, characteristics of the value space Θ are crucial. When softening the assumption of additive separability, describing the role of the type space beyond wealth inequality becomes less concrete. For more general preference spaces, the efficiency of the results would depend on the range of the willingness to pay and the willingness to accept in dependence on wealth. Additionally, the density of the preference space would play a significant role in analyzing implementable SCFs.

Assigning probability shares. In our analysis, we take an ex-interim perspective by considering deterministic outcomes and assuming that any tie-breakers (like priorities or lotteries) are determined ex-ante. Extending the model by assigning probability shares of objects might improve ex-ante efficiency since lotteries allow to exploit cardinal information about preferences. The definition of discrimination-freeness can be modified as the probabilistic object assignment of an agent is independent of wealth.

While the general idea holds that larger wealth inequality leads to more trading incentives, the discussion of efficiency properties becomes more complex. This is because the willingness to pay to receive an object with probability π is concave in π. This characteristic gives reason to smooth access to the goods by not assigning an object with probability one to one agent but to assign probability share to several agents (see Huesmann (2017)).

Two-sided Markets. We consider a one-sided market where only the agents that receive the objects have preferences and might act strategically. Whenever providers of the objects are strategic players (like it can be the case for kidney donations) our notion of discrimination-freeness can be applied for the other side of the market as well.

7 Conclusion

We study an assignment problem in which agents have heterogeneous wealth endowments and their preferences involve positive income effects. First, we introduce discrimination-freeness as a constraint that addresses the concern that wealthier agents may have better access to the goods. Second, we find that for large wealth inequalities, transfer-free and ordinal-efficient SCFs are inefficient but are at the efficient frontier of discrimination-free SCFs. Discrimination-freeness then implies the same restrictions on the assignment of objects as banning monetary transfers. However, for low wealth inequalities, ordinal and transfer-free assignments can be efficient or even not at the efficient frontier of discrimination-free SCFs. Depending on the structure of the preferences, monetary transfers might be used to assign the objects without violating discrimination-freeness. Our results contribute to explaining the severe restrictions on monetary transfers in certain markets like school choice procedures or organ donations if wealth inequality is high. For small wealth inequalities, monetary transfers might induce more severe restrictions on the assignments of objects than discrimination-freeness does. Countries with quite equitable distribution of wealth or high redistribution instruments have greater flexibility in allowing transfers than countries with substantial wealth inequalities.

A market designer might consider using incentives beyond money to account for preference intensities. For instance, in a setting where each agent receives several objects, endowing all agents with the same amount of tokens as play money can facilitate preference elicitation. Also, the willingness to wait can be interpreted as a sign of preference intensities. For instance, admission to public hearings at Capitol Hill in the US is typically granted on a first-come, first-served base (Vox, 2019). One could expect that those queuing earlier have a stronger preference for attending than those queuing later.

However, even if money is not used to allocate resources, money might help to influence factors outside a market designer's control. Banning transfers thus might not sufficiently address discrimination concerns. In the above example of queuing lines, it turned out that lobbyists paid homeless people to queue for them (Vox, 2019). Also, in the example of school choice, if better schools are allocated in more expensive neighborhoods, living in a rather expensive neighborhood implies better access to schools (Black, 1999). This raises valid concerns about unequal access. The chairman of the Black Alliance for Educational Option wrote: *"If access to high-performing schools has to come down to a number, better it be a lottery number than a ZIP code."*[15]

Li (2017) argues that a market designer's task is not to ultimately solve the question of how to design a market but to study the link between design and consequences to inform the respective decision-makers. This includes, for instance, considering whether a market design is fair or increases welfare. In this spirit, our work does not provide specific advice on whether to ban monetary transfers but helps to understand the consequences of wealth inequality on the assignment of resources and the implications of addressing inequality concerns by requiring discrimination-free access to goods.

There is a branch of questions for further research that arises from and extends our results. On which markets is discrimination-freeness a desideratum and why? What are the trade-offs between discrimination-freeness and efficiency? What are further moral concerns beyond discrimination-freeness that might be important in certain markets? For instance, slippery slope effects are often feared in the context of the introduction of monetary transfers that may lead to commercialisation of the goods. Another concern is

[15]See New York Times (2011).

the exploitation of people in the sense that financial distress might make people unable to decide in their best interest and they might thus regret a decision later.[16]

[16] Zargooshi (2001) surveyed people in Iran who sold a kidney after some years. A striking 85% percent of the questioned people indicated that they regret the donation.

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A Proofs

Preliminaries

We start the section of the proofs with some characteristics of the *wtp* and the *wta* that will be useful for the proofs.

Lemma 7. *For any* $x > 0$, $\lim_{e\to\infty} k(x,e) = c(x,e) = \infty$ *and* $\lim_{e\to-\infty} k(x,e) = c(x,e) = 0$ *hold.*

Proof of the Lemma. By assumption, $h'(e)$ is strictly decreasing with $h'(e) \to 0$ for $e \to \infty$. It implies that for all $\epsilon > 0$ there exists some κ with $h'(\kappa) < \epsilon$. Then, $k(x, \kappa + \frac{x}{\epsilon}) > \frac{x}{\epsilon}$. Since $\frac{x}{\epsilon} \to \infty$ for $\epsilon \to 0$ it also holds that $k(x,e) \to \infty$ for $e \to \infty$. Since $k(x,e) < c(x,e)$, $\lim_{e\to\infty} c(x,e) = \infty$ holds as well.

Proof of Proposition 1

Define $e^c = h^{-1}(h(\underline{e}) + \inf\Theta)$. First, we show that for any $x > 0$ it holds that

$$\overline{e} \leq e^c \Leftrightarrow k(x,\overline{e}) \leq c(x,\underline{e}). \tag{A.1}$$

For this, we use that $k(x,e) = e - h^{-1}(h(e) - x)$ and

$$\overline{e} \leq h^{-1}(h(\underline{e}) + x) \Leftrightarrow \underline{e} \geq h^{-1}(h(\overline{e}) - x) \tag{A.2}$$

We then obtain that $\overline{e} \leq e^c$ is equivalent to

$$\begin{aligned}
c(x,\underline{e}) &= c(x, \overline{e} - (\overline{e} - \underline{e})) && \text{(A.3)}\\
&\geq c(x, \overline{e} - (\overline{e} - h^{-1}(h(\overline{e}) - x))) && \text{(A.4)}\\
&= c(x, \overline{e} - k(x,\overline{e})) && \text{(A.5)}\\
&= k(x,\overline{e}) && \text{(A.6)}
\end{aligned}$$

Therefore, the equivalence (A.1) holds.

To show the proposition we first assume that $\overline{e} > e^c$. It is to show that φ^c is inefficient.

$\overline{e} > e^c$ is equivalent to

$$\inf \Theta < h(\overline{e}) - h(\underline{e}). \tag{A.7}$$

It implies that there exists $\theta \in \Theta$, $\omega', \omega \in \Omega$ such that $\theta(\omega') - \theta(\omega) < h(\overline{e}) - h(\underline{e})$. Assume that $\theta_i = \theta$ for all agents i. Furthermore, let i be the agent receiving ω' and j be the agent receiving ω. By the introducing discussion, it implies

$$k(\theta(\omega') - \theta(\omega), \overline{e}) > c(\theta(\omega') - \theta(\omega), \underline{e}). \tag{A.8}$$

Therefore, if $e_j = \underline{e}$ and $e_i \leq \overline{e}$ is large enough, the two agents have an incentive to trade. This implies that φ^c is inefficient. The same holds for $\overline{e} = \infty$.

Now assume that $\overline{e} \leq e^c$ which is equivalent to $\inf \Theta \geq h(\overline{e}) - h(\underline{e})$. It is to show that it implies efficiency of $\varphi^c = (\sigma^c, 0)$. $\varphi^c = (\sigma^c, 0)$ is efficient if and only if for all object utility profiles $(\theta)_{i \in I}$ and object allocations σ

$$\sum_{i \in I^+} k(\theta_i(\omega_i) - \theta_i(\omega_i^c), \overline{e}) \leq \sum_{i \in I^-} c(\theta_i(\omega_i^c) - \theta_i(\omega_i), \underline{e}). \tag{A.9}$$

Here, ω_i^c and ω_i are the objects that agent i receives under σ^c and σ, respectively. I^+ and I^- are the sets of agents for whom the object assignment under σ (compared to σ^c) improves and worsens, respectively.

Take any value profile $(\theta)_{i \in I}$ and any object allocation $\hat{\sigma}$. Define $X_i = |\theta_i(\omega_i) - \theta_i(\omega_i^c)|$. Since σ^c maximizes the sum of object utilities we have $\sum_{i \in I^+} X_i \leq \sum_{i \in I^-} X_i$. Furthermore, for $\delta = h(\overline{e}) - h(\underline{e})$ it holds that $k(\delta, \overline{e}) \leq c(\delta, \underline{e})$. It implies

$$\sum_{i \in I^+} X_i k(\delta, \overline{e}) \leq \sum_{i \in I^-} X_i c(\delta, \underline{e}). \tag{A.10}$$

Since $\inf \Theta \geq \delta(\underline{e}, \overline{e})$ we have $X_i \geq \delta$ for all i. Positive income effects then imply $k(X_i, \overline{e}) \leq \frac{X_i}{\delta} k(\delta, \overline{e})$ as well as $c(X_i, \underline{e}) \geq \frac{X_i}{\delta} c(X_i, \underline{e})$. Combining it with A.10 yields

$$\sum_{i \in I^+} k(X_i, \overline{e}) \leq \sum_{i \in I^+} \frac{X_i}{\delta} k(\delta, \overline{e}) \leq \sum_{i \in I^+} \frac{X_i}{\delta} c(\delta, \underline{e}) \leq \sum_{i \in I^+} c(X_i, \underline{e}). \tag{A.11}$$

This corresponds to (A.9) which implies that $\varphi^c = (\sigma^c, 0)$ is efficient.

$e^c(\inf \Theta, \underline{e}) = h^{-1}(h(\underline{e}) + \inf \Theta)$ strictly increases in $\inf \Theta$ since $h(\cdot)$ strictly increases.

Furthermore, $\lim_{\inf\Theta\to\infty} e^c(\inf\Theta, \underline{e}) = \lim_{\inf\Theta\to\infty} h^{-1}(h(\underline{e}) + \inf\Theta) = \infty$ and $e^c(\inf\Theta, \underline{e}) = h^{-1}(h(\underline{e}) + 0) = \underline{e}$.

Finally, we show that φ^c is at the efficient frontier of Θ_{DF}. Assume that a set of agents has an incentive to trade. Since φ^c maximizes the sum of object utilities, the trading incentive disappears once the buyers are poor and the sellers are rich. Therefore, any Pareto-improvement of φ^c discriminates.

Proof of Proposition 2

Let $\varphi^o = (\sigma^o, 0)$ be an implementable SCF at the efficient frontier of Φ_{TF}. It implies that φ^o is independent of wealth endowments and does only depend on the rank order of objects.

Case 1: $\overline{e} = \infty$. Assume that all agents have the same type $(\theta, e) \in \Theta \times E$. Assume agent i receives object ω and agent j receives object ω' with $\theta(\omega) - \theta(\omega') > 0$. Since $\overline{e} = \infty$ and $k(\theta(\omega) - \theta(\omega'), e) \to \infty$ for $e \to \infty$, there exists some $e' \in E$ such that

$$k(\theta(\omega) - \theta(\omega'), e') > c(\theta(\omega) - \theta(\omega'), e) > 0. \tag{A.12}$$

Now assume all agents except agent j have the type (θ, e) while agent j has the type (θ, e'). Since φ^o does not depend on wealth endowments, the object allocation does not alter. Then, agent i and agent j have an incentive to trade which implies that φ^o is inefficient.

To proof that φ^o is at the efficient frontier of Φ_{DF} assume that for the type profile $(\theta_i, e_i)_{i\in I}$ the agents have an incentive to trade. It implies that there exists $\varphi = (\sigma, m) \in \Phi_{DF}$ with

$$\sum_{i\in I^+} k(\theta_i(\omega_i) - \theta_i(\omega_i^o), e_i) > \sum_{i\in I^-} c(\theta_i(\omega_i^o) - \theta_i(\omega_i), e_i). \tag{A.13}$$

ω_i^o and ω_i are the objects which agent i receives under σ^o and σ, respectively. I^+ and I^- are the sets of agents for whom the object assignment under σ (compared to σ^o) improves and worsens, respectively.

Now assume that wealth endowments change for the agent in I^-. Since both φ^o and φ are discrimination-free, their object allocations do not depend on wealth. $c(x, e) \to \infty$ for $e \to \infty$ and $x > 0$ implies that there exists some $e' \in E$ such that if all agents $i \in I^-$

have wealth e', keeping all other type parameters fixed, inequality A.13 does not hold any more which implies that φ cannot be a discrimination-free Pareto-improvement of φ^o. Therefore, φ^o is at the efficient frontier of Φ_{DF}.

Case 2: $\inf \Theta = 0$ **(and** $\overline{e} < \infty$**).** For $\inf \Theta = 0$ there exists a sequence $(\theta_n)_{n \in \mathbb{N}}$ with $\theta_n \in \Theta$ and $R(\theta_n) = R(\theta_m)$ for all $n, m \in \mathbb{N}$ such that

$$|\theta_n(\omega') - \theta_n(\omega)| \to 0 \quad \text{for some} \quad \omega', \omega \in \Omega \tag{A.14}$$

Such a sequence because Ω is finite and therefore the number of rank orders that can exist is finite as well. If there does exists on sequence as described above, $\inf \Theta = 0$ cannot hold. Without loss of generality we assume that $\theta_n(\omega') > \theta_n(\omega)$.

(A.16) implies that there some $\theta, \theta' \in \Omega$ with $R(\theta) = R(\theta')$ such that

$$k(\theta(\omega') - \theta(\omega), \underline{e}) - c(\theta'(\omega') - \theta'(\omega), \overline{e}) > 0 \tag{A.15}$$

Let $\varphi^o = (\sigma^o, 0)$ be an implementable SCF at the efficient frontier of Θ_{TF}. Assume that for all agents object utility θ realizes and that $\sigma^o_i = \omega$ and $\sigma^o_j = \omega'$. Note that implementability implies that φ^o does not depend on wealth. Now assume that for all agents object utility θ realizes except for agent j who has object utility θ'. Then, the object allocation under φ^o is unchanged since it only depends on rank order. By (A.15), agents i and agent j have an incentive to trade for any wealth realization e. Therefore, φ^o is not efficient. Furthermore, providing i with ω' and providing j with ω plus a transfer for compensation from i to j is a Pareto-improvement. While the transfer may depend on wealth realizations, the new object allocation does not. This implies that the Pareto-improvement is discrimination-free and φ^o is therefore not at the efficient frontier of Φ_{TF}.

Case 2: $\sup \Theta = \infty$ **(and** $\overline{e} < \infty$**).** For $\sup \Theta = \infty$ analogous arguments as for $\inf \Theta = 0$ hold. More specifically, there exists a sequence $(\theta_n)_{n \in \mathbb{N}}$ with $\theta_n \in \Theta$ and $R(\theta_n) = R(\theta_m)$ for all $n, m \in \mathbb{N}$ such that

$$|\theta_n(\omega') - \theta_n(\omega)| \to \infty \quad \text{for some} \quad \omega', \omega \in \Omega \tag{A.16}$$

Again, without loss of generality, $\theta_n(\omega') > \theta_n(\omega)$. It implies that there some $\theta, \theta' \in \Omega$ with $R(\theta) = R(\theta')$ such that (A.16) holds. The remainder of the proof is then the same as for $\inf \Theta = 0$.

A.1 Proof of Proposition 3

Furthermore, $h'(e) \to -\infty$ for $e \to -\infty$. Then, for all $\epsilon > 0$ there exists some e with $h'(e) > \frac{1}{\epsilon}$. It implies that $k(x,e) < x\epsilon$. Therefore, $k(x,e) \to 0$ for $e \to -\infty$. Since $k(x,e) = c(x, e - k(x,e))$, it holds that $c(x,e) \to 0$ for $e \to -\infty$ as well.

Proof of the Proposition. Let $\varphi^o = (\sigma^o, 0)$ be a SCF at the efficient frontier of Φ_{TF}. φ^o then only depends on the rank order of objects. φ^o is efficient if and only if for all object utility profiles $(\theta_i)_{i \in I}$ and object allocations σ it holds that

$$\sum_{i \in I^+} k(\theta_i(\omega_i) - \theta_i(\omega_i^o), \overline{e}) \leq \sum_{i \in I^-} c(\theta_i(\omega_i^o) - \theta_i(\omega_i), \underline{e}). \tag{A.17}$$

ω_i^o and ω_i are the objects agent i receives under σ_i^o and σ_i, respectively. I^+ and I^- are the sets of all agents for whom the object assignment under σ, compared to σ^o, would improve and worsen, respectively. Since σ^o is ordinal efficient, $I^- \neq$. For every object allocation σ with $I^+ \neq$ define $\tilde{e}((\theta_i)_{i \in I}, \sigma)$ such that for $\overline{e} = \tilde{e}((\theta_i)_{i \in I}, \sigma)$, inequality (**??**) is satisfied with equality. $\tilde{e}((\theta_i)_{i \in I}, \sigma, \underline{e})$ exist and is well defined since $k(x,e)$ strictly increases in e with $k(x,e) \to \infty$ for $e \to \infty$ and $k(x,e) \to 0$ for $e \to -\infty$. Then define

$$f(e) = \inf_{\sigma} \inf_{(\theta_i)_{i \in I}} \tilde{e}((\theta_i)_{i \in I}, \sigma, e) \tag{A.18}$$

Then, if $\overline{e} \leq f(\underline{e})$, (A.26) holds and φ^o is efficient. For $\overline{e} > f(\underline{e})$, (A.26) does not hold and φ^o is thus inefficient. Note that f is increasing in e since $c(x,e)$ is increasing in e. By definition of the function f depends on φ^o and Θ but does not depend on the wealth space E.

Now define $\hat{e} = f^{-1}(\underline{e})$. Then, $\overline{e} > \hat{e}$ if and only if for all object utility profiles $(\theta_i)_{i \in I}$ and object allocations σ

$$\sum_{i\in I^+} k(\theta_i(\omega_i) - \theta_i(\omega_i^o), \underline{e}) < \sum_{i\in I^-} c(\theta_i(\omega_i^o) - \theta_i(\omega_i), \overline{e}). \tag{A.19}$$

If (A.19) holds, φ^o is tat the efficient frontier of discrimination-free SCFs because any potential trading incentives disappear once the buyers are poor enough and the seller are rich enough. Conversely, if (A.19) does not hold, there exists some type profile $(\theta_i)_{i\in I}$ and object allocation σ such that allocation the object according to σ and compensating those receiving worse object on the cost of those receiving better objects is a discrimination-free Pareto-improvement.

Proof of Proposition 4

We start with a technical lemma.

Lemma 8. *For any $n \geq 1$, $V \geq 1$ consider the function $F : \mathbb{R}_+ \to \mathbb{R}_+0$ defined as*

$$F(x) = nk(xV, e_1) - c(x, e_2). \tag{A.20}$$

For $e_1 > e_2$

(i) $F'(0) > 0$

(ii) $F''(x) < 0$ *for all* $x \geq 0$

(iii) There exists a unique $x^* > 0$ such that $F(x) < 0$ if and only if $x > x^*$

Proof. Explicit calculation (TBD).

Definition and characteristics of ρ. We define $\rho(\inf \Theta, v_\Theta, \underline{e})$ implicitly by

$$k(v_\Theta \inf \Theta, \rho) - c(\inf \Theta, \underline{e}) = 0 \tag{A.21}$$

ρ is well defined because $k(x,e)$ strictly increases in e with $\lim_{e\to\infty} k(x,e) = \infty$ and $\lim_{e\to-\infty} k(x,e) = 0$ (see Lemma **??**). Furthermore, $k(x,e)$ strictly increases in x. The properties of $k(\cdot,\cdot)$ thus imply that $\rho(\inf \Theta, v_\Theta, \underline{e})$ strictly decreases in v_Θ and strictly increases in $\underline{e}$. Also, $k(v_\Theta \inf \Theta, \underline{e}) > c(\inf \Theta, \underline{e})$ holds for for v_Θ large enough such that

$\lim\limits_{v_\Theta \to \infty} \rho(\inf \Theta, v_\Theta, \underline{e}) < \underline{e}$. Furthermore, for $v_\Theta = 1$ the definition of ρ equals the definition of e^c such that $\rho(\inf \Theta, 1, \underline{e}) = e^c(\inf \Theta, \underline{e})$. This implies $\rho(\inf \Theta, v_\Theta, \underline{e}) \leq e^c(\inf \Theta, \underline{e})$.

For how ρ depends on $\inf \Theta$ note that Lemma 8 implies $\frac{\delta \rho(v_\Theta \inf \Theta, \underline{e}}{\delta \inf \Theta} > 0$. Furthermore, $\rho(\inf \Theta, v_\Theta, \underline{e}) \leq e^c(\inf \Theta, \underline{e})$ implies that $\lim_{\inf \Theta \to 0} \rho(\inf \Theta, v_\Theta, \underline{e}) \leq \lim_{\inf \Theta \to 0} e^c(\inf \Theta, \underline{e}) = \underline{e}$.

We proof that $e^* \leq \rho(\inf \Theta, v_\Theta, \underline{e})$. For this, we show that $\overline{e} > \rho(\inf \Theta, v_\Theta, \underline{e})$ implies inefficiency of any implementable SCF at the efficient frontier of Φ_{TF}. By definition of ρ we have

$$k(v_\Theta \inf \Theta, \overline{e}) - c(\inf \Theta, \underline{e}) > 0 \tag{A.22}$$

Then, there are two objects $\omega, \omega' \in \Omega$ and $\theta \in \Theta$ such that

$$k(v_\Theta \inf \Theta, \overline{e}) - c(\theta(\omega') - \theta(\omega), \underline{e}) > 0 \tag{A.23}$$

By definition of v_Θ there exists some θ' with the same rank order as θ such that $\theta'(\omega') - \theta'(\omega) > v_\Theta \inf \Theta$ holds. It implies that

$$k(\theta'(\omega') - \theta'(\omega), \overline{e}) - c(\theta(\omega') - \theta(\omega), \underline{e}) > 0 \tag{A.24}$$

Now consider any implementable SCF $\varphi^o = (\sigma^o, 0)$ at the efficient frontier of Φ_{TF}. Since $\overline{e} > e^*$ implies inefficiency of φ^o it is to show that $\overline{e} > \rho(\inf \Theta, v_\Theta, \underline{e})$ implies inefficiency of φ^o. Assume that type realizations are such that $\theta_i = \theta$ for all agents. Furthermore, let agent i be the agent receiving object ω and let agent j be the agent receiving ω'. Now consider change the type profile such that all agents keep their type except for agent j who now has object utilities θ'. Since θ' implies the same rank order of objects as θ does, implementability of φ^o implies that the object allocation is the same as for σ^o. By (A.24), there exist wealth realizations such that agent j and agent i have an incentive to trade and φ^o is not efficient.

Definition and characteristics of δ. Define $\delta(\inf \Theta, V_\Theta, \underline{e})$ implicitly by

$$(n-1)k(V_\Theta \inf \Theta, \delta) - c(\inf \Theta, \underline{e}) = 0 \tag{A.25}$$

The monotonicity properties of δ are implied by the same arguments as for ρ.

To show that $e^* \leq \delta$ we show that if $\overline{e} \leq \delta$, any implementable SCF at the efficient frontier of Φ_{TF} is efficient. For this, consider any implementable SCF $\varphi^o = (\sigma^o, 0)$ at the efficient frontier of Φ_{TF}. φ^o then only depends on the rank order of objects. φ^o is efficient if for all object utility profiles $(\theta_i)_{i \in I}$ and object allocations σ it holds that

$$\sum_{i \in I^+} k(\theta_i(\omega_i) - \theta_i(\omega_i^o), \overline{e}) \leq \sum_{i \in I^-} c(\theta_i(\omega_i^o) - \theta_i(\omega_i), \underline{e}). \tag{A.26}$$

ω_i^o and ω_i are the objects agent i receives under σ_i^o and σ_i, respectively. I^+ and I^- are the sets of all agents for whom the object assignment under σ (compared to σ^o) improve and worsens, respectively. Since σ^o is ordinal efficient, $I^- \neq \emptyset$.

Now fix any σ and $(\theta_i)_{i \in I}$. It holds that

$$\sum_{i \in I^-} c(\theta_i(\omega_i^o) - \theta_i(\omega_i), \underline{e}) \geq c(\inf \Theta, \underline{e}). \tag{A.27}$$

Since $\theta_i(\omega) - \theta_i(\omega') \leq V_\Theta \inf \Theta$ for any two objects ω, ω' and I^+ contains at most $n-1$ agents we have

$$\sum_{i \in I^+} k(\theta_j(\omega_i) - \theta_i(\omega_i^o), \overline{e}) \leq (n-1) k(V_\Theta \inf \Theta, \overline{e}) \tag{A.28}$$

Now consider $\overline{e} \geq \delta(\inf \Theta, V_\Theta, \underline{e})$. It then implies

$$\sum_{i \in I^+} k(\theta_j(\omega_i) - \theta_i(\omega_i^o), \overline{e}) \leq (n-1) k(V_\Theta \inf \Theta, \overline{e}) \leq c(\inf \Theta, \underline{e}) \leq \sum_{i \in I^-} c(\theta_i(\omega_i^o) - \theta_i(\omega_i), \underline{e}). \tag{A.29}$$

Estimates for $\hat{e}$. By Proposition 3, it holds that $\hat{e} = f^{-1}(\underline{e})$. The same argumentation can be used to show that the estimates for $\hat{e}$ are the inverse functions of the estimates of e^*.

Proof of Proposition 5

Assume $\varphi = (\sigma, m) \in \Phi_{DF}$ that is implementable but $\varphi_0 = (\sigma, \mathbf{0})$ is not. We show that this leads to a contradiction if E is unbounded or if $\overline{\Theta}$ is convex.

Discrimination-freeness implies that σ does not depend on wealth realizations. Consider any $e \in E$. If $\varphi = (\sigma, m) \in \Phi_{DF}$ is implementable but $\varphi_0 = (\sigma, \mathbf{0})$ is not there exists some agent i who has an incentive to misreport under φ_0. That is, for a fixed announcement of the other agents (omitted in the following), there exists $\theta^1, \theta^2 \in \Theta$ and $\omega^1, \omega^2 \in \Omega$ such that

$$\varphi_i(\theta^1, e) = (\omega^1, m^1) \quad \text{and} \quad \varphi_i(\theta^2, e) = (\omega^2, m^2)$$

while $\theta^1(\omega^2) > \theta^1(\omega^1)$. It implies that $m^1 > m^2$ as agent i needs to be compensated for receiving a worse object if announcing θ^1 instead of θ^2. Differently said, if announcing θ^1 he pays a price for receiving a more preferred object.

Case I: $\overline{e} = \infty$ Implementability and discrimination-freeness of φ implies that $\theta^1(\omega^2) + h(e + m^2) > \theta^2(\omega^1) + h(e + m^1)$ for all $e \in E$. However, if $\overline{e} = \infty$, $m^1 - m^2$ is not enough to compensate the agent once he is rich enough. This is a contradiction.

Case II: Θ is convex. We call a bundle (ω, m) of an object and money *reachable* for agent i if there exists some $\theta \in \Theta$ with $\varphi_i(\theta, e) = (\omega, m)$. Implementability implies that the number of reachable bundles is finite and restricted by $k + 1$. Implementability of φ implies that for announcing (θ, e), agent i is assigned to the bundle that type (θ, e) prefers most among all reachable bundles.

Our goals is now to find $\theta \in \overline{\Theta}$ with $\varphi(\theta, e_L) \neq \varphi(\theta, e_H)$ for some $e_L < e_H \in E$. The main step is to construct some $\theta^* \in \overline{\Theta}$ such that agent i with type (θ^*, e) with $e = \frac{e_H - e_L}{2}$ and $e_L, e_H \in E$ is indifferent between two reachable bundles of objects and money while preferring both over all other reachable bundles. Positive income effects then imply that $\varphi(\theta^*, e_L) \neq \varphi(\theta^*, e_H)$. If $\theta^* \in \Theta$ we take $\theta = \theta^*$. If $\theta^* \notin \Theta$, we can find some θ that is close enough to θ^* such that $\varphi(\theta, e_L) \neq \varphi(\theta, e_H)$ holds as well.[17]

We construct θ^* by using the convexity of $\overline{\Theta}$. Convexity of $\overline{\Theta}$ implies that there exists some $k \in [0, 1]$ such that for $\theta^3 = k\theta^2 + (1 - k)\theta^1$ it holds that $\theta^3 \in \overline{\Theta}$ and $\theta^3(\omega^2) + h(e + m^2) = \theta^3(\omega^1) + h(e + m^1)$. If there does not exist any (ω^3, m^3) with $\theta^3(\omega^2) + h(e + m^2) > \theta^3(\omega^2) + h(e + m^2) = \theta^3(\omega^1) + h(e + m^1)$, take $\theta^* = \theta^3$. Otherwise, there exists some (ω^3, m^3) with

$$\theta^3(\omega^3) + h(e + m^3) > \theta^3(\omega^2) + h(e + m^2) = \theta^3(\omega^1) + h(e + m^1) \tag{A.30}$$

[17] It is not necessary that $e \in E$ since e is only needed to construct indifference for e while wealth endowments $e_L, e_H \in E$ imply the contradiction to discrimination-freeness.

With the same argument we then can find some θ^4 as a linear combination of θ^3 and θ^2 such that

$$\theta^3(\omega^3) + h(e+m^3) = \theta^3(\omega^2) + h(e+m^2) > \theta^3(\omega^1) + h(e+m^1) \tag{A.31}$$

Again, if there does not exist any (ω^4, m^4) with $\theta^4(\omega^4)+h(e+m^4) > \theta^3(\omega^2)+h(e+m^2) = \theta^3(\omega^1) + h(e+m^1)$, we are done with $\theta^* = \theta^4$. So assume there exists some (ω^4, m^4) with

$$\theta^3(\omega^4)+h(e+m^4) > \theta^3(\omega^3)+h(e+m^3) = \theta^3(\omega^2)+h(e+m^2) > \theta^3(\omega^1)+h(e+m^1) \tag{A.32}$$

Following this procedure we construct object values $\theta^a(\omega^a)+h(e+m^a)$ that are indifferent between two bundles of objects and money and these bundles are at least the $k-a+1$th best bundle. After at most $k+1$ steps, we have

$$\theta^a(\omega^a) + h(e+m^a) = \theta^a(\omega^{a-1}) + h(e+m^{a-1}) > \theta^a(\omega) + h(e+m) \tag{A.33}$$

for all other reachable bundles (ω, m). $\theta^* = \theta^a$ then satisfies the desired criteria which shows that convexity of $\overline{\Theta}$ implies that φ cannot be discrimination-free and implementable if φ^0 is not.

Proof of Proposition 6

If $\overline{\Theta}$ is not convex, there exists some $\theta, \theta' \in \Omega$ and $k \in (0,1)$ such that $\theta^* = k\theta+(1-k)\theta' \notin \overline{\Theta}$. I implies that there exists some ω, ω' such that $\theta^*(\omega) = k\theta(\omega) + (1-k)\theta'(\omega) \notin \overline{\Theta}$. Since $\overline{\Theta}^C$ is an open set, and $k(\theta^*(\omega), e)$ is continuous in e and in $\theta^*(\omega)$, there exists some $\epsilon > 0$ and $\delta > 0$ such that for all $\theta \in \Theta$, $e \in B_\delta(e^*)$ it either holds that

$$k(\theta(\omega), e) < k(\theta^*(\omega), e^*) - \epsilon \quad \text{or} \quad k(\theta(\omega), e) > k(\theta^*(\omega), e^*) + \epsilon.$$

Therefore, if $E \subset B_\delta(e^*)$, all values are such that the willingness to pay is either below $k(\theta^*(\omega), e^*)-\epsilon$ or above $k(\theta^*(\omega), e^*)+\epsilon$ but never the same. Furthermore, by construction of θ^*, there exists one values such that it is lower, and one values such that it is higher.

Now consider a mechanism that is a serial dictatorship mechanism such that one after each other every agent is allowed to pick an object. Transfers are zero for all object, expect for ω, for which the price $k(\theta^*(\omega), e^*)$ has to be paid.

This mechanism is implementable. It is furthermore discrimination-free, since whether or not an agent selects ω does not depend on wealth but only on θ. Finally, this mechanism is not implementable without transfers. This is because all agents prefer to have ω over having the Null-object 0. However, an agent that did not pick ω because his object value was too small will pick ω if transfers are not admitted. So we found a mechanisms that is implementable and discrimination-free while the object allocation is not implementable without transfers.

Proof of Remark 1

First, we provide a formal definition of bribe-proofness in the spirit of Schummer (2000b).

Definition 2 (Bribing). *Let $\varphi = (\sigma, m)$ be a social choice function. Agent i has an incentive to bribe agent j if there is a profile $t \in T^n$, a corrupted type $t'_j \neq t_j \in T$, and a bribe amount $\tau \geq 0$ such that*

- $u_i(\sigma_i(t'_j, t_{-j}), e_i + m_i(t'_j, t_{-j}) - \tau) > u_i(\sigma_i(t), e_i + m_i(t))$ *and*
- $u_j(\sigma_j(t'_j, t_{-j}), e_j + m_j(t'_j, t_{-j}) + \tau) > u_j(\sigma_j(t), e_j + m_j(t))$.

φ is bribe-proof if no incentives to bribe exist.

Consider some implementable and discrimination-free SCF $\varphi \in \Phi_{DF}$. Obviously, $\varphi_i(t_i, t_{-i}) = \varphi_i(t_i, t'_{-i})$ for all i, $t_i \in T$ and $t_{-i}, t'_{-i} \in T^{n-1}$, φ is bribe-proof since misreporting of any agent (including agent i) cannot improve agent i's assignment.

So assume that φ is implementable, discrimination-free and bribe-proof. We first show that it implies that φ does not depend on wealth realizations at all. Disrimination-freeness implies that for $\varphi_i = (\sigma_i, m_i)$, σ_i does not depend on agent i's wealth realization. By implementability, m_i does not depend on wealth realizations, either, because otherwise agent i had an incentive to misreport wealth. Now assume that agent i's wealth report impacts on the assignment of another agent j. Discrimination-freeness implies that it must not impact on another agent's object assignment. It remains to show that it does not impact on the tranfers as well. However, if the wealth report had an impact on the transfer of some agent i, this agent had an incentive to bribe agent i to misreport wealth. Therefore, φ does not depend on wealth reports.

For an agent of type t_i denote strict preferences over outcomes by P_i, weak preferences by R_i, and indifferences by I_i. We first show that it holds that $\varphi_i(t_j, t_{-j}) I_i \varphi_i(t'_j, t'_{-j})$ for all j, t_j, t'_j and t_{-j}. Second, we show that it implies $\varphi_i(t_j, t_{-j}) = \varphi_i(t'_j, t'_{-j})$. $\varphi_i(t_i, t_{-i}) = \varphi_i(t'_i, t'_{-i})$ then follows by induction.

To show that bribe-proofness implies $\varphi_i(t_j, t_{-j}) I_i \varphi_i(t'_j, t'_{-j})$ assume the contrary, i.e. there exists t_j, t'_j and t_{-j} (omitted in the following) such that $\varphi_i(t_j) P_i \varphi_i(t'_j)$ (the same argumentation holds if $\varphi_i(t'_j) P_i \varphi_i(t_j)$). We show that we can find type profiles such that i has an incentive to bribe agent j. For this, vary agent i's wealth such that his willingness to pay for the misreport exceed agent j's willingness to accept for the misreport. We can find such a wealth level, because wealth does not impact on the agent j's assignment (see above). Thus, there exists a type profile such that a bribery incentive occurs and ϕ is not bribe-proof.

It remains to show that $\varphi_i(t_j, t_{-j}) I_i \varphi_i(t'_j, t'_{-j})$ implies $\varphi_i(t_j, t_{-j}) = \varphi_i(t'_j, t'_{-j})$. Again, assume the contrary, i.e.,

$$(\omega, m) = \varphi_i(t_i, t_j) \neq \varphi_i(t_i, t'_j) = (\omega', m').$$

If $\omega = \omega'$ (i.e., agent i evaluates them equally), it implies that $m = m'$. Thus, it has to hold that $\omega \neq \omega'$ and $m \neq m'$. If agent i's wealth now is large enough, agent i strictly prefers the bundle with the more preferred object. This contradicts that φ is independent of wealth and that $\varphi_i(t_j, t_{-j}) I_i \varphi_i(t'_j, t'_{-j})$.