Price Setting with Real Rigidities in the Calvo Model*

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Abstract

We study the propagation of monetary shocks in a sticky-price general-equilibrium economy where the firms' pricing strategy feature a complementarity with the decisions of other firms. In a dynamic equilibrium the firm's price-setting decisions depend on aggregates, which in turn depend on the firms' decisions. We model sticky prices using the widely used Calvo assumption of a constant adjustment hazard for price changes. We analytically solve the equilibrium of this simple model. We establish existence and uniqueness of the equilibrium and characterize the impulse response function (IRF) of output following an aggregate shock. We prove that strategic complementarities make the IRF larger at each horizon. We establish that complementarities may give rise to an IRF with a hump-shaped profile. As the complementarity becomes large enough the IRF diverges and at a critical point there is no equilibrium.

Key Words: Strategic Complementarities, Mean Field Games, Menu costs, Impulse response analysis, monetary shocks.

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1 Introduction

In spite of substantive progress in the theory and empirics of general equilibrium models with sticky-prices, the need for tractability leads most analyses to abstract from the interactions between firms' decisions in price setting. Yet such complementarities are appealing because they seem empirically relevant, as argued by e.g. Cooper and Haltiwanger (1996); Amiti, Itskhoki, and Konings (2014, 2019); Beck and Lein (2020), and because they amplify the non-neutrality of nominal shocks, as argued by Nakamura and Steinsson (2010) and Klenow and Willis (2016).

Existing general equilibrium analyses proceed by exploring these effects numerically, as in Nakamura and Steinsson (2010), Klenow and Willis (2016) and Mongey (2021), or abstracting from the decision about the timing of adjustments, as in Wang and Werning (2022), or abstracting from idiosyncratic shocks, as in Caplin and Leahy (1997). In this paper we develop a new analytic approach to study a general equilibrium where the dynamic path of aggregates influences individual decisions, and viceversa. The results provide a complete characterization of the sticky-price equilibrium in a Calvo model featuring both idiosyncratic shocks and strategic complementarities, or substitutabilities, in pricing decisions.

2 General Equilibrium setup and Complementarities

This section presents an economy where households maximize the present value of lifetime utility and firms maximize profits subject to costly price adjustments. We show that non-negligible complementarities between the price setting strategies of firms can arise through two channels, possibly coexisting. First, from consumers' preferences that yield a demand system with a non-constant price elasticity, a phenomenon the literature dubbed *micro-complementarities* as in Kimball (1995). Second, a production structure that features sticky-price intermediate goods, as in Klenow and Willis (2016) and Nakamura and Steinsson (2010), referred to as *macro-complementarities*. We establish that the effects of both channels on the firm's pricing strategy are summarized by a single parameter and that, at a symmetric equilibrium, the firm's problem is approximated by a quadratic return function that depends on the own price and the aggregate price, as in the classic work of Caplin and Leahy (1997).

Households: We consider a continuum of households with time discount ρ and utility $\int_0^\infty e^{-\rho t}\left(U(\mathcal{C}(t)) - \mathfrak{a}\, L(t) + \log \frac{M(t)}{P(t)}\right) dt$, where U denotes a CRRA utility function over the consumption composite $\mathcal{C}$, the labor supply is L, M is the money stock, P is the consumption deflator, and $\mathfrak{a} > 0$ is a parameter. The linearity of the labor supply and the log specification for real balances are convenient simplifications also used in Golosov and Lucas (2007) and many other papers. We follow Kimball (1995) in modeling the consumption composite $\mathcal{C}$ using an implicit aggregator over a continuum of varieties k as follows $1 = \left(\int_0^1 \Upsilon\left(\frac{c_k(t)}{\mathcal{C}(t)}\, A_k(t)\right) dk\right)$ where A_k denotes a preference shock for variety k, and $\Upsilon(1) = 1$, $\Upsilon' > 0$ and $\Upsilon'' < 0$. The Kimball aggregator defines $\mathcal{C}$ implicitly, yielding an elasticity of substitution that varies with the relative demand $c_k/\mathcal{C}$. The standard CES demand is obtained as a special case when Υ is a power function.

The representative household chooses c_k, money demand and labor supply to maximize lifetime utility subject to the budget constraint

$$M(0) + \int_0^\infty e^{-\int_0^t R(s)ds}\left[\tilde{\Pi}(t) + (1+\tau_L)W(t)L(t) - R(t)M(t) - \int_0^1 \tilde{p}_k(t)c_k(t)dk\right] dt = 0$$

where $R(t)$ is the nominal interest rates, $W(t)$ the nominal wage, τ_L a constant labor subsidy, $\tilde{\Pi}(t)$ is the sum of the aggregate (net) nominal profits of firms and the lump sum nominal transfers from the government, and $\tilde{p}_k$ the price of each variety.

Firms. There is a continuum of firms indexed by $k \in [0,1]$, that use a labor (L_k) and intermediate-good inputs (I_k) to produce the final good y_k with a constant returns to scale technology (omit time index) as follows: $y_k = c_k + q_k = (L_k \,/\, Z_k)^\alpha\, I_k^{1-\alpha}$. Note that final goods are used by consumers, c_k, and also as an input in the production of the intermediate good $Q = \int_0^1 I_k\, dk$ through the production function $1 = \int_0^1 \Upsilon\left(\frac{q_k}{Q}\, A_k\right) dk$. The aggregates Q and $\mathcal{C}$ have the same unit price, P, since they are produced with identical inputs and the same function Υ. The labor productivity of firm k is $1/Z_k$ and we assume that $Z_k = \exp(\sigma \mathcal{W}_k)$ where $\mathcal{W}_k$ are standard Brownian motions, independent and identically distributed across firms, so that the log of Z_k follows a diffusion with variance σ^2. The households' labor supply L is used to produce each of the k goods and the price-adjustment services L_p, so $L = \int_0^1 L_k dk + L_p$.

The demand for final goods. The first order conditions of consumers and intermediate good producers yield the demand system, whose form depends on the function Υ. Given a total expenditure E the demand for variety k, evaluated at a symmetric equilibrium, is

$$y_k = \frac{1}{\Upsilon^{-1}(1)} \frac{E}{PA_k} D\left(\frac{p}{P}\right) \quad \text{where} \quad D\left(\frac{p}{P}\right) \equiv (\Upsilon')^{-1}\left(\frac{p}{P}\,\Upsilon'\left(\Upsilon^{-1}(1)\right)\right) \quad \text{and} \quad p \equiv \tilde{p}/A \, .$$

The firm's profit function. Let the nominal wage W be the numeraire, and $\tilde{p}_k = pA_k$ be the firm's price. Notice that the firm's marginal (and average) cost is $\chi \equiv (Z_k W)^\alpha P^{1-\alpha}$ where P is the price of intermediate inputs. We can write the firm's (nominal) profit as $y_k \cdot (pA_k - (Z_k W)^\alpha P^{1-\alpha})$. Assuming that $Z_k^\alpha = A_k$, i.e. that preference shocks are proportional to marginal cost shocks, then we have that each firm maximizes $\Pi(p,P) = y_k A_k\, W\left(\frac{p}{W} - \left(\frac{P}{W}\right)^{1-\alpha}\right)$ so the profits of the individual firm do not depend on Z_k since $y_k A_k = \frac{E}{\Upsilon^{-1}(1)\,P} D\left(\frac{p}{P}\right)$. The notation emphasizes that the firm's decision depends on both the own price, p, and the aggregate price P, and that prices are homogenous in W.

Let us write the profit in terms of the demand $D(p/P)$ and the cost function $\chi = \chi(P)$ giving the marginal cost. We have $\frac{\Pi(p,P)}{W} = \frac{E}{P\Upsilon^{-1}(1)} D(p/P)\,(p - \chi(P))$. The first order condition for optimality implicitly defines an optimal pricing function: $p^*(P) = \frac{\eta(p/P)}{\eta(p/P)-1}\,\chi(P)$ where $\eta(p/P) \equiv -\frac{p}{D(p/P)}\frac{\partial D(p/P)}{\partial p}$ so η is the elasticity of the demand D with respect to the own price p. We have the following:

Proposition 1. Consider a value for P such that $p^*(\bar{P}) = \bar{P}$. Assume that D is decreasing and that $\Pi(p,P)$ is strictly concave at $(p^*(\bar{P}),\bar{P}) = (\bar{P},\bar{P})$. We have

$$\frac{\bar{P}}{p^*(\bar{P})}\frac{\partial p^*(\bar{P})}{\partial P} = \frac{1}{1+\frac{\eta'(1)}{\eta(1)(\eta(1)-1)}}\left[\underbrace{\frac{\eta'(1)}{\eta(1)(\eta(1)-1)}}_{\text{micro elasticity}} + \underbrace{\frac{P}{\chi(P)}\frac{\partial \chi(P)}{\partial P}}_{\text{macro elasticity}}\right] \tag{1}$$

where $\eta(1) > 1$ and $1 + \frac{\eta'(1)}{\eta(1)(\eta(1)-1)} > 0$. Expanding the profit function around $(\bar{P},\bar{P})$:

$$\frac{\Pi(p,P)}{\Pi(\bar{P},\bar{P})} = 1 - \frac{1}{2}B\left(\frac{p-\bar{P}}{\bar{P}} + \theta\frac{P-\bar{P}}{\bar{P}}\right)^2 + \iota(P) + o\left(\left\|\frac{p-\bar{P}}{\bar{P}},\frac{P-\bar{P}}{\bar{P}}\right\|^2\right) \tag{2}$$

where $\iota(\cdot)$ is a function that does not depend on p, and where:

$$B \equiv -\frac{\Pi_{11}(\bar{P},\bar{P})}{\Pi(\bar{P},\bar{P})}\bar{P}^2 = [\,\eta'(1) + \eta(1)(\eta(1)-1)\,] > 0 \quad \text{and} \quad \theta \equiv \frac{\Pi_{12}(\bar{P},\bar{P})}{\Pi_{11}(\bar{P},\bar{P})} = -\frac{\bar{P}}{p^*}\frac{\partial p^*}{\partial P}\Big|_{p^*=\bar{P}} .$$

A few remarks are in order. First, equation (2) shows that the profit maximization problem of the firm is approximated by the minimization of the quadratic period return $B(x+\theta X)^2$, where $x = \frac{p-\bar{P}}{\bar{P}}$ and $X = \frac{P-\bar{P}}{\bar{P}}$ denote the percent deviation from the symmetric equilibrium of the own and the aggregate price, respectively.

Second, the extent of strategic interactions between the own price and the other firms' prices is captured by a single parameter, θ. Notice that static profits are maximized by setting $x = -\theta X$. The parameter θ measures the presence of strategic interactions. The firm's strategy exhibits strategic complementarity if $\theta < 0$, and it exhibits strategic substitutability if $\theta > 0$. Clearly, if $\theta \neq -1$ the only static equilibrium is $X = 0$.

Third, in the absence of macro complementarity, e.g. if $\frac{\partial \chi}{\partial P} = 0$, we have $\theta = -\frac{\eta'}{\eta(\eta-1)+\eta'}$ so that $\theta < 0$ occurs if $\eta' > 0$. This condition has a clear economic interpretation: if $\eta' > 0$ a higher P lowers the demand elasticity, which induces the firm to raise its markup. Thus $\eta' > 0$ implies that the own price and the aggregate price are strategic complements. Note moreover that if $\frac{\partial \chi}{\partial P} = 0$ the strength of strategic complementarities is bounded, since $\theta > -1$. Instead, if $\frac{\partial \chi}{\partial P} > 0$, we can have $\theta < -1$, a case of interest in the discussion of the equilibrium characterization and existence (see Section 3).

Impulse response of Output to a monetary shock. Note that an increase in the aggregate nominal wage for all firms reduces the average deviation of markups from its optimal value, i.e. it lowers X. One of the most interesting objects is the path of $X(t)$ after a small displacement of the stationary distribution, given by the initial condition $m_0(x) = \tilde{m}(x+\delta)$, where $\tilde{m}$ is the stationary density. The value of $X(t)$ is inversely proportional to the deviation from steady-state output t periods after the monetary shock δ. Below we also consider a more general perturbation $m_0(x) = \tilde{m}(x) + \delta\nu(x)$.

3 Strategic complementarities in the Calvo model

This section discusses a problem with strategic complementarities and Calvo's (1983) pricing. Due to its tractability this is the most common case analyzed in the sticky-price literature. The model offers a simple setup to introduce the essential elements of the analysis and several key results, such as existence, uniqueness and the non-monotone impulse response profiles, that will also appear in the state-dependent problem.[1]

The economy features a continuum of atomistic firms. Each firm takes as given the path of average deviation of markups $X(t)$ for all times $t \geq 0$. The firm can change its price *only* at random times $\{\bar{\tau}_k\}$, given by a Poisson process with parameter ζ. We refer to these times as *adjustment opportunities*, and to the state chosen at those times as the *optimal reset value*. After resetting its price at time t, the firm deviation of its markup $x(t)$ evolves as a drift-less Brownian motion with variance σ^2. The markup jumps right after a price change at $t = \bar{\tau}_k$ by the amount $\bar{J}_k$, thus the firm's deviation of its markup evolves as:

$$x(s) = x(t) + \sigma \left[\mathcal{W}(s) - \mathcal{W}(t)\right] + \sum_{k:\, \bar{\tau}_k \leq s} \bar{J}_k \quad \text{for all } s \in [t, T] \tag{3}$$

where $\mathcal{W}$ is a standard Brownian motion.

We assume that the strategic complementarities are at work only up to horizon T, and allow T to be finite or infinite. In particular for $t < T$ the period flow cost is $B\ (x + \theta X)^2$ with $B > 0$, which feature strategic interactions, corresponding to the description in Proposition 1. Each firm minimizes the expected discounted value of the flow cost – with discount rate ρ– taking the path $X(t)$ for $t \in [0, T)$ as given. At time $t = T$ a firm with state x has a continuation $u_T(x)$, independent of θ and $X(t)$. We will assume that u_T equals the steady state value function $\tilde{u}(x) = \frac{B}{(\rho+\zeta)}\left(\frac{\sigma^2}{\rho} + x^2\right)$, see Appendix A for the proofs.

Optimal price setting. For $t \in [0, T)$, the state of the firm is (x, t), the value function is $u(x, t)$, and the optimal reset value at t is $x^*(t)$. The firm takes as given $u_T : \mathbb{R} \to \mathbb{R}$ and

[1] We are thankful to an anonymous referee for suggesting us to study the Calvo problem.

$X : [0,T) \to \mathbb{R}$, and its value function $u : \mathbb{R} \times [0,T) \to \mathbb{R}$ solves:

$$u(x,t) = \min_{\{\bar{J}_k\}_{k=1}^{\infty}} \mathbb{E}\left[\int_t^T e^{-\rho(s-t)} B\left(x(s) + \theta X(s)\right)^2 \, ds \,+\, e^{-\rho(T-t)} u_T(x(T)) \,\Big|\, x(t) = x\right] \tag{4}$$

where the state evolves subject to equation (3).

Lemma 1. The value function $u : \mathbb{R} \times [0,T) \to \mathbb{R}$ solves the p.d.e.:

$$\rho u(x,t) = B \,\left(x + \theta X(t)\right)^2 + \frac{\sigma^2}{2} u_{xx}(x,t) + u_t(x,t) + \zeta\left(\min_z u(z,t) - u(x,t)\right) \tag{5}$$

with terminal condition $u(x,T) = \hat{u}(x)$ for all x, and the optimal reset $x^* : [0,T) \to \mathbb{R}$ solves $x^*(t) = \arg\min_z u(z,t)$, and it is given by

$$x^*(t) = -(\rho+\zeta)\theta \int_t^T \, e^{-(\rho+\zeta)(s-t)} X(s)\, ds \text{ for all } t \in [0,T) \quad . \tag{6}$$

The value function $u(x,0)$ is finite for all x if and only if

$$\int_0^T e^{-\rho t} B\left(x^*(t) + \theta X(t)\right)^2 dt < \infty \quad . \tag{7}$$

Thus, the optimal policy at the times when $t = \bar{\tau}_k$ is for x to jump to $x^*(t)$, i.e. $\bar{J}_k = x^*(t) - x(t^-)$. Three features will hold, with appropriate modifications, in the general case: the p.d.e. in equation (5), the condition that $0 = u_x(x^*(t),t)$, and that the optimal decision rule $x^*(t)$ is a (linear) function of the path of future X's. One difference with the state dependent model is that neither B nor σ^2 affect the optimal reset price in equation (5). If the condition in equation (7) is violated, expected discounted profits for the firm are minus infinity; this condition, given equation (6), restricts the path of $X(t)$. Furthermore, for future reference, note that the integral equation (6) is equivalent to the following o.d.e. and boundary condition:

$$\dot{x}^*(t) = (\rho+\zeta)\left(x^*(t) + \theta X(t)\right) \text{ for all } t \in [0,T) \text{ and } \lim_{t\to T} e^{-(\rho+\zeta)t} x^*(t) = 0 \tag{8}$$

Aggregation. For the models of interest, $X(t)$ is the cross sectional average of the x's. Consider a discrete time version with a (short) interval of length dt. In this interval a fraction of firm ζdt change its price, so their markup becomes $x^*(t)$. The remaining firms keep their (expected) value since x evolves as a drift-less Brownian Motion. Thus $X(t+dt) = (1-\zeta\, dt)X(t) + \zeta\, dt\, x^*(t)$. Taking the limit as $dt \to 0$ we obtain:

$$\dot{X}(t) = \zeta\Big(x^*(t) - X(t)\Big) \text{ for all } \in [0,T), \text{ with } X(0) = -\delta \text{ or equivalently} \tag{9}$$

$$X(t) = X(0)e^{-\zeta t} + \zeta \int_0^t e^{-\zeta(t-s)} x^*(s)ds \text{ for all } t \in [0,T) \tag{10}$$

Equilibrium Definition. An equilibrium for an initial condition $X(0) = -\delta$, are two paths $\{x^*(t), X(t)\}$ for $t \in [0,T)$ that solve the integral equation (6), encoding optimality, the integral equation (10), encoding aggregation, and satisfy the finite-value condition in equation (7). Alternatively, one could replace the two integral equations with the o.d.e's and boundary conditions in equation (8) and equation (9).

A few comments are in order, which anticipate our general case. First, the initial condition $X(0) = -\delta$ has the interpretation of the impact effect of a once and for all shock to nominal wages, triggered by a monetary shock. Second, optimal decisions are "forward looking", as usual, and are solved backward from the terminal condition $x^*(T) = 0$. Aggregation is "backward looking", and is solved forward given the initial condition $X(0)$. Third, both of these (integral) equations are linear, so the equilibrium is the fixed point of a linear operator. Fourth, for the case of $T = \infty$, there is a constraint on the square discounted integral of the paths.

The next lemma gives some features of the linear operator, such as the dominant eigenvalue μ_1, that are needed to study the equilibrium existence and its characterization. The proof is given for the general case in Lemma **??**.

We define the effect on the steady state deviation of output after a monetary shock of size one to be $Y_\theta(t,T) = -X(t)$, since output deviation is, up to first order, the negative of markups deviations. We summarize the main properties of the equilibrium in the next lemma.

3.1 Solving equilibrium by solving two dimensional o.d.e. system

To recap, the two o.d.e. equation (8) and equation (9) can be written as the linear two dimensional system:

$$\dot{y}(t) \equiv \begin{bmatrix} \dot{x}^*(t) \\ \dot{X}(t) \end{bmatrix} = \begin{bmatrix} (\rho+\zeta) & (\rho+\zeta)\theta \\ \zeta & -\zeta \end{bmatrix} \begin{bmatrix} x^*(t) \\ X(t) \end{bmatrix} \equiv A(\theta)\, y(t) \tag{11}$$

when $S(\theta)$ in invertible, as $\dot{y}(t) = S(\theta)\,\Lambda(\theta)\,S(\theta)^{-1}y(t)$, where $\Lambda(\theta)$ is the matrix with the eigenvalues $\lambda_1(\theta), \lambda_2(\theta)$ in the diagonal and $S(\theta)$ is the matrix of eigenvectors. Both Λ and S are functions of θ. To simplify we write

$$A(\theta) = \frac{\zeta}{\gamma} \begin{bmatrix} 1 & \theta \\ \gamma & -\gamma \end{bmatrix} \text{ where } \gamma \equiv \frac{\zeta}{\zeta+\rho} \tag{12}$$

The eigenvalues of $A(\theta)$ are given by:

$$\lambda_1(\theta) = \zeta\frac{(1-\gamma-\Delta(\theta))}{2\gamma}\,,\; \lambda_2(\theta) = \zeta\frac{(1-\gamma+\Delta(\theta))}{2\gamma} \text{ where } \Delta(\theta) \equiv \sqrt{(1+\gamma)^2 + 4\gamma\theta} \tag{13}$$

There is a critical value of θ^* which solves $\Delta(\theta^*) = 0$ given by

$$-1 \geq \theta^* = -\frac{(1+\gamma)^2}{4\gamma} = -1 - \frac{1}{4}\left(\frac{\rho}{\zeta}\right)^2 + o\left(\left(\frac{\rho}{\zeta}\right)^2\right) \tag{14}$$

For $\theta \geq \theta^*$ both eigenvalues of real, otherwise they are complex conjugates. If $\theta \neq \theta^*$, then the elements of the matrix $S(\theta)$ of eigenvectors are given by

$$s_{11}(\theta) = \frac{1+\gamma-\Delta(\theta)}{2\gamma}\,,\; s_{12}(\theta) = \frac{1+\gamma+\Delta(\theta)}{2\gamma}, \text{ and } s_{21}(\theta) = s_{22}(\theta) = 1 \tag{15}$$

In the case of $\theta = \theta^*$, the eigenvalues are repeated $\lambda_1(\theta) = \lambda_2(\theta) = \rho/2$, and hence to find the solution we use the Jordan form to represent $A(\theta)$ as $A(\theta) = S(\theta)J(\theta)S^{-1}(\theta)$ where $J(\theta)$ has $\rho/2$ on the diagonal, $J_{1,2}(\theta) = 1$ and zero otherwise. In this case $S(\theta)$ is given by:

$$s_{11}(\theta) = \frac{1+\gamma}{2\gamma}\,,\; s_{12}(\theta) = \frac{1}{\zeta}\,,\; s_{21}(\theta) = 1 \text{ and } s_{22}(\theta) = 0 \tag{16}$$

If $\theta \neq \theta^*$ the two eigenvalues are different, so we can solve $y(t) = S(\theta)z(t)$ where $\dot{z}(t) = \Lambda(\theta)z(t)$, or $z_j(t) = e^{\lambda_j(\theta)t}z_j(0)$ for $j = 1, 2$. The boundary conditions for $T < \infty$ can be written as

$$0 = x^*(T) \equiv y_1(T) = \left[s_{11}(\theta)z_1(0)e^{\lambda_1(\theta)T} + s_{12}(\theta)z_2(0)e^{\lambda_2(\theta)T}\right] \tag{17}$$

$$-1 = X(0) \equiv y_2(0) = s_{21}(\theta)z_1(0) + s_{22}(\theta)z_2(0) \tag{18}$$

From this system we obtain $z(0) = (z_1(0), z_2(0))$. Note that this include the case where the eigenvalues $\Lambda(\theta)$ and the matrix $S(\theta)$ are complex.

If $\theta = \theta^*$, we have $y(t) = S(\theta)z(t)$ where $\dot{z}(t) = J(\theta)z(t)$, so $z_2(t) = e^{\frac{\rho}{2}t}z_2(0)$ and $z_1(t) = e^{\frac{\rho}{2}t}z_1(0) + e^{\frac{\rho}{2}t}\, t\, z_2(0)$. for $j = 1, 2$.

$$0 = x^*(T) \equiv y_1(T) = e^{\frac{\rho}{2}T}\left[s_{11}(\theta)\left(z_1(0) + Tz_2(0)\right) + s_{12}(\theta)z_2(0)\right] \tag{19}$$

$$-1 = X(0) \equiv y_2(0) = s_{21}(\theta)z_1(0) + s_{22}(\theta)z_2(0) \tag{20}$$

From this system we obtain $z(0) = (z_1(0), z_2(0))$.

Lemma 2. Fix $\rho \geq 0$ and $\zeta > 0$ and let $T < \infty$. If the system of two o.d.e's and boundary conditions equation (8) and equation (9) has a solution $Y_\theta(t; T)$

$$Y_\theta(t; T) = (1 + \mathfrak{c}(\theta, T))e^{\lambda_2(\theta)t} - \mathfrak{c}(\theta, T)e^{\lambda_1(\theta)t} \quad \text{for} \quad t \in (0, T) \tag{21}$$

$$\mathfrak{c}(\theta, T) = \frac{(1 + \gamma + \Delta(\theta))\, e^{\lambda_2(\theta)T}}{(1 + \gamma - \Delta(\theta))\, e^{\lambda_1(\theta)T} - (1 + \gamma + \Delta(\theta))\, e^{\lambda_2(\theta)T}} \tag{22}$$

for the case of $\theta \neq \theta^*$ and

$$Y_{\theta^*}(t; T) = e^{\frac{\rho}{2}t}\left(1 - \frac{\zeta t}{\zeta T + \frac{2\gamma}{1+\gamma}}\right) \quad \text{for} \quad t \in (0, T) \tag{23}$$

if $\theta = \theta^*$. Then

1. If $\theta > \theta^*$, then $\Delta(\theta), \lambda_1(\theta), \lambda_2(\theta)$ are real. $\mathfrak{c}(\theta, T)$ is finite, and well defined. The solution is given by equation (21).

2. If $\theta = \theta^*$, then $\Delta(\theta^*) = 0$, and $\lambda_1(\theta^*) = \lambda_2(\theta^*) = \rho/2$. The solution is given by

equation (23).

3. If $\theta < \theta^*$, then $\Delta(\theta)$ is purely complex, the roots $\lambda_1(\theta), \lambda_2(\theta)$ are complex conjugates. The solution is given by equation (23). $\mathfrak{c}(\theta, T)$ is finite if and only if $\theta \neq \theta_j$ where the sequence $\{\theta_j\}_{j=1}^{\infty}$ is given by

$$\theta_j = \theta^* - \frac{(\Delta_j)^2}{4\gamma} \text{ where } \Delta_j \text{ solves } \Delta_j = -(1+\gamma)\tan\left(\Delta_j \frac{\zeta T}{2\gamma}\right) \text{ and}$$

$$\theta_1 \approx \theta^* - \gamma\left(\frac{\pi}{\zeta T + 2\frac{\gamma}{(1+\gamma)}}\right)^2 \text{ for large } \zeta T \text{ and } \theta_1 \to \theta^* \text{ as } \zeta T \to \infty$$

Proposition 2. Fix $\rho \geq 0$ and $\zeta > 0$. Let $\lambda_1(\theta), \lambda_2(\theta), \gamma, \Delta(\theta), \theta^*$ and $\{\theta_j\}_{j=1}^{\infty}$ be defined as in Lemma 2. We look for a solution of the two o.d.e's and boundary conditions equation (8) and equation (9), which satisfies the inequality equation (7). The equilibrium output $Y_\theta(t;T)$ is given by equation (21) or equation (23) depending on θ. We have:

1. Whenever an equilibrium exists it is unique.

2. If $T < \infty$:

   (a) If $\theta > \theta^*$ the equilibrium exists and $Y_\theta(t,T)$ is monotone in t and given by equation (21). Moreover, for any $t > 0$, then $Y_\theta(t,T) \to e^{\lambda_1(\theta)t}$ as $T \to \infty$.

   (b) If $\theta = \theta^*$ the equilibrium exists, $Y_\theta(t,T)$ is monotone in t, and it is given by equation (23).

   (c) If $\theta \in (\theta_1, \theta^*)$ the equilibrium exists, and it is given by equation (21). $Y_\theta(t,T)$ is hump-shaped in t.

   (d) If $\theta \leq \theta_1$ the equilibrium exists for all $\theta \neq \theta_j$ and $j = 1, 2, \ldots$. In this case $Y_\theta(t,T)$ oscillates with frequency $\frac{|\Delta(\theta)|(\rho+\zeta)}{4\pi}$, and amplitude $e^{\rho t/2}$. Moreover, for any $t > 0$, then $Y_\theta(t,T)$ does not converge as $T \to \infty$.

   (e) If $\theta = \theta_j$ for some j, there is no equilibrium. Fix any $t > 0$, the function $Y_\theta(t,T)$ has a pole at $\theta = \theta_j$, so it changes sign and satisfies $\lim_{\theta \to \theta_j} Y_\theta(t,T) = \pm\infty$.

3. If $T = \infty$:

(a) If $\theta > \theta^*$ there is a unique equilibrium given by $Y_\theta(t, \infty) = e^{\lambda_1(\theta)t}$. Fix any $t > 0$, then $Y_\theta(t, \infty)$ is strictly increasing and convex in $(-\theta)$, converges to 1 for all t as $\theta \to \theta^*$, and converges to zero for all t as $\theta \to \infty$.

(b) If $\theta \leq \theta^*$, there is no equilibrium.

3.2 Solving equilibrium by solving linear operator

LEMMA 3. The paths $\{x^*(t), X(t)\}$ are an equilibrium if and only if the path $\{X(t)\}$ satisfies $\int_0^T e^{-\rho t} X(t)^2 ds < \infty$ and solves the integral equation:

$$X(t) = X(0)e^{-\zeta t} + \theta \int_0^T K(t, s) X(s) ds \text{ for all } t \in [0, T) \tag{24}$$

for the kernel $K(t, s) \equiv \frac{\zeta(\rho+\zeta)}{2\zeta+\rho} \left(1 - e^{(2\zeta+\rho)\min\{t,s\}}\right) e^{-\zeta(t+s)-\rho s}$ for all$(t, s) \in [0, T)^2$. The kernel satisfies: (i) $K(t, s) \leq 0$, (ii) $K(t, s)e^{-\rho t}$ is symmetric in (t, s), (iii) for all T: $\sup_t \int_0^T |K(t, s)| ds \leq 1$; (iv) $\int_0^T \int_0^T K(t, s)^2 ds dt = \mathfrak{B}(T) < \infty$, and $\mathfrak{B}(T) \to \infty$ as $T \to \infty$; (v) for $T < \infty$ the kernel has countably many eigenvalues $\mu_j < 0$. They are ordered as $|\mu_1| > |\mu_2| > |\mu_3| \ldots$, and $|\mu_j| \to 0$ as $j \to \infty$. The corresponding eigenfunctions form an orthonormal base.

The lemma shows that the equilibrium kernel K does not depend on θ. This feature will also appear in the general problem. Thus, for instance, the dominant eigenvalue μ_1, which will be key to delimit the region where the equilibrium is well posed, does not depend on θ. Moreover, since $\mathfrak{B}(T) < \infty$ for finite T, then the operator is compact and we can employ a spectral decomposition of K to characterize the solution (see Proposition **??** for the proof and a more general case) in terms of eigenvalues and eigenfunctions, independent of θ.

We note that finding the equilibrium as the solution of the integral equation in equation (24) is less conventional in economics than using the equivalent system of two o.d.e's and boundary conditions. Nevertheless, we introduce this linear operator because it previews the use of a similar operator that will be used in the state-dependent case.

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Online Appendix:

A Proofs

Proof. (of Proposition 1). Define the markup $m(p/P) \equiv \frac{\eta(p/P)}{\eta(p/P)-1}$. Totally differentiating the first order condition $p^*(P) = m(p^*(P)/P)\,\chi(P)$ with respect to P, completing elasticities and evaluating at $p^* = P$ gives

$$\frac{P}{p^*}\frac{\partial p^*}{\partial P}\Big|_{p^*=P} = -\frac{m(1)\frac{\chi(P)}{p^*}}{1-m'(1)\frac{\chi(P)}{p^*}}\left(\frac{m'(1)}{m(1)}\right) + \frac{m(1)\frac{\chi(P)}{p^*}}{1-m'(1)\frac{\chi(P)}{p^*}}\left(\frac{P}{\chi(P)}\frac{\partial\chi(P)}{\partial P}\right)$$

and using that $\chi(P)/p^* = 1/m(1)$:

$$\frac{P}{p^*}\frac{\partial p^*}{\partial P}\Big|_{p^*=P} = \left[\frac{1}{1-\frac{m'(1)}{m(1)}}\right]\left[-\frac{m'(1)}{m(1)} + \frac{P}{\chi(P)}\frac{\partial\chi(P)}{\partial P}\ \right]$$

To get the expression in equation (1) note that $m(x) \equiv \frac{\eta(x)}{\eta(x)-1}$ so $m'(x) = \frac{\eta'(x)(\eta(x)-1)-\eta(x)\eta'(x)}{(\eta(x)-1)^2} = -\frac{\eta'(x)}{(\eta(x)-1)^2}$ and hence: $\frac{m'(1)}{m(1)} = -\frac{\eta'(1)}{(\eta(1)-1)^2}\frac{(\eta(1)-1)}{\eta(1)} = -\frac{\eta'(1)}{\eta(1)(\eta(1)-1)}$. That $\eta(1) > 1$ is implied by the first order optimality condition.

Next we show that $1+\frac{\eta'(1)}{\eta(1)(\eta(1)-1)} > 0$. Recall the second order condition for a maximum

$$\Pi_{11}(p^*,P) = D''(p^*/P)(p^*-\chi(P))/P^2 + 2D'(p^*/P)/P < 0$$

Note that $D' < 0$ and that $\chi/p^* = 1/m$ and rewrite the second order condition as

$$\frac{D''(p^*/P)}{D'(p^*/P)}\frac{p^*}{P}\left(1-\frac{1}{m}\right) + 2 > 0 \tag{25}$$

Next differentiate the elasticity $\eta(x) \equiv -\frac{\partial D(x)}{\partial x}\ \frac{x}{D(x)}$ and evaluate it at $x \equiv p^*/P = 1$. We get

$$\eta'(1) = -\frac{D''(1)}{D(1)} + \left(\frac{D'(1)}{D(1)}\right)^2 - \frac{D'(1)}{D(1)} = -\frac{D''(1)}{D(1)} + \eta^2 + \eta$$

where the second equality uses the elasticity definition. We can then write the second order condition equation (25) as $\frac{D''(1)}{D(1)}\ \frac{D(1)}{D'(1)}\ \frac{1}{\eta} + 2 > 0$ or, using the expression for D''/D and the elasticity definition $(\eta' - \eta^2 - \eta)\ \frac{1}{\eta^2} + 2 = \frac{\eta'+\eta(\eta-1)}{\eta^2} > 0$ which establishes that $1 + \frac{\eta'}{\eta(\eta-1)} > 0$, where all η are evaluated at $p^* = P$.

Finally, the expression for $B \equiv -\frac{\Pi_{11}(\bar{P},\bar{P})}{\Pi(\bar{P},\bar{P})}\bar{P}^2$, is obtained by direct computation evaluating the objects at $p^* = P \equiv \bar{P}$. We get

$$\frac{\Pi_{11}}{\Pi} = \frac{D''\left(1-\frac{1}{m}\right)\frac{p^*}{P^2} + 2\frac{D'}{P}}{D\,P\left(1-\frac{1}{m}\right)} = \frac{1}{P^2}\left(\frac{D''}{D} + 2\frac{D'}{D}\eta\right) = -\frac{1}{P^2}\left(\eta' + \eta(\eta-1)\right)\ .$$

Proof. (of Lemma 1). Equation (5) is the conventional Hamilton Jacobi equation giving the recursive formulation of the sequence problem in equation (4). As usual the flow value equals the period costs and the expected change in the value function, given by Ito's term and the possibility of a price adjustment. Likewise, the continuation value function for $t > T$ solves the HJB $(\rho + \zeta)\hat{u}(x) = Bx^2 + \frac{\sigma^2}{2}\hat{u}_{xx}(x) + \zeta\left(\min_z \hat{u}(z)\right)$, whose solution is given by $\hat{u}(x) = \frac{B}{(\rho+\zeta)}\left(\frac{\sigma}{\rho} + x^2\right)$.

Equation (6) follows by taking the first order condition of equation (4) with respect to x and using that the adjustment times are exponentially distributed with parameter ζ to compute the expectation. Equation (7) holds since T is finite, the continuation value function, $e^{-\rho T}\hat{u}(x)$, is bounded and since $u(x,t)$ is a quadratic function of x (as can be shown by equation (5)).

Proof. (of Proposition 2). Recall that $Y(t) = -X(t)$. The solution in equation (21) is obtained by solving the system of differential equations equation (8) and equation (9) with boundary conditions $x^*(T) = 0$ and $X(0) = -1$. This is a canonical 2 by 2 system whose solution $\dot{y} = Ay$ is readily obtained by a factorization of the matrix $A = S\Lambda S^{-1}$ into a diagonal matrix Λ of eigenvalues λ_j, $j = 1, 2$, given in the proposition, and the matrix of eigenvectors $S \equiv \begin{bmatrix} s_{11} & s_{12} \\ 1 & 1 \end{bmatrix}$ where $s_{11} \equiv \frac{1+\gamma-\Delta}{2\gamma}$, $s_{12} \equiv \frac{1+\gamma+\Delta}{2\gamma}$ and $\Delta \equiv \sqrt{(1+\gamma)^2 + 4\gamma\theta}$ is the discriminant of the characteristic equation. The eigenvalues are real if $\Delta \geq 0$. Using the boundary conditions the constant $\mathfrak{c}(\theta, T)$ appearing in the solution is $\mathfrak{c}(\theta, T) \equiv \frac{s_{12}e^{\lambda_2 T}}{s_{11}e^{\lambda_1 T} - s_{12}e^{\lambda_2 T}}$.
$\square$

Proof. (of Proposition **??**). Here we argue that, if $\theta \neq -1$, then the stationary solution displayed above is unique. On the other hand, if $\theta = -1$, then any value X_{ss} corresponds to a steady state. Define $w \equiv x + \theta X_{ss}$. Consider the value function $\hat{u}$ corresponding to the control problem:

$$\hat{u}(w) = \min_{\{\tau_i, \Delta w_i\}} \mathbb{E}\left[\int_0^\infty e^{-\rho t} B w^2(t) dt + \sum_{i=1}^{\infty} \psi 1_{\{\tau_i \neq t_i\}} e^{-\rho \tau_i} \,|\, w(0) = w\right]$$

where $dw = \sigma dW$ for $t \in [\tau_i, \tau_{i+1})$ and $w(\tau_i^+) = w(\tau_i^-) + \Delta w_i$ and where t_i are the realizations of the exogenously given times at which the fixed cost is zero, which are exponentially distributed with parameter ζ.

We start making two claims about this problem, and then a third claim about the stationary distribution. First, the value function $\hat{u}$ is symmetric around zero, i.e. $\hat{u}(w) = \hat{u}(-w)$ for all w. This follows because the flow cost Bw^2 is symmetric around zero, and because a standard Brownian motion has, for any collection of times, increments that are normally distributed, and hence symmetric around zero. Second, if the solution of the value function is C^2 then it must satisfy: $(\rho+\zeta)\hat{u}(w) = Bw^2 + \hat{u}_{ww}(w)\frac{\sigma^2}{2} + \zeta u(w^*)$ for all $w \in [-\underline{w}, \bar{w}]$ with boundary conditions: $\hat{u}(\bar{w}) = \hat{u}(\underline{w}) = \hat{u}(w^*) + \psi$ and $0 = \hat{u}_w(\bar{w}) = \hat{u}_w(\underline{w}) = \hat{u}_w(w^*)$. Thus, since $\hat{u}$ is symmetric, it must be the case that $\bar{w} = -\underline{w}$ and $w^* = 0$.

Third, and finally, using the symmetry of the thresholds $\{\underline{w}, w^*, \bar{w}\}$, we can find the stationary density $\hat{m}(w)$ which is the unique solution of

$$0 = \hat{m}_{ww}(w)\tfrac{\sigma^2}{2} - \zeta\hat{m}(w) \text{ for all } w \in [\underline{w}, w^*) \cup (w^*, \bar{w}]$$

with boundary conditions: $0 = \hat{m}(\bar{w}) = \hat{m}(\underline{w})$, $\lim_{w\uparrow w^*} \hat{m}(w) = \lim_{w\downarrow w^*} \hat{m}(w)$, and $1 = \int_{\underline{w}}^{\bar{w}} \hat{m}(w)dw$. Importantly, the density $\hat{m}$ must be symmetric, centered at $w^* = 0$.[2] Hence, $\int_{\underline{w}}^{\bar{w}} w\, \hat{m}(w)dw = 0$. Thus, a stationary equilibrium solution of the original problem requires: $x^*_{ss} = w^* - \theta X_{ss}$, $\underline{x}_{ss} = \underline{w} - \theta X_{ss}$, $\bar{x}_{ss} = \bar{w} - \theta X_{ss}$,

$$X_{ss} = \int_{\underline{w}}^{\bar{w}} \hat{m}(w)\left(w - \theta X_{ss}\right) dw = \int_{\underline{w}}^{\bar{w}} \hat{m}(w) w dw - \theta X_{ss} \int_{\underline{w}}^{\bar{w}} \hat{m}(w) dw$$

and thus we can construct a stationary state if and only if: $X_{ss} = -\theta X_{ss}$. Hence if $\theta \neq -1$, then $X_{ss} = 0$ is the only stationary state, and if $\theta = -1$ one can construct a stationary state for any X_{ss}. $\square$

[2] This can be shown since for $[\underline{w}, 0]$ and $[0, \bar{w}]$, the density is a linear combination of the same two exponentials. Using the boundary conditions at $\underline{w}$ and $\bar{w}$ we express each the density in each segment as function of one constant of integration. Finally by continuity at $w = 0$ we find that the distribution must be symmetric.