

# Dynamic Bidding in Second Price Auction

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## Abstract

We consider equilibrium bidding behavior in a dynamic second price auction where agents have the option to increase bids at random times and values follow a Markov process. We prove that equilibrium exists and is unique and give an algorithm to solve for bids as a function of time and values. Equilibrium bids equal the expected final value conditional on the bid placed being the final one, meaning that either the agent doesn't get another opportunity to rebid or chooses not to increase this bid if given the option. This results in adverse selection with respect to a bidder's own future strategy, and as a result bids are shaded relative to the bidder's expected value. This is true in spite of values being independent across bidders. Under mild conditions, desired bids increase as time increases and the close of the auction is approached. Our results are consistent with repeated bidding and sniping, two puzzling observations in eBay auctions. We estimate the model by matching moments from Ebay auctions and consider a series of counterfactuals.

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...if something went over my limit early on, I might re-evaluate my budget and make a higher bid because I want the product. If it was the last few minutes I wouldn't have the time to consider if I can afford it.

...work + college = does not give you the right time you need to baby an auction.

The wife wanted to go to lunch right at the same time as the end of the auction, so I decided to drop an early bid an hour before the close.

## 1 Introduction

Various auction mechanisms happen in a dynamic setting while most of the theoretical models of auctions are static. For example, on eBay, auction listings usually last between three and seven days. Bidders can bid at anytime during the active time of the auction and also increase their bids at any other time prior to the end. As the first quote suggests<sup>1</sup>, information might arrive during the auction that can change a bidder's value. The second and third quotes suggest bidders might not have full control of the timing of future bids.<sup>2</sup> In this paper we develop a model of a dynamic auction that captures these two features.

In our model, bidders arrive randomly and possibly at multiple times in a second price auction set in continuous time for a fixed interval. At each time of arrival bidders observe a new signal and choose whether to place a bid or increase a previous one. At the end of the auction, the winner is the highest bidder and the price equal to the second highest bid. We characterize the equilibrium of this dynamic auction.

Our paper is motivated by bidding behavior in eBay auctions, which are close to second price auctions, where bidders can place bids at any point in time up until the end of the auction. Several authors have emphasized what seem to be anomalies in bidding behavior, like submission of multiple bids throughout the auction and a higher concentration of bids towards the end of the auction, as we document below. This type of behavior cannot be

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<sup>1</sup>The first two quotes were obtained from a blog <http://www.neowin.net/forum/topic/587154-why-do-people-bid-so-early-on-ebay/>. The second one is from <http://www.kenrockwell.com/tech/ebay/early-bidding.htm>.

<sup>2</sup>Sniping programs are an incomplete solution as they do not condition bids on possible changes of valuation that could arise from new information.

rationalized through the lens of a static model, where bidders should bid their valuation only once and at the time of arrival to the auction. Our model provides a way to rationalize this behavior combining two ingredients: new information can change the optimal bid throughout the auction and the possible lack of a future opportunity to bid provides a rationale for early bidding.

In most of the paper, we consider the case of independent private values, where signals and final values are independent across bidders, and where bidding times are exogenous.<sup>3</sup> We model this through a—otherwise unrestricted—joint Markov process of signals and bidding times that are also sufficient statistics for the final expected value. We prove equilibrium is unique and provide a dynamic programming algorithm to derive the optimal bids as functions of the state variable—signal and time.

Bidding behavior has an intuitive analog to a standard second price auctions' behavior. Consider first the case where bidders have a single opportunity to bid prior to the end of the auction. In that case, they bid their expected final value as in a standard second price auction. In contrast, when bidders can return to the auction prior to its end with positive probability and rebid, the expected value is modified taking into account that the current bid will only apply if the bidder chooses not to exercise that option. The optimal bid takes this into account and as we show at each bidding time bidders bid their expected value conditional on that bid being the final one.

The possibility of rebidding is a source of *adverse selection* as the current bid only applies when the bidder chooses not to exercise future options of rebidding and this is correlated with lower future value.<sup>4</sup> As a consequence, bidders shade bids below their unconditional expected final value. In particular, as the probability of bidding at the very end of the auction converges to one, any prior bid goes to zero as would happen with sniping.

There is an important difference to the case of adverse selection with correlated values. While bids are shaded relative to expected values, the *expected final bid* still equals the expected final value and is thus unbiased. These two results are consistent since bids, while

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<sup>3</sup>Section 9.1 analyzes a simplified model with endogenous bidding and Section 9.2 analyzes a simplified model with correlated values among bidders.

<sup>4</sup>An analogue result is found in Harris and Holmstrom [1982], where initially worker's wages are shaded below marginal products, as the wage is effective in the future only if it is less or equal than the realized marginal product of the worker.

shaded, are still unbiased in the set where they *apply*. As a comparison, an auction that admits bidders to retract on bids will have no shading, since a bid only applies when no rebidding takes place or when the final expected value is the same at a new bidding time.

To derive further properties, we consider the specialized case where bidding times are Poisson, with possibly time dependent arrival rates, while signals and final values are independent of arrivals. Assuming the expected final value conditional on a signal is non-decreasing in time, we show that the bidding function increases over time. This provides a rationale why bidders might increase their bids over time independently of competitive pressures. The intuition for this result is straightforward and goes back to the incentives for shading: as the end of the auction approaches and the option of rebidding becomes less likely, the adverse selection problem mentioned above and the incentives for shading tend to disappear.

The problem of solving for the optimal bidding function is not a simple one, as current bids depend on all future bidding behavior. The dynamic programming problem we define provides an indirect method for deriving the optimal bidding functions, defined implicitly as zeros of the value function. We use this method to derive explicitly a closed form solution to the bidding function in a special case we call the *bad news* model, where bidders initial value can become zero if a Poisson shock arrives. This scenario can be motivated by exogenous events, e.g., the impossibility of attending a concert or bidders changing their mind or finding a better alternative for their purchase. In addition to this case, we also specialize the model to signals that follow a Brownian motion with drift, independently of Poisson arrivals for bidding times.<sup>5</sup>We derive a partial differential equation that can be used to solve for equilibrium bidding function.

Matching a series of moments from the data, we estimate parameter values for the Brownian motion case. We then consider as a counterfactual a similar bidding mechanism to our baseline but where bidders can retract bids at any of their random bidding times. The possibility of retraction eliminates the source of adverse selection mentioned earlier and thus the incentives for bid shading. As a consequence, at any bidding time the bidder will choose

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<sup>5</sup>We conjecture there exist differential equations for optimal bids as a function of time and current signals that we have not yet derived.

a bid equal to the expected final value. As a reference point we also compute the efficient allocation and prices, i.e. that corresponding to a second price auction taking place at the end of period. The correlation between final values and final bids increases moderately (from 0.85 to 0.88) when comparing the baseline scenario to the case of bid-retraction. As a result the expected final value increases, closing 1/3 of a fairly moderate gap of 3.5% between the baseline model and the efficient allocation. While the gap in values is moderate, the gap in prices between baseline and the efficient level is more than twice as high. This is precisely the consequence of bid shading in our baseline scenario. As a comparison, the price gap is in the order of only 2% for the bid retraction case. As a consequence, allowing for bid-retraction would increase revenues of sellers by approximately 6% while decreasing the expected utility of buyers by 18%.

The paper is organized as follows. Section 2 discusses the evidence on eBay bidding behavior and reviews related literature. Section 3 provides a simple example that conveys the main intuition and results in the paper. Section 4 describes the general model and defines an equilibrium. Section 5 proves existence and uniqueness of equilibrium, characterizes bidding behavior and provides the dynamic programming algorithm. Section 6 gives properties for the case where values are independent of Poisson arrivals for bidding times and solves the two special cases described above. Section 7 shows that the bidding function derived before is still applicable under arbitrary assumptions about the information a bidder observes on the past bids of other bidders. Section 8 provides the estimates and counterfactuals. Sections 9.1 and 9.2 consider, respectively, the cases of endogenous bidding and correlated signals.

## 2 Evidence and Related Literature

The model presented in this paper closely mimics eBay auctions. eBay is an online auction and shipping website introduced in 2005. Sellers can sell their items either through an auction or by setting a fixed price for their item, an option called “Buy it Now.” The auction mechanism is similar to a second price or Vickery auction. A seller sets the starting bid of an auction and bidders can bid repeatedly for the item until the end of the auction. Each bidder observes all previous bids, except for the current highest bid. A bidder should

bid an amount higher than the current second highest bid, plus some minimum increment.<sup>6</sup> If this value is higher than the current highest bid, the bidder becomes the new highest bidder. Otherwise, he becomes the second highest bidder. The winner has to pay the second highest bid, plus the increment or his/her own bid, whichever is smaller. Auctions last for one to ten days and they have a pre-determined and fixed ending time that cannot be changed once the auction is active.

On eBay, bidders' bid-placement, as noted in the literature, does not follow the prediction of a static model of auction, for instance a disproportional share of bids are placed in the last few seconds of an auction. As a recent paper by [Backus et al. \[2013\]](#) shows, about third of winning bids are placed in the last ten seconds of an auction. [Hayne et al. \[2003\]](#) report that about 15 percent of bids placed are within the last 60 minutes of an auction and also about 61 percent of bids placed by bidders who submitted more than one bid. [Gonzalez et al. \[2009\]](#) similarly report 11 percent of bids placed within the last 60 minutes of auction and 77 percent of bidders placed more than one bids. [Hayne et al. \[2003\]](#) further note that on average there are 6.78 bids placed and about 3.98 unique bidders per auction. Moreover, they show that late bidding has a much higher success rate, about 75.17 percent, much more than early bidding, 7.33 percent, or bidding in between, 40.50 percent. Using a static model of auctions, strategic late bidding or bidding multiple times is not rational.

Several papers have tried to rationalize sniping behavior and incremental bidding. [Bajari and Hortacsu \[2003\]](#) incorporate a model of common value auction to explain this phenomenon. Having informed and uninformed participants, the informed bidders do not bid before the last period since that would reveal their private information to other potential bidders. Hence in the equilibrium, all bidders only place a bid at the very last period of the auction, leading to sniping by everyone. In another set of papers, to explain sniping [Ockenfels and Roth \[2002\]](#) and [Ockenfels and Roth \[2006\]](#) compare auctions mechanisms at eBay and Amazon which have hard ending and soft ending, respectively. Hard ending refers to a fixed ending time for an auction which cannot be extended by the seller or the marketplace; whereas soft ending refers to a tentative ending time: placing a bid within the last few minutes of the auction extends the duration of the auction for another ten minutes.

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<sup>6</sup>The increment is a function of the second highest bid, fixed for all auctions, and is set by eBay.

They argue that along with the hard-ending rule at eBay, there are incremental bidders on eBay whom increase their maximum bid as they get outbid by other bidders. Therefore, a strategic bidder facing incremental bidders places bids in the last possible moment not to give incremental bidders time to react and place higher bids, giving rise to sniping.

Gray and Reiley [2004] and Ely and Hossain [2009] independently run experiments evaluating the benefit of sniping. The former does not find a statistically significant effect while reconfirming results in Ockenfels and Roth [2002] and Ockenfels and Roth [2006] regarding the prevalence of sniping; in their dataset 50 percent of bids are placed within the last minute of an auction. On the other hand, the latter find a significant value to sniping about \$1 per auction for various DVD listings. Backus et al. [2013] in a more recent paper consider the effect of sniping on the return rate of new buyers to the marketplace. In all of these papers, the concentration is mainly on rationalizing sniping, and neither early bidding nor bidding multiple time is rationalized.

In this paper, we concentrate mainly on dynamics within the span of an auction. However, we are borrowing the result from Zeithammer [2006], Said [2011], Hendricks and Sorensen [2014] and Backus and Lewis [2012] implicitly. These papers model the dynamic value of not winning the current auction and the opportunity cost of participating in the next available auction. There are usually many closely substitutable items simultaneously available for auction on eBay and if bidders do not win a particular auction they have a chance of participating in the next available one. This will lead to a reservation price for the bidders below the static valuation of the item for them. Changes in the available alternative items can change the reservation price for bidders over time, and as they get closer to the end of the auction their valuation of the item at that instant becomes closer to their valuation at the end of the auction. While our paper misses some of the interesting features arising from the link between these auctions, it provides a very tractable reduced form.

### 3 A simple example

There are two periods  $t = \{0, 1\}$ . Bidders can submit a bid for sure in the first period and with probability  $p$  in the second period. Bidders have no information in the first period and

draw values  $v \in [0, 1]$  independently from distribution  $F$  in the second period, prior to bidding time. Since the auction is second price, it follows that the second period bid will equal  $v_i$  for all bidders. Let  $b_{-i}$  denote the maximum final bid among all bidders, excluding  $i$  and let  $G$  denote its cdf. The first period, expected value for bidder  $i$  is:

$$\begin{aligned}
U_i &= (1-p) \int_0^{b_0} (Ev - b_{-i}) dG(b_{-i}) \\
&+ p \int_0^{b_0} \int_0^{b_0} (v - b_{-i}) dF(v) dG(b_{-i}) + p \int_{b_0}^1 \int_0^v (v - b_{-i}) dF(b) dG(b_{-i})
\end{aligned}$$

Taking derivate with respect to  $b$  and equating to zero:

$$(1-p)(Ev - b_0) dG(b_0) + p \int_0^{b_0} (v - b_0) dG(b_0) dF(v) = 0$$

$$b_0 = \frac{(1-p)Ev + p \int_0^{b_0} v dF(v)}{(1-p) + pF(b_0)} \quad (1)$$

This has a natural interpretation: the optimal initial bid equals the expected value conditional on no rebidding. Given a current bid  $b_0$ , the final bid prevailing at the end of the auction is then  $b_0$  with probability  $(1-p)$  and  $\max(b_0, v)$  with probability  $p$ . The initial bid  $b_0$  binds in two cases: 1) there is no opportunity to rebid and 2) there is an opportunity to rebid but  $v \leq b_0$  therefore the bidder decides not to bid. The second term in the above equation represents an adverse selection effect. It follows that  $b_0 < E(v)$  as  $\int_0^{b_0} v dF(v) / F(b_0) < E(v)$ . Indeed, as  $p \rightarrow 1$  it is easy to see that  $b_0$  decreases monotonically to zero.

The probability distribution  $\tilde{F}$  for the final bids of a player is given by:

$$\tilde{F}(b) = \begin{cases} 0 & \text{if } b < b_0 \\ (1-p) + pF(b) & \text{if } b \geq b_0 \end{cases} \quad (2)$$

with mean

$$E(b) = b_0 [p + (1-p)F(b_0)] + p \int_{b_0}^1 v dF(v)$$



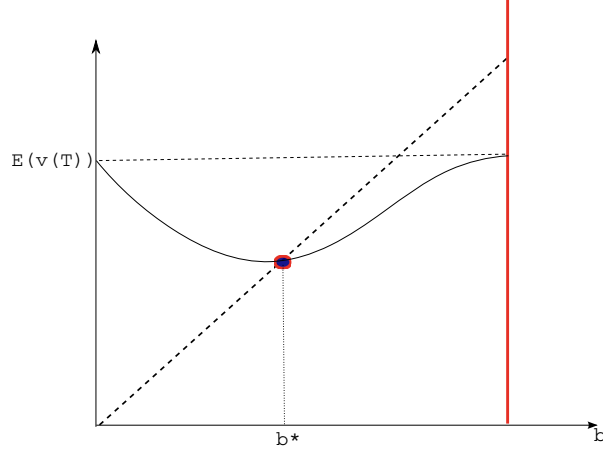


Figure 1: Optimal bid

Substituting the first term using (1) it follows that:

$$E(b) = (1 - p) Ev + p \int v dF(v) = Ev$$

so the expected final is unbiased, equaling the expected final value.

It is also interesting to compare the above results to a case where bidders can retract their bids. It is easy to see that with retraction there is no incentive to shade, so  $b_0$  is equal to  $Ev$ ; the final bid equals  $Ev$  with probability  $(1 - p)$  and equals the final value with probability  $p$ . The distribution for final bids is thus:

$$\tilde{F}(b) = \begin{cases} pF(b) & \text{if } b < Ev \\ (1 - p) + pF(b) & \text{if } b \geq Ev \end{cases}$$

Compared to (2) it is easy to see that this is a mean preserving spread of bidding with no retraction. The higher dispersion is the simple result of a higher correlation with final values.

## 4 The Model

There are  $i=1, \dots, N$  potential bidders in an auction. The auction is sealed bid second price and takes place in time interval  $[0, T]$  where bids are submitted. As shown below in Section 7 equilibrium bidding functions are also applicable to sequential auctions -such as in eBay-

with publicly available information on past bides.

Each bidder has the option of submitting bids only at random times  $\tau_1, \tau_2, \dots$ . Bids can only be increased at any of these random bidding times and cannot be retracted. The valuation of the bidder is modeled as a stochastic process  $v_i(t)$  of signals where without loss of generality  $v(T)$  corresponds to the final value. Since these signals are only relevant at bidding times  $\tau_n$  we restrict attention to the corresponding signals  $v_n$  at these dates. Assume  $\{v_n, \tau_n\}$  follows a joint Markov process with transition function  $P(\tau_{n+1}, v_{n+1} | \tau_n, v_n)$ . Define the *state* of a bidder as the pair  $(\tau, v)$  at the last bidding time.<sup>7</sup> We assume this process is independent across bidders.<sup>8</sup> Note that the model allows for random entry and rebidding, as well as a random number of bidders.

A bidding function for bidder  $i$  specifies at each possible bidding time  $\tau_n$  and given signal  $v_n$  a desired bid  $B_i(\tau_n, v_n)$ . Given that bids can only be increased,  $b_i(t) = \max\{B_i(\tau_m, v_m) | \tau_m \leq t\}$  is the bid that prevails at time  $t$  and in particular  $b_i(T)$  is the final bid. Let  $b_{-i}(T)$  denote the maximum bid over the remaining bidders at time  $T$  and  $F_{-i}(b)$  its distribution. Utility for bidder  $i$  is given by:

$$U(v(T), b(T)) = \int_0^{b(T)} (v(T) - u) dF_{-i}(u).$$

An optimal bidding function  $B_i$  for bidder  $i$  is the one that maximizes expected utility at the time the bidder enters the auction, i.e.,

$$\max_{B(\cdot)} E(U(v(T), b(T)) | \tau_1, v_1) \tag{3}$$

where  $\tau_1$  is the time at which the bidder enters the auction and  $v_1$  the initial value.

**Definition.** An *equilibrium* for the auction is a vector of bidding functions  $B_i$  and final distributions  $F_{-i}$  for each player  $i$ , such that for every bidder  $i$  the bidding function  $B_i$  is the best responses to  $F_{-i}$  and bidding functions are consistent with the final distribution of bids.

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<sup>7</sup>The process for the value  $v_i(t)$  can be considered a continuous Markov process sampled at random stopping times  $\tau_n$  that are Markov with respect to last stopping time and valuation at that time.

<sup>8</sup>Section 9.2 considers a simplified model with correlation.

## 5 Equilibrium Bidding

In this section, we characterize the equilibrium bidding functions and provide a dynamic programming problem that can be used to solve them. We show that the optimal bid  $B(\tau_n, v_n) = E[v(T) | \tau_n, v_n, b(T) = B(\tau_n, v_n)]$ , namely the expected final value for bidder  $i$  conditional on the current state and the event that no higher bids are placed by the same bidder at a later instance so that the final bid  $b(T)$  equals the current bid. Intuitively, this mimics the notion that in a second price auction it is weakly dominant strategy to bid the valuation or and when it is random, the expected valuation.

**Proposition 1.** *The optimal bid  $B(\tau_n, v_n) = E[v(T) | \tau_n, v_n, b(T) = B(\tau_n, v_n)]$ .*

*Proof.* (sketch) We provide here a variational argument to characterize bids. Take a candidate optimal bidding function  $B_i$  for bidder  $i$  and consider bid  $b$  in state  $\tau_n, v_n$ . Let  $H(b)$  denote all paths  $\omega = \{\tau_m, v_m\}_{m>n}$  starting from the current history where  $B(\tau_m, v_m) \leq b$  including  $\tau_{n+1} > T$  (i.e. no rebidding opportunity.) The expected value of bidding  $b$  equals:

$$V(\tau_n, v_n, b) = \int_{H(b)} \int^b (v(T) - u) dF_{-i}(u) dP(\omega) + \int_{H(b)^c} \int^{b(T, \omega)} (v(T) - u) dF_{-i}(u) dP(\omega)$$

We claim that the optimal bid is  $b = B(\tau_n, v_n) = E_{H(b)}v(T)$ . First, note that the boundary of the set  $H(b)$  consists of all those paths starting from  $(\tau_n, v_n)$  for which the final bid  $b(T)$  is equal to  $b$ . So when considering the derivative of the above we can ignore the effect of the change in the supports of the two integrals. The first order condition is then  $\partial V / \partial b = \int_{H(b)} (v(T) - b) dF_{-i}(b) dP(\omega) = 0$  which is equivalent to the statement that  $b = E_{H(b)}v(T)$ .  $\square$

While Proposition 1 characterizes bidding at a given history, it also shows that current bidding behavior depends on the whole strategy for future bidding, making the problem of calculating equilibrium bids potentially very complicated. However, there is a natural recursive structure to this problem which we exploit to define a dynamic programming problem that will help derive the optimal bidding function and establish uniqueness.

Define recursively the following function<sup>9</sup>:

$$W(b, v, t) = \int_t^T \min(W(b, v', \tau), 0) dP(v', \tau | v, t) + P(\tau > T | v, t) (E[v_T | v, t, \tau > T] - b) \quad (4)$$

where  $\tau$  denotes the following bidding time.

**Assumption 2.** *Assume  $P(\tau > T | v, t) > \delta > 0$  for all  $(v, t)$ . In addition, assume all conditional probabilities and expectations are continuous in the state.*

By Assumption 2 and using standard dynamic programming arguments, it follows that there is a unique function satisfying functional equation (4), that it is strictly decreasing in  $b$  and continuous. Moreover, it is greater or equal to zero when  $b = 0$  and negative for large  $b$ . It follows, by the intermediate value theorem, that there is a unique value  $B(t, v)$  such that  $W(B(t, v), v, t) = 0$ . Given a new bidding time  $\tau$ , value  $v(\tau)$  and outstanding bid, this function also defines the rebidding region  $\{(b, v(\tau), \tau) | W(b, v(\tau), \tau) > 0\}$ . We will next show that this bidding function maximizes the agents expected utility.

**Proposition 3.** *The function  $B(\tau, v)$  defined implicitly by  $W(B(\tau, v), v, \tau) = 0$  satisfies  $E[v(T) | \tau, v, b(T) = B(\tau, v)]$  and is thus the optimal bidding function.*

*Proof.* See Appendix. □

## Bid Shading

Our leading example suggests that bidders will shade bids as a consequence of adverse selection given by the option of future rebidding. In this section we prove this is true for an arbitrary process. We first establish the following result:

**Lemma 4.**  *$W(b, v, t) \leq E[v_T | v, t] - b$  with strict inequality if  $P(\tau \leq T | v, t) > 0$ .*

*Proof.* See Appendix. □

Since  $W(b, v, t)$  is decreasing in  $b$ , it follows that:

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<sup>9</sup>When conditioning with respect to  $\tau = t$  and  $v(\tau) = v$  we will write for short  $P[. | v, t]$ .

**Proposition 5.**  $B(v, t) \leq E[v_T | v, t]$  with strict inequality if and only if rebidding occurs with positive probability.

### Unbiased expected final bids

Bids are shaded as the result of conditioning on the adverse selected set of no-rebidding. However, in the complement of this set the current bid is replaced by a higher one. What is the overall effect on the final expected bid? The answer is given in the following Proposition.

**Proposition 6.** *Suppose  $v$  follows a Martingale. Consider a bidding time  $t$  with value  $v$  and bid  $b = B(v, t)$ . Then the expected final bid equals  $v$ , i.e.  $E(B(T) | v, t, b = B(v, t)) = v$ .*

*Proof.* Let  $H_0$  consist of all histories  $\omega$  starting from  $(v, t, b)$  where  $B(T) = b$ . By Proposition 3  $E(V(T) | H_0) = b$ . Now consider all histories  $\omega$  on the complement  $H_0^c$  where rebidding occurs with probability one. Let  $\tau$  be the stopping time corresponding to the first time where rebidding occurs. Assume, by way of induction, that  $E[B(T) | v(\tau), \tau, b = B(v(\tau), \tau)] = v(\tau)$ . By the optional stopping theorem it follows that  $E(B(T) | H_0^c) = E(v(\tau) | H_0^c) = E(v(T) | H_0^c)$ .  $\square$

This Proposition implies, in particular, that when a bidder places the initial bid, the expected final bid is unbiased and equals the expected final value. This result is consistent with bid shading since bids, while shaded, are still unbiased in the set where they *apply*.

## 6 Properties

This section derives general properties on equilibrium bidding behavior and considers some special cases. We specialize the model here to a case where the process for rebidding times  $\tau_n$  is Poisson with intensity  $\rho(t)$  that is independent of the signals and value  $v(T)$ . The main result in this section is that the bidding function  $B(t, v)$  is increasing in  $t$ .

**Proposition.** *Assume  $E[v_T | v, t]$  is weakly increasing in  $t$ . Then  $\partial W(b, v, t) / \partial t \geq 0$  and bid  $B(t, v)$  increases with  $t$ .*

*Proof.* See Appendix.  $\square$

The above result implies that bids increase over time even in the absence of competitive pressure.

## 6.1 The Bad News Model

The agent starts with a valuation  $v(0) > 0$ . There is a Poisson process that can turn that valuation to zero forever, with arrival rate  $\lambda$ ; otherwise it remains unchanged. The random bidding times  $\tau_1, \tau_2, \dots$  are determined by a Poisson process with arrival  $\rho$ . This special case captures the idea that a bidder might receive information during the auction that makes the object auctioned unattractive, e.g. the bidder might find there is another show she wishes to attend that day or engages in other commitments. It immediately follows that  $W(b, 0, t) = -b$  for all  $t$ . Moreover, the unconditional expectation at  $T$  equals  $\exp(-\lambda(T-t))v - b$ . Substituting in functional equation (4) and using Proposition 3 we can derive an explicit bidding function.

**Proposition 7.** *The bidding function for the bad news model is given by:*

$$B(v, t) = \frac{(\rho + \lambda) \exp(-(\rho + \lambda)(T - t)) v}{\lambda + \rho \exp(-(\rho + \lambda)(T - t))}.$$

*Proof.* See Appendix. □

Note that when  $\lambda = 0$  or  $t = T$  this gives  $B(t) = v$  and when  $\rho = 0$  this gives the expected value  $\exp(-\lambda(T-t))v$ , as should be. To connect to our intuition that bids are the expectation conditional on no rebidding, the two events to consider are: (1) no rebidding occurred and the value did not go to zero, that has probability  $\exp(-(\rho + \lambda)(T - t))$  and value  $v$ . The other event is that the value went to zero before the next rebid. This has probability  $\lambda \int_t^T \exp(-(\rho + \lambda)s) ds = \frac{\lambda}{\rho + \lambda} (1 - \exp(-(\rho + \lambda)(T - t)))$  and value zero. Note that the sum of these two events:

$$\exp(-(\rho + \lambda)(T - t)) + \frac{\lambda}{\rho + \lambda} (1 - \exp(-(\rho + \lambda)(T - t))) = \frac{\lambda + \rho \exp(-(\rho + \lambda)(T - t))}{\rho + \lambda}$$

Dividing the weighted value  $\exp(-(\rho + \lambda)(T - t))v$  by this sum of probabilities gives the bidding function above.

## 6.2 Brownian Motion

Suppose that  $v(t)$  follows a Brownian motion with drift  $\mu$  and variance  $\sigma^2$  and the process for rebidding is Poisson with arrival  $\rho$ , as in the previous case. The functional equation (4) is now:

$$W(b, v, t) = \rho \int_t^T \exp(-\rho(\tau - t)) \int \min(0, W(b, v + \mu(\tau - t) + \sqrt{\tau - t}\sigma\varepsilon, \tau)) d\Phi(\varepsilon) d\tau + \exp(-\rho(T - t))(v + \mu(T - t) - b) \quad (5)$$

Note that for this case,  $E[v(T) | v, t] = v + \mu(T - t)$  so if the drift  $\mu \leq 0$  then  $E[v(T) | v, t]$  is increasing with  $t$ , so by Proposition 6,  $B(v, t)$  is increasing in  $t$ .

It is easy to show that  $W(b, v, t) = W(0, v - b, t)$

**Lemma 8.**  $W(b, v, t) = W(0, v - b, t)$ .

*Proof.* The proof is by induction. So suppose the right hand side of (5) satisfies this condition. Then,

$$\begin{aligned} W(b, v, t) &= \rho \int_t^T \exp(-\rho(\tau - t)) \int \min(0, W(0, v - b + \mu(\tau - t) + \sqrt{\tau - t}\sigma\varepsilon, \tau)) d\Phi(\varepsilon) d\tau \\ &\quad + \exp(-\rho(T - t))(v - b + \mu(T - t)) \\ &= W(0, v - b, t) \end{aligned}$$

□

As a consequence of this Lemma, we can write the value function  $W_\mu(x, t)$  where  $x = v - b$ ,

$$W_\mu(x, t) = \rho \int_t^T \exp(-\rho(\tau - t)) \int \min(0, W_\mu(x + \mu(\tau - t) + \sqrt{\tau - t}\sigma\varepsilon, \tau)) d\Phi(\varepsilon) d\tau + \exp(-\rho(T - t))(x + \mu(T - t)) \quad (6)$$

### Scaling

Here we show is that if we scale  $\sigma$  and  $\mu$  by a factor  $\lambda$ , we get  $W(\lambda x; \lambda\mu, \lambda\sigma) = \lambda W(x, \mu, \sigma)$ . The proof is by induction, so assume the right hand side of (6) satisfies this property. It verifies immediately that  $W(\lambda x, t; \lambda\mu, \lambda\sigma) = \lambda W(x, t, \mu, \sigma)$ . This also implies that the shading function  $x(t; \lambda\mu, \lambda\sigma) = \lambda x(t; \mu, \sigma)$  or equivalently  $B(\lambda v, t; \lambda\mu, \lambda\sigma) = \lambda B(v, t; \mu, \sigma)$ .

**Drift** The effect of drift on the value function can also be easily determined as shown by the following Lemma.

**Lemma 9.** *Let  $W_\mu$  denote the value function with drift  $\mu$  and  $W$  the value function with drift zero. Then  $W_\mu(x, t) = W(x + \mu(T - t), t)$ .*

*Proof.* This is proved inductively using:

$$\begin{aligned}
W(x + \mu(T - t), t) &= \rho \int_t^T \exp(-\rho(\tau - t)) \int \min(0, W(x + \mu(T - \tau) + \mu(\tau - t) + \sqrt{\tau - t}\sigma\varepsilon, \tau)) d\Phi(\varepsilon) d\tau \\
&\quad + \exp(-\rho(T - t))(x + \mu(T - t)) \\
&= \rho \int_t^T \exp(-\rho(\tau - t)) \int \min(0, W_\mu(x + \mu(\tau - t) + \sqrt{\tau - t}\sigma\varepsilon, \tau)) d\Phi(\varepsilon) d\tau \\
&\quad + \exp(-\rho(T - t))(x + \mu(T - t)) \\
&= W_\mu(x, t)
\end{aligned}$$

□

It also follows that the bid shading function is of the form  $x(t) + \mu(T - t)$  where  $x(t)$  is the shading function when  $\mu = 0$ . In what follows we assume  $\mu = 0$ .

**Solving Bellman equation** To find the pde for the Bellman equation (the Hamilton-Jacobi equation), first subtract  $W(x, t)$

$$\begin{aligned}
0 &= \rho \int_t^{t+\Delta} \exp(-\rho(\tau - t)) \int [\min(0, W(x + \sqrt{\tau - t}\sigma\varepsilon, \tau)) - W(x, t)] d\Phi(\varepsilon) d\tau \\
&\quad + \exp(-\rho\Delta) \left[ \int W(x + \sqrt{\Delta}\sigma\varepsilon, t + \Delta) - W(x, t) \right] d\Phi(\varepsilon)
\end{aligned} \tag{7}$$

Taking derivative with respect to  $\Delta$  and evaluating at  $\Delta = 0$

$$\begin{aligned}
0 &= \rho [\min(0, W(x, t)) - W(x, t)] + \frac{1}{2}\sigma \frac{\partial^2}{\partial x^2} W(x, t) + \frac{\partial}{\partial t} W(x, t) \\
\rho \max(W(x, t), 0) &= \frac{1}{2}\sigma \frac{\partial^2}{\partial x^2} W(x, t) + \frac{\partial}{\partial t} W(x, t)
\end{aligned}$$

This partial differential equation allows to solve for the value function and calculate the bidding/shading function.



## 7 Bidding with Partially Observed Competing Bids and Censoring

The above assumed that the auction is sealed bid, so there is no information on competing bids. In this section, we show that this assumption is really immaterial and the bidding function derived above still holds, with a slightly different interpretation. Throughout this section we maintain our assumption of independent values across bidders. Let  $B_i(v, t)$  denote the maximum bid that bidder  $i$  is willing to place at time  $t$ . The information on other bids during the auction has an effect on the expected utility of the agent as given by (3) as it affects the distribution of the highest competing final bid. But as we have seen, the optimal bid is independent of this distribution. Therefore, it remains an equilibrium for bidders to “ignore” other bidders bids in choosing their own bid  $B(v, t)$ .<sup>10</sup>

However, when taking the model to the data one must take into account that when a bidder observes an outstanding bid that is greater than  $B(v, t)$ , he will choose not to bid. Hence, the process for observed bids is censored and this censoring depends on the available information. In eBay auctions, bidders can see the outstanding second highest value thus defining the threshold below which bids are censored.

### Timing of Bids

The model has implications for the observed timing of bids. If at time  $t$  a bidder has a high probability of returning to the auction right before its end, its desired bid  $B(v, t)$  will be very low and thus it is likely to be censored by an existing higher second bid. As a consequence, it is unlikely to see any bids from this bidder until the end of the auction, consistent with the observed sniping behavior.

There is likely to be asymmetry in the frequency of bidding times for different bidders. The observation of sniping—defined as bids that are overtaken in the last few minutes of the auction—is usually interpreted as indication that many bidders follow this kind of strategy. As the following example shows, this might not be true. Suppose there are  $n$  bidders. One of these bidders can bid with probability one at the end of the auction, while the other  $n - 1$

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<sup>10</sup>We do not know if there are other equilibria.

bidders only bid at the beginning of the auction. To make things extreme, suppose the final value is uniform between  $[0, 1]$  and all the  $n - 1$  initial bidders have no information and thus approximately the same expectation equal to  $1/2$  and so the initial winning bid is also around  $1/2$ . It follows that the probability that the remaining “sniping” bidder wins the auction is  $1/2$ . So in this auction,  $1/2$  of the times the auction will be *sniped* while the share of snipers is only  $1/n$ . Note also that given the information structure sniping is still efficient as in absence of correlated information the expected value of the sniped bidder is  $1/2$  and thus lower than the value for the sniper.

## 8 Estimation and Counterfactual (preliminary)

In this section we provide estimates and some counterfactual exercises for the Brownian motion setting described in section 6.2. The diffusion has standard deviation  $\sigma$ , bidding times given by an independent Poisson process with arrival rate  $\rho$ , and first arrival rates of bidders at Poisson rate  $\lambda$ . To simplify the estimation procedure we fix  $\lambda = 5$  (representing an average of 5 bidders per auction) and assume bidders initial values are uniform between 0 and 1.

We consider the following three moments for the estimation:

- Average number of bidders-defined as those that placed either a highest or second highest bid at some point during the auction. We refer to these as *recorded* bids.
- Average number of bids per bidder.
- % of winning bids placed in the last 10% of time

As  $\rho$  increases, we can expect the number of bids to increase and also the number of bids per bidder. For winning bids, this is an immediate result that as  $\rho$  is increased, there is stronger bid shading initially and thus more likely that winning bids occur towards the end when there is less shading. An increase in  $\sigma$  will also lead to more shading of bids and it thus likely to increase the % of winning bids placed closer to the end. The effect on the number of bidders and bids per bidder is unclear. The increased shading might contribute

Table 1: Matching of moments

	#of bidders	# bids	% late winning bids
Data	6	2	80%
Fitted values	4.5	3.3	53%

to increasing the number of bids/bidders as early bids are more likely to get outbid by newer ones.

To estimate these moments, we proceed as follows:

1. Solve the pde for the shading function as described in section 6.2 for a fixed set of parameters  $(\sigma, \rho)$ .
2. Simulate all bids taking into account censoring, so only consider a bid for our moment calculation if it was either first or second highest at the time it was placed.
3. Construct moments from  $n = 1000$  simulations.
4. Find highest scoring on a grid.

Table 1 gives the moments corresponding to our estimated parameter values:  $\sigma = 0.3$  and  $\rho = 6$ . While the matching is far from perfect, it is a first step in this preliminary version. In the future we expect to improve it considerably by choosing the distribution of initial bids, arrival of bidders and most likely allowing for some heterogeneity across bidder parameter values.

## Counterfactuals

We consider two counterfactuals: 1) final efficient allocation, as would be given by a second price auction at the end of time and 2) equilibrium bidding with the possibility of retracting bids. For the latter we assume that bidders can choose to retract/lower bids during any one of their bidding times but not outside these bidding times. This is a natural restriction, since when bidders are allowed to retract at the end, bidding would become meaningless. The possibility of retraction eliminates the source of adverse selection mentioned earlier and thus the incentives for bid shading. Given that values follow a Martingale, at every bidding time  $\tau$  a bidder will choose a bid equal to the current value  $v_\tau$ .

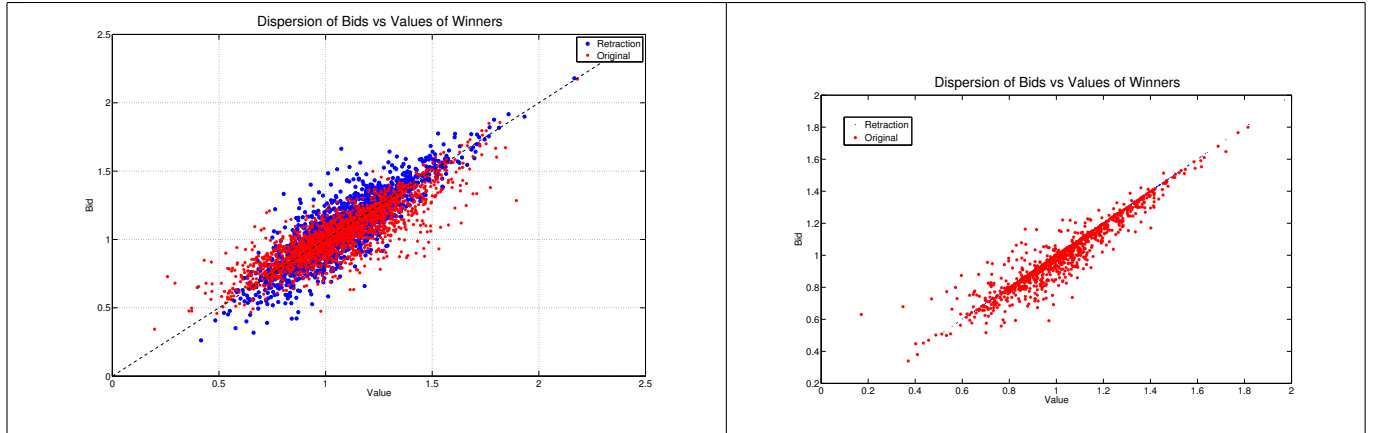


Figure 2: Correlation between bids and values

Figure 2 compares the correlation between values and bids for the baseline and no-retraction case. In the left hand we consider simulated final values and final bids in both scenarios. There is substantial dispersion in both cases, as bids are placed some random time before the end of the auction giving rise to some residual uncertainty as of final value. The correlation between bids and value is 0.85 for the baseline case and 0.88 for the no-retraction case. The right hand side figure plots the correlation between bids and the expected value at the time of the last bid/retraction opportunity. Here by definition bids equal to the value with no-retraction, so the figure illustrates more clearly the added dispersion in the baseline scenario, which is consistent with the lower correlation indicated above.

Table 2 gives the average performance of these bidding scenarios. The expected value in the baseline is 96.5% of the one corresponding to an efficient final allocation, which is considerably close. Bid retraction reduces this gap by 1/3. The gap in expected highest bids is twice of large when comparing the baseline scenario and the efficient allocation while it remains unchanged for the case of bid retraction. The latter follows immediately from the fact bids are unbiased in the case of bid retraction. In contrast, bids are shaded for our baseline scenario which explains the larger gap. Interestingly, similar effect is found when considering the gap in prices. As a result of shading of the second highest bid, the gap between the price in the baseline scenario and the efficient (second highest value) one is over 7.5%, while the corresponding gap for the case of bid-retraction is slightly over 2%. Allowing for bid-retraction would increase revenues of sellers by approximately 6% while decreasing

Table 2: Revenues and Efficiency

	Baseline	Bid-retraction	Efficient
Expected value	1.064	1.077	1.102
Expected highest bid	1.03	1.077	1.102
Expected price	0.835	0.885	0.904
Expected utility	0.220	0.180	0.198

the expected utility of buyers by 18%.

## 9 Extensions

This section considers two extensions to our model. The first one is endogenous rebidding, where there is a cost to rebid and bidders exercise this option optimally. The environment is considerably more difficult as the value of rebidding is highly dependent of the distribution of other players' bids, in contrast to the simple bidding derived above. The second extension is to allow for correlation between the signals received by bidders. While we still assume that conditional on these signals values are independent across bidders, the adverse selection problem becomes more severe: winning the auction when placing an early bid is correlated with negative signals observed by later bidders. As we show below, this results in a larger incentive to shade bids.

### 9.1 Endogenous Rebidding

We consider here a very simple model. There are  $N = 2$  bidders and 2 periods. Let  $v_i$  denote the value for bidder  $i$  in the first period drawn from distribution  $G(v)$  and  $v'_i$  the value in the second period, drawn from conditional distribution  $F(v'|v)$ . Both bidders can bid freely in the first period but must pay a cost  $c > 0$  to rebid in the second period. We assume bids are sealed.

Strategies for the bidders can be defined as follows: a bidding function  $B_i(v)$  for the first period and a rebidding set  $R_i(v)$  for the second period. Let  $N_i(v)$  denote the complement of

$R_i(v)$ . Given the strategy for the other player, player  $i$ 's expected utility is given by:

$$U(v) = \int_{N_i(v)} Q_i(B_1(v))(v' - P_i(B_i(v))) dF(v'|v) \\ + \int_{R_i(v)} -c + Q_i(v')(v' - P_i(v')) dF(v'|v)$$

where  $Q_i$  is the probability of winning function and  $P_i$  the expected payment conditional on winning. A symmetric equilibrium  $(Q, R)$  is a Nash equilibrium in these strategies.

**Example 10.** Both bidders draw independently their initial value  $v \in [0, 1]$  from distribution  $G$  and with probability  $1 - \rho$  get zero value next period and with probability  $\rho$  the value remains equal to  $v$ . Conjecture a threshold  $v^*$  so that:

$$B_1(v) = \rho v \text{ if } v < v^* \text{ and zero otherwise}$$

$$R(v) = \{\} \text{ for } v < v^* \text{ and } R(v) = \{v\} \text{ otherwise}$$

Consider the player with  $v = v^*$ . Bidding first or second period doesn't change his probability of winning since for  $\rho v^* \leq b \leq v^*$ ,  $Q(b) = G(v^*) + (1 - \rho)(1 - G(v^*))$ . The expected payment is also the same in both cases equal to:

$$\frac{\rho \int_0^{v^*} v dG(v)}{G(v^*) + (1 - \rho)(1 - G(v^*))}$$

The difference is that if he chooses not to rebid, he pays this expected value for sure while if he chooses to rebid he pays it with probability  $\rho$ . The difference in expected payment is then:

$$(1 - \rho) \frac{\rho \int_0^{v^*} v dG(v)}{G(v^*) + (1 - \rho)(1 - G(v^*))}$$

to which we need to add that he pays an expected cost  $\rho c$ . So  $v^*$  must be such that:

$$c = \frac{(1 - \rho) \int_0^{v^*} v dG(v)}{\rho G(v^*) + (1 - \rho)}. \quad (8)$$

The derivative of the right hand side with respect to  $v^*$  equals in sign to

$$v^* [\rho G(v^*) + (1 - \rho)] - \rho \int_0^{v^*} v dG(v) > 0.$$

The last step is to show that for all  $v > v^*$  it is optimal to rebid and conversely for those not in this set. The difference between rebidding and not is

$$\begin{aligned} & \rho \int_0^v (v - x) dF_{-i}(x) - \int_0^{\rho v} (\rho v - x) dF_{-i}(x) \\ = & \int_0^v (\rho v - \rho x) dF_{-i}(x) - \int_0^{\rho v} (\rho v - x) dF_{-i} \end{aligned}$$

Using the envelope condition and taking derivatives with respect to  $v$  (keeping bids fixed)

$$\rho F_{-i}(v) - \rho F_{-i}(\rho v) > 0.$$

This establishes that the gains from rebidding are increasing in  $v$  so the threshold  $v^*$  defined by (8) is an equilibrium and it is unique.

One might expect that similar results will hold with more bidders. Moreover, it is natural to conjecture that the threshold for rebidding increases with the number of bidders, as expected payoffs decrease. Following this conjecture, with endogenous bidding one might expect that the number of bids per player decreases too.

## 9.2 Correlated Information

In many cases, it is likely that information or signals observed are correlated across bidders. For example, the arrival of a competing auction with a similar product is an event that creates an opportunity cost and is likely to affect in a correlated way the value of all bidders that keep track of that information. Our results extend easily to the case of pure common values, where all agents have the same values but different bidding windows. For more general cases we do not have a general result so we restrict to a simplified scenario.

### 9.2.1 Pure common values

The setting is identical to that described in our general model, but where all bidders values  $v_i$  are identical, bidders observe the same signals but have independent bidding windows. As before, it follows that the optimal bid at state  $(v, t)$  satisfies  $b_i = E(v(T) | v, t, b(T) = b_i)$

where now  $b(T) = \max_{ij} b_i(v_j, t_j)$  is the maximum over all bids, including those of other bidders. The recursive representation given before still holds,

$$W(b, v, t) = \int_t^T \min(W(b, v', \tau), 0) dP(v', \tau|v, t) + P(\tau > T|v, t) (E[v_T|v, t, \tau > T] - b)$$

where the interpretation of arrivals now is for the arrival to any bidder. As an example, if opportunities for bidding are Poisson with arrival  $\rho$  then the total arrival rate that would be used in this dynamic programming equation is  $N\rho$ , where  $N$  is the number of bidder. The effect on bidding behavior is similar to an increase in  $\rho$  that intuitively should result in greater shading of bids. This follows naturally the interpretation of increased adverse selection as now the current bid will win not only when that bidder decides not to bid any higher but when any other bidder chooses not to do so.

### 9.2.2 Imperfectly correlated information

Consider the following simplified scenario. Suppose time is discrete and there are two periods  $\{1, 2\}$  where 2 corresponds to the end of the auction. Each bidder receives an independent initial value  $v_i$ , drawn from distribution  $F(v)$ , that we interpret as the unconditional expected final value in absence of further information. They simultaneously bid in the first period. With probability  $\rho$ , they have a chance to bid in the second period. Second period valuations are drawn from distribution  $G(v'|v, \theta)$  where  $\theta$  is a common observable shock that is independent of the initial values. Here  $v$  (resp.  $v'$ ) denotes the initial (resp. final) value for bidder one and  $v_2$  ( $v'_2$ ) the corresponding values for bidder two.

An agent that bids in the second period will choose  $b_2(v') = v'$ . Let  $B_1(v)$  denote the bid in the first period. This bid should equal the expected value conditional on the union of the following events: (1) none of the two agents get to rebid and  $B_1(v) > B_1(v_2)$ ; (2) agent one gets to rebid but  $v' \leq B_1(v)$  and  $v_2 \leq v$ ; (3) other agent rebids but  $v'_2 < B_1(v)$  and  $B_1(v_2) < B_1(v)$ , and (4) both get to rebid but  $v', v'_2 < B_1(v)$  and  $B_1(v_2) < B_1(v)$ . In a symmetric equilibrium with monotone bidding functions, the condition  $B_1(v) \geq B_1(v_2)$  can be substituted by  $v \geq v_2$ .

Consider a symmetric equilibrium Given a bid  $b$  for the first player, we claim the following:



**Proposition 11.**  $B_1(v)$  is an equilibrium if and only if  $E(v'|v, B_1(v)) = B_1(v)$ , where

$$E(v'|v, b) = \frac{F(v)(1-\rho)^2 v_1(b, v) + F(v)\rho(1-\rho)v_2(b, v) + \rho(1-\rho)v_3(b, v) + F(v)\rho^2 v_4(b, v)}{\pi(v, b)} \quad (9)$$

where:

$$\begin{aligned} \pi(v, b) &= F(v)(1-\rho)^2 + F(v)\rho(1-\rho)G(b|v) + \rho(1-\rho)P(v_2 \leq v, v'_2 \leq b) \\ &\quad + F(v)\rho^2 \int \chi_{v' \leq b, v'_2 \leq b} dP(v', v'_2 | v, v_2 \leq v) \end{aligned}$$

$$v_1(b, v) = E[v'|v]$$

$$v_2(b, v) = E[v'|v' \leq b, v]$$

$$v_3(b, v) = \int v' dG(v'|v, \theta) dP(\theta | v'_2 \leq b \& B_1(v_2) < b)$$

$$v_4(b, v) = \int \chi_{v' \leq b, v'_2 \leq b} v' dP$$

and  $H(v'_2, v_2)$  is the joint distribution that is assumed to be independent of  $v$ .

*Proof.* Note that the denominator is also  $\pi(v, b)$  the probability of winning with a bid equal to  $b$ . Suppose the bidding function satisfies this condition for all  $v$ . By symmetry and monotonicity of the bidding function,  $B_1(v_2) \leq B_1(v)$  if and only if  $v_2 \leq v$ . Hence conditioning on  $v_2 \leq v$  is equivalent to conditioning on  $B_1(v_2) \leq b$  for  $b = B_1(v)$ . It is now easy to verify the expression given in equation (9) for  $b = B_1(v)$  is precisely the conditional expectation described above.  $\square$

Suppose  $v'_i = \theta v_i$  where  $\theta$  is the common component. Rewriting equation (9),

$$E(v'|v, b) = \frac{F(v)(1-\rho)^2 v \int \theta dG(\theta) + F(v)\rho(1-\rho) v \int^{b/v} \theta dG(\theta) + \rho(1-\rho) v \int \chi_{\{v_2 \leq v, \theta \leq b/v_2\}} \theta dG(\theta) dF(v_2)}{F(v)(1-\rho)^2 + F(v)\rho(1-\rho)G(b/v) + \rho(1-\rho) \int \chi_{\{v_2 \leq v, \theta \leq b/v_2\}} dG(\theta) dF(v_2)} \quad (10)$$

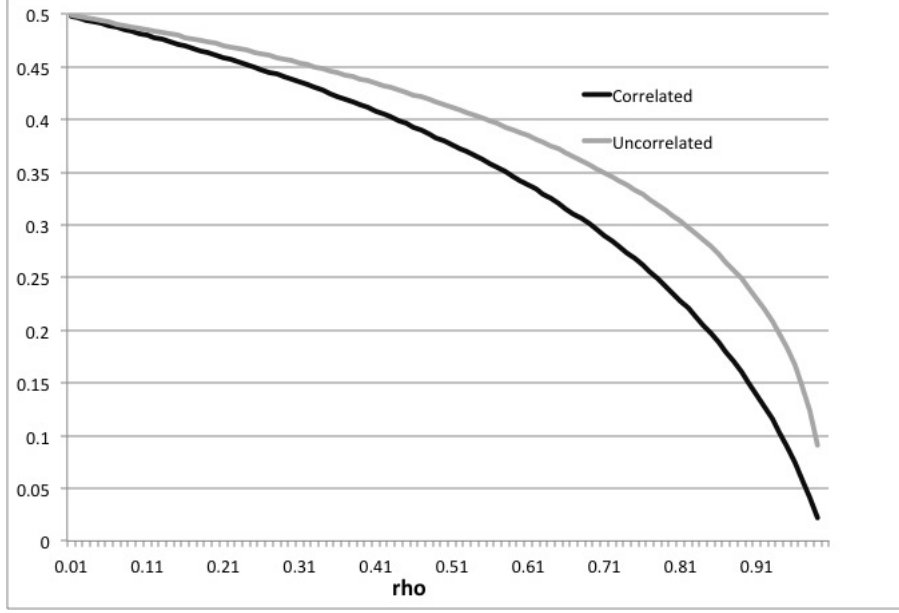


Figure 3: Bidding functions: effect of correlation

Suppose in addition that  $v$  and  $\theta$  are both uniform  $[0, 1]$ . Closed form expressions can be found when  $\theta$  is uniform  $[0, 1]$  and  $v_2$  is also uniform.

$$E(v'|v, b) = \frac{v^2(1-\rho)^2/2 + \rho(1-\rho)b^2/2 + \rho(1-\rho)v \int^v \min\left(\frac{1}{2}, \frac{b^2}{2v_2^2}\right) dv_2 + \frac{\rho^2 vb^2}{2}}{v(1-\rho)^2 + \rho(1-\rho)b + \rho(1-\rho) \int^v \min(1, b/v_2) dv_2 + \rho^2 b} \quad (11)$$

$$= \frac{v^2(1-\rho)^2/2 + \rho(1-\rho)b^2/2 + \rho(1-\rho)v\left(b - \frac{b^2}{2v}\right) + \frac{\rho^2 vb^2}{2}}{v(1-\rho)^2 + \rho(1-\rho)b + \rho(1-\rho)(b + b \ln(v/b)) + \rho^2 b} \quad (12)$$

$$= \frac{v^2(1-\rho)/2 + \rho vb + \frac{\rho^2 vb^2}{2(1-\rho)}}{v(1-\rho) + \rho(2b + b \ln(v/b)) + \frac{\rho^2 b}{(1-\rho)}} \quad (13)$$

We solve for the fixed point above numerically and compare the bidding function with the one obtained for the uncorrelated case, where  $\theta$  is independently drawn for the two players from a uniform distribution. Figure 3 plots bidding functions in both scenarios for an initial value  $v = 1$ . The  $x$  axis shows different probabilities of rebidding  $\rho$  and the  $y$  axis the corresponding bids. Consistent with our findings, as  $\rho \rightarrow 1$  bids go to zero in both cases and at the other extreme, when  $\rho = 0$  bids equal the unconditional mean of  $\theta = 1/2$ . More importantly, when information is correlated bidders shade their bids even more.

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## 10 Proofs

### Proof of Proposition 3

Let  $Q(b, v, t)$  denote the probability that  $b(T) = b$  conditional on  $\tau = t$  and  $v(\tau) = v$ . This can be also expressed as the compound lottery

$$Q(b, v, t) = \int_t^T \chi_{W(b, v', \tau) \leq 0} Q(b, v', \tau) dP(v', \tau | v, t) + P(\tau > T | v, t)$$

We now show recursively that

$$W(b, v, t) / Q(b, v, t) = E[v(T) | \tau = t, v(\tau) = v, b(T) = b] - b. \quad (14)$$

Substituting in (4)

$$\begin{aligned} \frac{W(b, v, t)}{Q(b, v, t)} &= \left\{ \int_t^T \min(W(b, v', \tau), 0) dP(v', \tau | v, t) + P(\tau > T | v, t) (E[v_T | v, t] - b) \right\} \\ &\quad * Q^{-1}(b, v, t) \\ &= \left\{ \int_t^T Q(b, v', \tau) \min\left(\frac{W(b, v', \tau)}{Q(b, v', \tau)}, 0\right) dP(v', \tau | v, t) + P(\tau > T | v, t) (E[v_T | v, t] - b) \right\} \\ &\quad * Q^{-1}(b, v, t) \\ &= \left\{ \int Q(b, v', \tau) \min(E[v(T) | \tau, v', b(T) = b] - b, 0) dP(v', \tau | v, t) \right. \\ &\quad \left. + P(\tau > T | v, t) (E[v_T | v, t] - b) \right\} * Q^{-1}(b, v, t) \\ &= \left\{ \int_t^T \chi_{\{W(b, v', \tau) \leq 0\}} Q(b, v', \tau) (E[v(T) | \tau, v', b(T) = b] - b) dP(v', \tau | v, t) \right. \\ &\quad \left. + P(\tau > T | v, t) (E[v_T | v, t] - b) \right\} * Q^{-1}(b, v, t) \\ &= E(v(T) | \tau = t, v(\tau) = v, b(T) = b) - b. \end{aligned}$$

## Proof of Lemma 4

First note that by the law of iterated expectation,  $E(E([v_T|v', \tau] - b|v', \tau) | v, t) = E[v_T|v, t] - b$  and that

$$\begin{aligned}
E(E([v_T|v', \tau] - b|v', \tau) | v, t) &= \int_t^T E([v_T|v', \tau] - b|v', \tau) dP(v', \tau|v, t) \\
&= \int_t^T E([v_T|v', \tau] - b|v', \tau) dP(v', \tau|v, t) \\
&\quad + \int_T^T E([v_T|v', \tau] - b|v', \tau) dP(v', \tau|v, t) \\
&= \int_t^T E([v_T|v', \tau] - b|v', \tau) dP(v', \tau|v, t) \\
&\quad + P(\tau > T|v, t) E([v_T|v', \tau] - b|v, t, \tau > T).
\end{aligned}$$

We now prove the Lemma by induction. So suppose that  $W(b, v', \tau) \leq E(v(T) | v', \tau) - b$ .

Then

$$\begin{aligned}
W(b, v, t) &= \int_t^T \min(W(b, v', \tau), 0) dP(v', \tau|v, t) + P(\tau > T|v, t) (E[v_T|v, t, \tau > T] - b) \\
&\leq \int_t^T W(b, v', \tau) dP(v', \tau|v, t) + P(\tau > T|v, t) (E[v_T|v, t, \tau > T] - b) \\
&\leq \int_t^T E([v_T|v', \tau] - b|v', \tau) dP(v', \tau|v, t) + P(\tau > T|v, t) (E[v_T|v, t, \tau > T] - b). \\
&= E[v_T|v, t] - b
\end{aligned}$$

where the first inequality is strict if  $P(\tau \leq T|v, t) > 0$ , thus completing the proof.

## Proof of Proposition 6

Let  $S(t, \tau) = \exp(-\int_t^t \rho(s) ds)$  be the probability of no arrival of bidding time in the interval  $[t, t + \tau]$ . For the specific process considered here, the value function (4) specializes

to:

$$W(b, v, t) = \int_t^T \rho(\tau) S(t, \tau) \left[ \int \min(W(b, v', \tau), 0) dP(v'|v) \right] d\tau + S(t, T) [E(v_T|v, t) - b]$$

We prove inductively that  $W(b, v, t)$  is weakly increasing in  $t$ .

$$\partial W(b, v, t) / \partial t = -\rho(t) \int \min(W(b, v', t), 0) dP(v'|v) \tag{15}$$

$$+ \int_t^T \rho(\tau) \rho(t) \left[ \int \min(W(b, v', \tau), 0) dP(v'|v) \right] d\tau + \rho(t) [E(v_T|v, t) - b] + S(t, T) \frac{\partial}{\partial t} E(v_T|v, t) \tag{16}$$

$$> -\rho(t) \int \min(W(b, v', t), 0) dP(v'|v) + \rho(t) [E(v_T|v, t) - b] + S(t, T) \frac{\partial}{\partial t} E(v_T|v, t) \tag{17}$$

$$\geq S(t, T) \frac{\partial}{\partial t} E(v_T|v, t) \geq 0$$

where the last inequality follows from Lemma 4. This completes the inductive proof. The second claim of the Proposition follows from the first one, the fact that  $W(B(t, v), v, t) = 0$  and that  $W$  is decreasing in  $b$ .

## Proof of Proposition 7

Using functional equation (4)

$$\begin{aligned}
W(b, v, t) &= \rho \int_0^{T-t} \exp(-\rho\tau) [\exp(-\lambda\tau) \min(W(b, v, t + \tau), 0) - (1 - \exp(-\lambda\tau)) b] d\tau \\
&\quad + \exp(-\rho(T-t)) (\exp(-\lambda(T-t)) v - b) \\
&= \rho \int_0^{T-t} \exp(-\rho\tau) \exp(-\lambda\tau) \min(W(b, v, t + \tau), 0) d\tau \\
&\quad - \rho \int_0^{T-t} \exp(-\rho\tau) (1 - \exp(-\lambda\tau)) b d\tau \\
&\quad + \exp(-\rho(T-t)) (\exp(-\lambda(T-t)) v - b) \\
&= \rho \int_0^{T-t} \exp(-\rho\tau) [\exp(-\lambda\tau) \min(W(b, v, t + \tau), 0) - (1 - \exp(-\lambda\tau)) b] d\tau \\
&\quad - \frac{b}{\rho + \lambda} [\lambda + \exp(-(\rho + \lambda)(T-t))] + \exp(-(\rho + \lambda)(T-t)) v
\end{aligned}$$

Now make the guess that  $W(b, v, t)$  is increasing in  $t$ . Given that  $W(B(v, t), v, t) = 0$  it follows that  $W(B(v, t), v, t + \tau) > 0$  for all  $\tau > 0$ . This implies that:

$$\begin{aligned}
0 &= W(B(v, t), v, t) \\
&= -\frac{B(v, t)}{\rho + \lambda} [\lambda + \rho \exp(-(\rho + \lambda)(T-t))] + \exp(-(\rho + \lambda)(T-t)) v
\end{aligned}$$

Solving for  $B(v, t)$  this gives

$$B(v, t) = \frac{(\rho + \lambda) \exp(-(\rho + \lambda)(T-t)) v}{\lambda + \rho \exp(-(\rho + \lambda)(T-t))}.$$