Caballero-Engel meet Lasry-Lions: A uniqueness result*

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Abstract

We prove the uniqueness of the equilibrium in Mean Field Games (MFG) in a class of problems where the decision makers control (and reset) the state at optimally chosen times. We consider a MFG in which each decision maker cares about the cross sectional distribution of the state across agents, and where the dynamics of the distribution is generated by the optimal decisions of the agents. This setup accommodates several problems featuring non-convex adjustment costs, and complements the well known drift-control case studied by Lasry-Lions. Example of these problems are described by Caballero and Engel in several papers, which introduce the concept of the generalized hazard function of adjustment. We extend the analysis to a general "impulse control problem" by introducing the concept of the "Impulse Hamiltonian". Under the monotonicity assumption (a form of strategic substitutability), we establish the uniqueness of equilibrium. In this context, the Impulse Hamiltonian and its derivative play a similar role to the classical Hamiltonian that arises in the standard drift control case.

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1 Introduction

Several economic problems feature a continuum of agents subject to idiosyncratic shocks and aggregate dynamics. The equilibrium of these economies is a fixed point problem in which individual decisions depend on aggregates, and aggregate dynamics depend on individual decisions. The feedback between individual decision and aggregates raises non-trivial questions about the uniqueness of the equilibrium, that cannot be addressed computationally. Such fixed point problems can be represented by the mathematical structure of Mean Field Games (MFG). The hallmark of MFG is a system of two partial differential equations with appropriate boundary conditions: a "forward-looking" Hamilton-Jacobi-Bellman equation, describing individual choices, coupled with a "backward-looking" Kolmogorov equation describing the dynamics of the cross sectional distribution. The MFG formulation rigorously defines the problem to analyze the uniqueness of the equilibrium.

For a broad class of MFG problems, in which the decision maker controls *the drift* of a diffusion process, Lasry and Lions (2007) establish an important uniqueness result. These problems are akin to e.g., a dynamic savings problem, as described by Aiyagari (1994) and analyzed as a MFG by Achdou et al. (2021). A remarkable feature of the Lasry-Lions result is that the uniqueness is ensured by a property of the period return function, the so called Lasry-Lions monotonicity condition and an appropriate separability assumption. This property is a form of "strategic substitutability" when applied to a game defined by the period return function. To be clear, this implies the uniqueness of the equilibrium of a dynamic game for an arbitrary initial condition.

In this paper we present a complementary uniqueness result for a class of problems where the control by the decision maker consists in choosing the stopping times for resetting the state, rather than in controlling the drift. In other words, we examine the uniqueness of the equilibrium in a Mean Field Games (MFG) characterized by a form of "Impulse Control" instead of "drift control". These problems are akin to e.g., optimal investment or price setting problems in the presence of fixed costs, pioneered by Caballero and Engel (1999). Our setup assumes that the uncontrolled state follows a diffusion process, and that the decision maker can take (costly) actions that affect the probability to control the state. When such a control opportunity arises, the decision maker can reset the state to any desired level. Consequently, the state follows a diffusion process with jumps, and the intensity and size of these jumps are optimally determined.[1] We demonstrate that under standard monotonicity conditions a MFG possesses a unique classical equilibrium.

We now describe three decision problems that fit within the formulation we are considering. We note that these problems consider the case where the decision maker faces a time-invariant environment, and that they are not embedded into a MFG. In the first problem the decision maker draws a fixed cost of adjustment from an arbitrary distribution with a constant probability per unit of time. If the decision maker pays the fixed cost she can exercise control and reset the state to any desired value. Otherwise the state remains uncontrolled and continues to follow a diffusion process until the next (random) time when the decision maker has the opportunity to change the state. This type of decision problem was first proposed in Caballero and Engel (1993b), analyzed in Caballero and Engel (1999, 2007), and recently further characterized in Alvarez, Lippi, and Oskolkov (2021), where it has been applied to optimal investment and pricing decisions of firms. In these problems, the decision maker selects a *time invariant probability of adjustment* as a function of the state, referred to as the *generalized hazard function*. Additionally, upon adjusting, the state's is optimally reset to a time-invariant value, known as the return point. The second problem stems from papers in applied mathematics that consider continuous-time setups where the decision maker controls a diffusion process only at random Poisson times. Some notable contributions in this area include Wang (2001), Dupuis and Wang (2002), and Menaldi and Robin (2016).[2] These papers develop the appropriate variational inequality that applies to the decision maker's value function. Finally, the third type of problems are economic mod-

[1] As mentioned, a key aspect of the MFG concept is that the decision maker's flow return depends on the cross-sectional distribution of the state.

[2] Menaldi and Robin (2016) considers a more general specifications for the distribution of the random times where the decision maker can control the state, and a different specification of the adjustment cost.

els such as the one in Costain and Nakov (2011), based on Woodford (2009), where agents choose a probability of adjustment per unit of time subject to a cost that increases with this probability. Instead in this paper we consider these decision problems as part of a MFG. In particular we analyze the uniqueness of the MFG equilibrium, i.e. the uniqueness of the fixed point, where the decision maker's flow return depends on the cross-sectional distribution, and the cross-sectional distribution evolves according to the optimal decisions of the agents.

We consider a MFG without aggregate uncertainty, i.e. in the language of MFG the case with "no aggregate noise". Consequently, the optimal decisions can be summarized by two objects. The first is the optimal return, which represents the value of the state that the decision maker will choose if she has an opportunity to make an adjustment. This optimal return is a function of time. The second is the probability per unit of time of an adjustment opportunity, which is also a function of the state and time. These two objects depend on time because the cross sectional distribution evolves through time, and the decision maker is forward looking. Equivalently, using the terms introduced by Caballero and Engel, the generalized hazard rate and the optimal return point are time dependent and endogenous. They are time dependent because the future path of cross sectional distribution is time varying. They are endogenous because the cross sectional distribution depends on the optimal decision rules.

To analyze the problems described above we introduce a function which we label Impulse Hamiltonian. The Impulse Hamiltonian gives the optimized expected reduction in the cost, as a function of the difference between the value function at the current time and state, and the minimized value function at the current time. The Impulse Hamiltonian serves a similar purpose than the Hamiltonian in the case of drift control. The Impulse Hamiltonian has convexity properties which are analogous to the ones of the standard Hamiltonian. Moreover, Impulse Hamiltonian has the property analogous to the one in the drift control in which its derivative gives the drift of the diffusion used in the Kolmogorov-Fokker-Planck equation. In particular, in the case of the Impulse Hamiltonian, the probability per unit of time of a change of the state equals (the negative) of its derivative. It turns out that, as in the standard case, the coupling of the Hamilton-Jacobi-Bellman equation with the Kolmogorov-Fokker-Planck equation can be done using the Impulse Hamiltonian and its derivative.

Our main result are sufficient conditions for the uniqueness of equilibrium of the MFG. The conditions for uniqueness of the MFG equilibrium are similar to the ones in the drift control case, which essentially require monotonicity of the flow cost and terminal value function, and some uniformity in the convexity of the Hamiltonian. Recall that monotonicity of the period return function has the the same interpretation that strategic substitution in static games. In our case, besides the same monotonicity condition on the period return, we require some uniformity on first and second derivative of the Impulse Hamiltonian. We can translate the conditions of the Impulse Hamiltonian to the distribution of random adjustment cost. For instance, the uniformity of the first derivative is equivalent to have mass point of zero cost of adjustment, which in economic models translate to have some adjustment as in the Calvo model, or in the Calvo$^+$ model. The uniformity of the second derivative, is equivalent to a density of the fixed cost bounded from below.

We consider two extensions. The first extension is one where only part of the state is controlled, as often occurs in many applications in economics. The second one is a version where the controlled state is multidimensional. Examples of multidimensional control are the multiproduct pricing models of Midrigan (2011), Alvarez and Lippi (2014), and Bhattarai and Schoenle (2014). In the multidimensional case we obtain the main uniqueness result by considering a relaxed version where the decision maker, upon exercising control, instead of setting the state at the optimal value it sets to that value plus a zero mean noisy, which can be taken to have an arbitrarily small variance.

Our contributions relates to the literature in MFGs. In particular, there is a large literature on uniqueness on MFGs based on the seminal contribution by Lasry and Lions (2007) and the literature that follows it. The vast majority of this literature focuses in the case of drift control, and has extended their initial results to different set ups. Our work is related to the one by Bertucci (2020), and our own work in Alvarez, Lippi, and Souganidis (2023). In the models in these two papers, at any time agents chose to pay a fixed cost and change the value of the state. As a consequence, the optimal decision rule at each time has the form of dividing the state space in two regions, one where there is inaction, and one where control is exercised, i.e. for a given state either adjustment occurs or not. Instead, in the context of the model in current paper, at each time the optimal adjustment occurs with a probability rate per unit of time, which varies with the value of the state. There are two more differences of our paper. In Alvarez, Lippi, and Souganidis (2023) we consider only a perturbation from the MFG equilibrium, but we allow both for the case of monotonicity and "anti-monotonicity", i.e. strategic substitutability and complementarity. In Bertucci (2020), like in our current work, we consider the uniqueness of the equilibrium, not just a perturbation.

Organization. In Section 2 we set up the equilibrium of the mean field game in the simplest case of one dimensional variable. We define the decision maker problem, the aggregation, and how the equilibrium relates them as a fixed point. In this section we introduce the concept of an Impulse Hamiltonian. In Section 3 we write two decision problems that give rise to an Impulse Hamiltonian. In Section 4 we develop the result of uniqueness of the equilibrium for the mean field game. In Section 5 we introduce an exogenous random variable, in addition to the idiosyncratic state that the decision maker controls. This is closest to the cases used in economic. We use this set up to illustrate how in price setting models one may obtain the Lasry-Lions monotonicity. We show the main result that leads to uniqueness for this case. In Section 6 we outline the extension of the set up and uniqueness result to a multidimensional case. For this case we use a version with noisy control, but the extent of the noise can be taken to be arbitarily small.

2 Set Up

We describe the elements that define a MFG. Let $\mathcal{P} \equiv \{m : \mathbb{R} \to \mathbb{R}_+, \int m(x)dx = 1\}$ be the space of densities, and $0 < T < \infty$ be the time horizon. A MFG is defined by $\{\rho, F, H, \mu, \sigma, m_0, u_T\}$ where: (i) $\rho \geq 0$ is the discount rate, (ii) $F : \mathbb{R} \times \mathcal{P} \to \mathbb{R}$ is the flow cost with global coupling, (iii) $H : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_-$ is the Impulse-Hamiltonian, (iv) $\mu : \mathbb{R} \to \mathbb{R}$, and $\sigma : \mathbb{R} \to \mathbb{R}_+$ are the drift and diffusion coefficients of the uncontrolled state, (v) $m_0 \in \mathcal{P}$ is the initial cross sectional density, (vi) $u_T : \mathbb{R} \times \mathcal{P} \to \mathbb{R}$ is the terminal value function.

The equilibrium of the MFG is given by triple $\{\bar{x}, u, m\}$: (a) a path for the optimal return $\bar{x} : [0, T] \to \mathbb{R}$, (b) a value function $u : \mathbb{R} \times [0, T] \to \mathbb{R}$, and (c) a cross-sectional density $m : \mathbb{R} \times [0, T] \to \mathbb{R}_+$. This triplet has to solve the coupled Hamilton-Jacobi-Bellman equation, and the corresponding Kolmogorov Forward equation. We describe each one in turn.

Hamilton-Jacobi-Bellman equation. Given a path $\{m(x, t)\}$, the value function u and the path $\bar{x}$ solves the following HJB equation and boundary conditions:

$$\begin{aligned}
\rho u(x, t) = & H\left(u(x, t) - u(\bar{x}(t), t), x\right) + F(x, m(t)) + \mathcal{L}\left(u(x, t)\right) \\
& + \partial_t u(x, t) \text{ for all } t \in [0, T], x \in \mathbb{R} \qquad (1) \\
u(x, t) \geq & \; u(\bar{x}(t), t) \text{ for all } t \in [0, T], x \in \mathbb{R} \qquad (2) \\
u(x, T) = & \; u_T(x, m(T)) \text{ for all } x \in \mathbb{R} \qquad (3)
\end{aligned}$$

where $\mathcal{L}$ gives the expected change on the value function per unit of time due to the change in x, and is defined as

$$\mathcal{L}\left(f\right)(x, t) = \mu(x)\partial_x f(x, t) + \tfrac{1}{2}\sigma^2(x)\partial_{xx} f(x, t)$$

for any function $f(\cdot, t)$ that is twice differentiable. The equation (2) defines $\bar{x}(t)$ as the optimal return point at t, i.e. it is the optimal decision for an agent that can adjust her state. Finally, equation (3) specifies when the games end at $t = T$, the decision makers receives the given terminal reward u_T. Next we give the interpretation of the Impulse Hamiltonian H.

Interpretation of Impulse-Hamiltonian. The Impulse-Hamiltonian $H : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_-$ gives the expected change (decrease) in the value function per unit of time when the highest difference between the current value function and the minimum of the value function is v, i.e. $v = u(x, t) - u(\bar{x}(t), t)$, conditional on the current state x. This change occurs as consequence of an optimal impulse –i.e. a discrete adjustment– that changes the state from x to $\bar{x}(t)$, i.e. from the current value x to the value that minimizes $u(\cdot, t)$. We let $\lambda^*(v, x)$ the probability (per unit of time) of an adjustment given v. We impose that $H(0, x) = 0$ for all x, since at the optimal there is nothing that can be gain. We postulate that λ^* is positive, and increasing in v, i.e.:

$$\lambda^*(v, x) \geq 0, \lambda^*(0, x) = 0, \text{ and } \lambda^*_v(v, x) > 0$$

Furthermore, we postulate that H_v is (minus) the probability per unit of adjustments, i.e.:

$$H_v(v, x) = -\lambda^*(v, x) \text{ and } H_{vv}(v, x) = -\lambda^*_v(v, x) < 0$$

To comments are in order. First, as of now it is not clear why (minus) the derivative of the Impulse Hamiltonian has to be the probabilty per unit of time of an impulse in the state. In Section 3 we present two examples of explicit problems a decision maker solves were we *derive* the Impulse Hamiltonian with all the properties we assumed here. Second, if we evaluate the derivative of the Impulse Hamiltonian gives the extension of what Caballero and Engel (1993a) and Alvarez, Lippi, and Oskolkov (2021) call the *generalized hazard function* $\Lambda(x, t) \equiv \lambda^*(u(x, t) - u(\bar{x}(t), t)) = -H_v(u(x, t) - u(\bar{x}(t), t), x)$

Fokker-Planck-Kolmogorov Forward equation. Given the value function $\{u(x,t)\}$ and path $\{\bar{x}(t)\}$ the cross sectional density m solves the following partial differential equation:

$$\partial_t m(x,t) = \mathcal{L}^* (m) (x,t) - \lambda^* (u(\bar{x}(t), x) - u(x,t), x) \; m(x,t) \tag{4}$$

for all $t \in [0,T]$, all $x \in \mathbb{R}, x \neq \bar{x}(t)$, where $\mathcal{L}^*$ is defined at x for a function $f(\cdot, t)$ that is twice differentiable as:

$$\mathcal{L}^* (f) (x,t) = -\partial_x (\mu(x) f(x,t)) + \tfrac{1}{2} \partial_{xx} \left(\sigma^2(x) f(x,t) \right)$$

Few comments are in order. First, the term $\lambda^* (u(\bar{x}(t), x) - u(x,t), x) \; m(x,t)$ is the probability flux that leaves the state (x,t) as a consequence of the optimal adjustment. Second, this flux is the product of the probability of adjustment $\lambda^* (u(\bar{x}(t), x) - u(x,t), x)$ per unit of time, multiplied by the density at that point $m(x,t)$. Third, this equation does *not* apply to $(x,t) = (\bar{x}(t), t)$. In this point there is a flux of probability coming in from all the other points (x,t) with $x \neq \bar{x}(t)$, so its evolution is not local. We return to this in Lemma 1.

We can rewrite the time evolution of m using the assumed property of the Impulse Hamiltonian, namely that $\lambda^*(v,t) = -H_v(v,x)$ is the probability of an adjustment, per unit of time, if $v = u(x,t) - u(\bar{x}(t), t)$. Using this property we can rewrite the time evolution as:

$$\begin{aligned} \partial_t m(x,t) &= \mathcal{L}^* (m) (x,t) \\ &\quad + H_v (u(\bar{x}(t), x) - u(x,t), x) \; m(x,t) \text{ for all } t \in [0,T], x \in \mathbb{R}, x \neq \bar{x}(t) \end{aligned} \tag{5}$$

Finally, to completely determine the time evolution we require that m must preserve the probability and that it is initialized by m_0, namely:

$$1 = \int_{-\infty}^{\infty} m(x,t) dx \text{ for all } t \in [0,T] \tag{6}$$

$$m_0(x) = m(x,0) \text{ for all } x \in \mathbb{R} \tag{7}$$

Relative value function. Note that in both H in the Hamilton-Bellman-Jacobi equation and in H_v Kolmogorov forward equation the argument is $u(x,t) - u(\bar{x}(t),t)$. Motivated by this we define v as:

$$v(x,t) = u(x,t) - u(\bar{x}(t),t) \text{ for all } x \text{ and for all } t \in [0,T] \tag{8}$$

We are ready to define an equilibrium for a MFG.

Definition 1. Fix an initial density m_0 and a terminal value u_T. A classical equilibrium of the MFG is a triplet of functions $\{u, m, \bar{x}\}$ where (a) $\{u, \bar{x}\}$ satisfies the p.d.e. 1, the boundary conditions 2, and terminal condition 3 given m, and where (b) m satisfies the p.d.e. 5, the condition 6, and the initial condition 7 given $\{u, \bar{x}\}$.

3 Two examples of Impulse Hamiltonians

In this section we present two decision problems where we can define the impulse Hamiltonian with the properties we used above. Wang (2001) and Dupuis and Wang (2002) also study similar decision problems, which can be written using an Impulse Hamiltonian. Menaldi and Robin (2016) study a decision problem with in some dimension, a more general setup which cannot be accommodated by using an Impulse Hamiltonean.

3.1 Costly probabilistic adjustment

In this example, the decision maker pays a flow cost $c(\lambda, x)$ and obtain a probability per unit of time of changing the state, λ, i.e. she decides the Poisson arrival rate of an opportunity to change the state. The cost depends on the current level of the state x and on the chosen probability λ. If the opportunity materializes, which occur with probability λ, the decision maker can chose the level of the state freely, and it will do so to minimize $u(\cdot, t)$. The model here is a generalization of the one in Costain and Nakov (2011). In Appendix 8 we write the discrete time version of this problem. The value function $u(x,t)$ must then solve:

$$
\begin{aligned}
\rho u(x,t) &= u_t(x,t) + \mathcal{L}(u)(x,t) + F(x,m(t)) \qquad (9)\\
&\quad + \min_{\lambda\geq 0}\left[c(\lambda,x) - \lambda\left(u(x,t) - \min_z u(z,t)\right)\right] \text{ for all } x\in\mathbb{R} \text{ and } t\in[0,T] \text{ and}\\
u(x,T) &= u_T(x,m(T)) \text{ for all } x\in\mathbb{R}\,. \qquad (10)
\end{aligned}
$$

We assume that the cost function depends on the probability λ and the current value of the state x, i.e. $c:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$. We assume that for each $x\in\mathbb{R}$ then $c(\cdot,x)$ is differentiable with :

$$
c(\lambda,x)\geq 0,\ c(0,x)=0,\ c_\lambda(\lambda,x)\geq 0,\ c_\lambda(0,x)=0, \text{ and } c_{\lambda\lambda}(\lambda,x)>0 \text{ if } c_\lambda(\lambda,x)>0
$$

We now rewrite this problem introducing the optimal choice $\bar{x}(t)$ as before $\bar{x}(t)=\arg\min_z u(z,t)$ for all $t\in[0,T]$. Then we can define the impulse Hamiltonenan any $v\geq 0$ as:

$$
\begin{aligned}
H(v,x) &= \min_{\lambda\geq 0} c(\lambda,x) - \lambda v \text{ and its optimal choice} \qquad (11)\\
\lambda^*(v,x) &= \arg\min_{\lambda\geq 0} c(\lambda,x) - \lambda v \qquad (12)
\end{aligned}
$$

with the following properties:

1. $H(0,x)=0$.
2. $H(v,x)\leq 0$, and $H_v(v,x)<0$ for $v>0$.
3. $H_v(v,x) = -\lambda^*(v,x)$ solving $v = c_\lambda\left(\lambda^*(v,x),x\right)$ for $v>0$.
4. $H_{vv}(v,x) = -\lambda^*_v(v,x) = -1/c_{\lambda\lambda}\left(\lambda^*(v,x),x\right)<0$ for $v>0$.

Uniform bounds. For future reference note that if there is an $\underline{\ell}>0$ such that $c(\lambda,x)=0$ for all $\lambda\in[0,\underline{\ell}]$ and x, then $H_v(v,x)\leq \bar{H}_v\equiv -\underline{\ell}<0$. Also, assume there is an $\underline{L}>0$ such that $c_{\lambda\lambda}(\lambda,x)>\underline{L}>0$ for all $\lambda>\underline{l}$ and x, then $H_{vv}(v,x)\leq\bar{H}_{vv}\equiv -\underline{L}<0$ for all v,x.

3.2 Random Fixed Costs

In this case, with probability $\kappa(x) > 0$ per unit of time, the decision maker draws a fixed cost of adjustment ψ from a distribution with a mass point $G(0, x) \geq 0$ for $\psi = 0$, and with a density $g(\psi, x)$ for $\psi > 0$. The mass point G, and density g is allowed to depend on the state x, so $g : (0, \infty) \times \mathbb{R} \to [0, \infty)$. It is assumed that successive draws of the fixed cost, conditional on x are independently distributed. A decision maker with state (x, t) and a realization ψ of the cost at hand can either pay the cost ψ and adjust, changing its value function from $u(x, t)$ to $\psi + u(\bar{x}(t), x)$ where $\bar{x}(t) = \arg\min_x u(x, t)$, or not pay the fixed cost and let the the state x evolve as uncontrolled. This is a continuous time version of the decision problem in Caballero and Engel (1991, 1993c,b, 2007), as characterized in Alvarez, Lippi, and Oskolkov (2021). In Appendix 9, we describe the discrete time version of this problem, and derive its continuous time limit displayed in this section.

Again, the firm takes as given the path $m(t)$ for $t \geq 0$.

$$\rho u(x,t) = u_t\left(x,t\right) + \mathcal{L}\left(u\right)\left(x,t\right) + F(x, m(t)) + \kappa(x) G(0,x)\left(u(\bar{x}(t),t) - u(x,t)\right) \tag{13}$$

$$+ \kappa(x) \int_0^\infty \min\left\{0\,,\ \psi\ + u(\bar{x}(t),t) - u(x,t)\right\} g(\psi,x) d\psi \text{ for all } x \in \mathbb{R} \text{ and } t \in [0,T] \text{ and}$$

$$u(x,T) = u_T(x, m(T)) \text{ for all } x \in \mathbb{R}\,. \tag{14}$$

For this case we can define for any $v \geq 0$, the function $H : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$:

$$\begin{aligned} H(v,x) &= \kappa(x)\left[-G(0,x)v + \int_0^\infty \min\left\{0\,,\ \psi\ - v\right\} g(\psi,x)d\psi\right] \\ &= \kappa(x)\left[-G(0,x)v + \int_0^v \left(\psi - v\right) g(\psi,x)d\psi\right] \end{aligned} \tag{15}$$

$$H_v(v,x) = -\kappa(x)\left[G(0,x) + \int_0^v g(\psi,x)d\psi\right] \leq 0 \text{ so that}$$

$$\lambda^*(v,x) = \kappa(x)\left[G(0,x) + \int_0^v g(\psi,x)d\psi\right] \geq 0 \text{ and} \tag{16}$$

$$H_{vv}(v,x) = -\kappa(x)g(v,x) \leq 0 \text{ so that } \lambda^*_v(v,x) = \kappa(x)g(v,x) \geq 0 \tag{17}$$

Uniform bounds. For future reference note that if there is an $\underline{\ell} > 0$ such that $\kappa(x)G(0,x) \geq \underline{\ell} > 0$ for all x, then $H_v(v,x) \leq \bar{H}_v \equiv -\underline{\ell} < 0$. Moreover, if there is an $\underline{L} > 0$ such that $\kappa(x)g(v,x) \geq \underline{L}$, then $H_{vv}(v,x) \leq \bar{H}_{vv} \equiv -\underline{L} < 0$.

4 Uniqueness of MFG: benchmark case

In this section we show the uniqueness of a classical MFG in one dimension. We start by stating the assumptions that we use on the flow return F, the terminal value function u_T, the drift and volatility μ, σ^2, and the impulse Hamiltonian H. Then we show two lemmas, the second one being the key for the proof. Finally, we state the main result.

1. **Monotonicity Assumption.** F and u_T are *weakly monotone* if

$$\int \big(F(x,m^a) - F(x,m^b)\big)\big(m^a(dx) - m^b(dx)\big) \geq 0 \tag{18}$$

$$\int \big(u_T(x,m^a) - u_T(x,m^b)\big)\big(m^a(dx) - m^b(dx)\big) \geq 0 \tag{19}$$

   for any $m^a, m^b \in \mathcal{P}$. F or u_T are *strictly monotone* if the inequality holds strictly every time that $\int (m^a - m^b)^2 dx > 0$.

2. **Single peaked and boundedness of F:** We assume that F is continuously differentiable and that:

$$\exists\, B > 0 \text{ s.t. for all } m \in \mathcal{P} \text{ and } x \in \mathbb{R} : |F(x,m)| \leq B \tag{20}$$

$$\forall\, m \in \mathcal{P} \text{ there is a unique } x^*(m) : \partial_x F(x^*(m), m) = 0 \tag{21}$$

3. **Regularity of drift and volatility:** We assume that $\mu(\cdot)$ is once continuous differ-

entiable, and $\sigma^2(\cdot)$ is twice continuous differentiable with:

$$\begin{aligned}&\exists\, B \text{ s.t. : } ||\sigma^2||_\infty \leq B,\ ||\mu||_\infty \leq B,\ ||\partial_x \mu||_\infty,\ ||\partial_x \sigma^2||_\infty \leq B \text{ and}\\ &\exists\, \underline{s} > 0 \text{ s.t. : } \underline{s}^2 \leq \sigma^2(x) \text{ all } x\end{aligned} \tag{22}$$

4. **Impulse Hamiltonian.** We assume that the impulse Hamiltonian is twice continuously differentiable and it satisfies the following properties for all x:

$$H : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_-, H(0, x) = 0, \text{ and } H_v(v, x) \leq 0, H_{vv}(v, x) \leq 0 \text{ for all } v > 0. \tag{23}$$

Finally, we add some regularity conditions to the triplet $\{u, m, \bar{x}\}$ defining a MFG.

Definition 2. Let $\{u, m, \bar{x}\}$ be a classical equilibrium of the MFG, specified as in Definition 1. We say that $\{u, m, \bar{x}\}$ is a classical regular equilibrium if in addition:

1. $\bar{x}$ is a continuously differentiable function of time,
2. u is once continuously differentiable with respect to t, twice continuously differentiable with respect to x, and there is a function $M : [0, T] \to \mathbb{R}_+$ such that for all $t \in (0, T)$:

$$||u(\cdot, t)||_\infty \leq M(t), ||\partial_x u(\cdot, t)||_2 \leq M(t), ||\partial_{xx} u(\cdot, t)||_2 \leq M(t) \tag{24}$$

3. m is continuous on (x, t), once continuously differentiable with respect to t for all t, and twice continuously differentiable with respect to x for $(x, t) \neq (\bar{x}(t), t)$ and there is a function $M : [0, T] \to \mathbb{R}_+$ such that for all $t \in (0, T)$:

$$||m(\cdot, t)||_2, ||\partial_x m(\cdot, t)||_2 \leq M(t),\ ||\partial_{xx} m(\cdot, t)||_2 \leq M(t) \tag{25}$$

As a final preliminary definition we let the constants $\bar{H}_v$ and $\bar{H}_{vv}$ be the bounds for the first and second derivative of the Impulse Hamiltonian:

$$H_v(z,x) \le \bar{H}_v \le 0 \text{ for all } z \in \mathbb{R} \text{ and all } x \tag{26}$$

$$H_{vv}(z,x) \le \bar{H}_{vv} \le 0 \text{ for all } z \in \mathbb{R} \text{ and all } x \tag{27}$$

The next lemma uses the conservation of probability to obtain a useful equality between the probabilities flows of adjustment.

Lemma 1. Assume that m solves the Kolmogorov Forward Equation given $\{v, \bar{x}\}$ as described by 5, 6 and 7. Assume that m satisfies the integrability conditions of a classical regular equilibrium in (25), and that σ^2, μ satisfies the assumptions in (22). Then for all $t \in [0,T]$ we have:

$$\tfrac{1}{2}\sigma^2(\bar{x}(t))\left[\partial_x m(\bar{x}(t)_-, t) - \partial_x m(\bar{x}(t)_+, t)\right] = -\int_{-\infty}^{\infty} m(x,t) H_v(v(x,t),x) dx > 0 \tag{28}$$

For each t the left hand side of equation (28) is the probability inflow at $\bar{x}(t)$, which equals the probability outflow everywhere else. Note that this requires that $\partial_x m(\cdot, t)$ has a different right and left limit.

Given two classical regular equilibria of the MFG $\{u^a, m^a, \bar{x}^a,\}$ and $\{u^a, m^b, \bar{x}^b\}$ we define

$$K(t) \equiv \int_{-\infty}^{\infty} \left(u^a(x,t) - u^b(x,t)\right)\left(m^a(x,t) - m^b(x,t)\right) dx \text{ for all } t \in [0,T] \tag{29}$$

for all $t \in [0,T]$.

The next Proposition obtains the key result to establish uniqueness, in a similar manner than the classical Lasry-Lions inequality.

Proposition 1. Assume that F satisfy the weak monotonicity conditions given by 18, and that H satisfy the conditions given by equation (23). Furthermore, assume that μ and σ^2 are once and twice continuously differentiable in x. Suppose that $\{u^a, m^a, \bar{x}^a,\}$ and $\{u^a, m^b, \bar{x}^b\}$

are two classical regular equilibria of the MFG. Let v^a, v^b be defined as in equation (8). Then

$$
\begin{aligned}
&\frac{d}{dt}K(t) = \rho K(t) + S(t) \text{ for all } t \in [0,T] \text{ where} \\
&\quad S(t) \leq \bar{S}(t) \equiv \bar{H}_v \left[v^b(\bar{x}^a(t),t) + v^a(\bar{x}^b(t),t) \right] \\
&\qquad\qquad + \bar{H}_{vv} \int_{-\infty}^{\infty} \left[m^a(x,t) + m^b(x,t) \right] \left(v^a(x,t) - v^b(x,t) \right)^2 dx \;\; \text{all} \;\; t \in (0,T)
\end{aligned} \tag{30}
$$

The strategy of the proof of Proposition 1 is similar to the one pioneered by Lasry and Lions, i.e. reminiscent of an energy method. There are three differences. First, $m(\cdot,t)$ cannot be differentiable at $x = \bar{x}(t)$. Second the p.d.e.'s for the HBJ uses of Impulse Hamiltonian H which depends on $v(x,t) = u(x,t) - u(\bar{x}(t),t)$, as opposed to the Hamiltonian which depends on the the space derivative of u in the drift control case. Third, the p.d.e. for the KFE is coupled using the level of the Impulse Hamiltonian H, as opposed to the divergence in the case of the drift control case. Nevertheless, the convexity properties of the Impulse Hamiltonian allows a similar proof strategy as the classical result by Lasry and Lions. Finally, the integrability conditions assumed in Definition 2 are used here for $K(t)$ and its time derivative. To be concrete, letting $\tilde{u}^i, \tilde{m}^i$ be the value functions and distributions for the case of drift control, and let $\tilde{H}_{pp}$ be a lower bound on the second derivative of the corresponding Hamiltonian. The classical Lasry-Lions inequality for the drift control case is:

$$
\begin{aligned}
&\tilde{K}(t) \equiv \int \left(\tilde{u}^a(x,t) - \tilde{u}^b(x,t) \right) \left(\tilde{m}^a(x,t) - \tilde{m}^b(x,t) \right) = \int_0^t e^{\rho(t-\tau)} \tilde{S}(\tau) d\tau \text{ and} \\
&\tilde{S}(\tau) \leq \tilde{H}_{pp} \int \left(\tilde{m}^a(x,\tau) + \tilde{m}^b(x,\tau) \right) \left(\tilde{\partial}_x u^a(x,\tau) - \tilde{\partial}_x u^b(x,\tau) \right)^2 dx \leq 0
\end{aligned}
$$

This expression is to be compared with equation (30) for our case.

Note that in equation (30), v^a and v^b are both positive, and $\bar{H}_v \leq 0$ and $\bar{H}_{vv} \leq 0$. Then, the together with the monotonicity assumption of u_T, the next lemma obtains that $S(t) = 0$.

Lemma 2. Let $\{u^a, m^a, \bar{x}^a\}$ and $\{u^b, m^b, \bar{x}^b\}$ be two classical regular equilibria of the MFG, with initial density m_0 and terminal condition u_T. Let K defined as in equation (29). Assume, in addition to the assumptions for Proposition 1 that u_T satisfies the weak monotonicity condition 19. Then: $K(t) = S(t) = 0$ for all $t \in [0, T]$, where S is defined in equation (30).

The next lemma uses the conclusions of the previous one, to obtain two intermediate results used to establish uniqueness. In particular, Lemma 3, shows that the optimal return is the same in any classical equilibrium of the MFG.

Lemma 3. Let $\{u^a, m^a, \bar{x}^a\}$ and $\{u^b, m^b, \bar{x}^b\}$ be two equilibria of the MFG, with initial density m_0 and terminal condition u_T. Assume that $K(t) = S(t) = 0$. Furthermore, assume that $\bar{H}_v < 0$ and that $F(\cdot, m)$ has a unique minimum as in condition (21). Then $\bar{x}^a(t) = \bar{x}^b(t)$ for all $t \in [0, T]$. Furthermore, if $\bar{H}_{vv} < 0$, then $v^a(x,t) = v^b(x,t)$ for all x and $t \in [0, T]$.

Using the previous two lemmas we can show the main result, i.e. the uniqueness of the classical MFG equilibrium.

Theorem 1. Assume that (i) F is strongly monotone and u_T weakly monotone as defined in 18 and 19, that (ii) H satisfy the conditions given by equation (23) with $\bar{H}_{vv} < 0$ and $\bar{H}_v < 0$, that iii) μ, σ^2 satisfies (22), and that (iv) $F(\cdot, m)$ has a unique minimum as in condition (21). Let $\{u^a, m^a, \bar{x}^a\}$ and $\{u^b, m^b, \bar{x}^b\}$ two classical regular MFG equilibrium as in Definition 2 for the initial distribution m_0 and terminal value u_T. Then $m^a = m^b$, $u^a = u^b$, and $\bar{x}^a = \bar{x}^b$, i.e. a classical regular equilibrium is unique.

The previous theorem is the main result for the paper. We now discuss the role of strong vs weak monotonicity on this theorem. Theorem 1 uses a strong form of Monotonicity of F while the previous results, i.e. Lemma 1, Proposition 1, Lemma 2 and Lemma 3 only use weak monotonicity of F. We note that without strong monotonicity, there is uniqueness of the value function $u(x,t)$, of the path optimal return point $\bar{x}$, and of the generalized hazard function $-H_v(u(x^*(t),t) - u(x,t))$. In particular the values of $F(x, m(t))$ are unique too. Yet, we have not shown, without the strong monotonicity assumption, that m is unique. Note that in this case, if there were more that one m, this lack of uniqueness would *not* be due to strategic considerations, and will have no effect on the values for agents. We next write down an assumption that, almost by fiat, eliminates this lack of uniqueness.

Fix an optimal return $\bar{x}$ and a relative value function v, so we can define the Poisson rate $\lambda^*(x,t) = -H_v(v(x,t),x)$. Fix a λ^* function and consider equation (4) for a given initial condition m_0.

Assumption: uniqueness of m given policy. Assume that given a function λ^* and an initial condition m_0, the functions μ are σ are such that there is a unique solution m of the Kolmogorov forward equation (4).

We can then relax the strong monotonicity of F, and instead add the assumption of uniqueness of m given a policy. In this case we note that Lemma 1, Proposition 1, Lemma 2 and Lemma 3 only use weak monotonicity of F. Thus adding the assumption of uniqueness of m for a given pollicy we have the following variation on the main theorem.

Theorem 2. Assume that (i) F is weakly monotone and u_T weakly monotone as defined in 18 and 19, that (ii) H satisfy the conditions given by equation (23) with $\bar{H}_{vv} < 0$ and $\bar{H}_v < 0$, that iii) μ, σ^2 satisfies (22), that (iv) $F(\cdot, m)$ has a unique minimum as in condition (21), and that (v) m is unique for a given policy. Let $\{u^a, m^a, \bar{x}^a\}$ and $\{u^b, m^b, \bar{x}^b\}$ two classical regular MFG equilibrium as in Definition 2 for the initial distribution m_0 and terminal value u_T. Then $m^a = m^b$, $u^a = u^b$, and $\bar{x}^a = \bar{x}^b$, i.e. a classical regular equilibrium is unique.

The proof is straightforward given Lemma 1, Proposition 1, Lemma 2 and Lemma 3.

5 Adding a exogenous state

In this section we consider the case where the state of the problem is given by a triplet (x, z, t), where x is the state that can be controlled and affects the flow cost, and where z is a state that can not be affected by the decision maker, but that affects the flow cost. In this problem, the decision maker controls the probability of an adjustment of x, given the state (x, z, t). If an adjustment takes place, then the state will go from (x, z) at time t to $(\bar{x}(z, t), z)$ so $\bar{x}(z, t)$ is the optimally chosen value of x. The value function u has arguments (x, z, t). The jump Hamiltonian depends on the decrease in cost, conditional on adjustment, which we denote by $u(x, z, t) - u(\bar{x}(z, t), z, t)$, as well as the value of (x, z).

The state z follows a diffusion $dz = \mu_z(z)dt + \sigma_z(z)dW_z$, where W_z is a standard Brownian motion. In this case the law of for the density of z can be written as:

$$\partial_t n(z, t) = -\partial_z \left(\mu_z(z) n(z, t)\right) + \partial_{zz} \left(\frac{\sigma_z^2(z)}{2} n(z, t)\right) \tag{31}$$

To simplify the analysis we assume that the cross sectional distribution of z is at steady state and we omit the t index and simply write

$$0 = -\partial_z \left(\mu_z(z) n(z)\right) + \partial_{zz} \left(\frac{\sigma_z^2(z)}{2} n(z)\right)$$

Hamilton-Jacobi-Bellman equation. Given a path $\{m(x, z, t)\}$, the value function u and the path $\bar{x}(z, t)$ solves the following HJB equation and boundary conditions:

$$\begin{aligned} \rho u(x, z, t) = & H\left(u(x, z, t) - u(\bar{x}(z, t), z, t), x, z\right) + F(x, z, m(t)) + \mathcal{L}(u)(x, z, t) \\ & + \partial_t u(x, z, t) \text{ for all } t \in [0, T], x \in \mathbb{R} \end{aligned} \tag{32}$$

$$u(x, z, t) \geq u(\bar{x}(t, z), z, t) \text{ for all } t \in [0, T], x \in \mathbb{R} \tag{33}$$

$$u(x, z, T) = u_T(x, z, m(T)) \text{ for all } x \in \mathbb{R} \tag{34}$$

where $\mathcal{L}$ gives the expected change on the value function per unit of time due to the change in x, z, and is defined as

$$\begin{aligned} \mathcal{L}(f)(x, z, t) = & \mu_z(x)\partial_x f(x, z, t) + \mu_x(x, z)\partial_x f(x, z, t) \\ & + \tfrac{1}{2}\sigma_x^2(x, z)\partial_{xx} f(x, z, t) + \tfrac{1}{2}\sigma_z^2(x)\partial_{zz} f(x, z, t) \end{aligned}$$

Note that the optimal return $\bar{x}(z,t)$ depends both on t and z. So, upon an adjustment at time t the state of the decision maker jumps from (x,z) to $(\bar{x}(z,t),z)$. Also, implicit in the notation for $\mathcal{L}$ is the simplified hypothesis that the brownian form x and z are independent, i.e. at times where there is no adjustment we assume that $dx = \mu_x(x,z)dt + \sigma_x(x,z)dW_x$ where $\mathbb{E}[dW_z, dW_x] = 0$.

In this section we define v as

$$v(x,z,t) = u(x,z,t) - u(\bar{x}(x,z),z,t) \tag{35}$$

Fokker-Planck-Kolmogorov Forward equation. Given the value function $\{u(x,z,t)\}$ and path $\{\bar{x}(z,t)\}$ the cross sectional density m solves the following partial differential equation:

$$\partial_t m(x,z,t) = \mathcal{L}^*(m(x,z,t)) + H_v\left(u(x,z,t) - u(\bar{x}(z,t),x),x,z\right)\, m(x,t)$$

$$\text{for all } t \in [0,T], (x,z) \in \mathbb{R}^2, x \neq \bar{x}(z,t) \tag{36}$$

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m(x,z,t)dxdz \text{ for all } t \in [0,T] \tag{37}$$

with initial condition $m(x,z,0) = m_0(x,z)$, and where $\mathcal{L}^*$ is defined at x,z for a function $f(\cdot,t)$ that is twice differentiable as:

$$\begin{aligned}\mathcal{L}^*(f)(x,z,t) = &- \partial_x\left(\mu_x(x,z)f(x,z,t)\right) + \tfrac{1}{2}\partial_{xx}\left(\sigma_x^2(x,z)f(x,z,t)\right)\\ &- \partial_z\left(\mu_z(z)f(x,z,t)\right) + \tfrac{1}{2}\partial_{zz}\left(\sigma_z^2(z)f(x,z,t)\right)\end{aligned}$$

As before, $-H_v\left(u(x,z,t) - u(\bar{x}(z,t),x),x,z\right)$, gives the optimally chosen probability of adjustment.

The initial condition for m has to satisfy:

$$m(x,z,0) = m_0(x,z) \text{ for all } x,z \text{ with } \int_{-\infty}^{\infty} m_0(x,z)dx = n(z) \text{ for all } z \tag{38}$$

We define the conditional probability as:

$$m(x,t|z) = m(x,z,t)/n(z,t) = m(x,z,t)/n(z) \tag{39}$$

and note that for each (z,t):

$$\int m(x,z,t)dx = n(z) \text{ and } \int \partial_t m(x,z,t)dx = 0 \tag{40}$$

5.1 Two Price Setting Examples

In this section we give two simple price setting examples where we describe the flow profits of the firm F. In both examples x denotes the price charged for the firm. In both cases, the firm can only change the price if it pays a cost, where this can be a random menu cost as described in Section 3.2 or a costly probability adjustment as in Section 3.1. In the first example the marginal cost of the firm depends on the output of the rest of the industry. In the second example the demand of the firm depends on the average price charged by the other firms. Both cases are simple enough to that checking monotonicity is trivial.

Example 1: Input price dependent on industry output. The demand is given by $D(x,z)$ where z is an exogenous shock. The marginal cost of the firm depend on the price of the input, which instead, depends on the the aggregate output of the industry. This aggregate output is given by $\bar{D} \equiv \int D(x',z')m(x',z')dx'dz'$, where m is the cross sectional distribution of firms indexed by x,z. The marginal cost is given by $c(\bar{D},z) = \gamma(z) + \theta\bar{D}$ where the constant $\theta > 0$. We let F be minus the profits, since we will be minimizing the objective function. Then the flow cost, or minus profits are

$$
\begin{aligned}
F(x,z,m) &= -\left[x - c\left(\bar{D}, z\right)\right] D(x,z) = -\left[x - \gamma(z) - \theta \bar{D}\right] D(x,z) \\
&= -xD(x,z) + \gamma(z)D(x,z) + \theta D(x,z) \int D(x',z')m(x',z')dx'dz
\end{aligned}
$$

We can easily check weak monotonicity in this example. Consider $m^a(x,z) = f^a(x|z)n(z)$ and $m^b(x,z) = f^b(x|z)n(z)$

$$
\begin{aligned}
&\int\int \left(F(x,z,m^a) - F(x,z,m^b)\right)(dm^a - dm^b) \\
&= \int n(z) \int \left(F(x,z,m^a) - F(x,z,m^b)\right)(f^a - f^b)dxdz \\
&= \theta \left[\int n(z) \left(\int D(x,z)f^a(x|z)dx - \int D(x,z)f^b(x|z)dx\right) dz\right]^2 \geq 0 \iff \theta > 0
\end{aligned}
$$

Thus monotonicity requires that the marginal cost increases with the industry output.

Example 2: Cross demand elasticity dependent on average price. The demand of the good for the firm depend on its own price x and the average price $X = \int x' \, dm(x',z')$. We consider the following demand $D(x,z,X) = D_0(x,z) + \theta(z)X$, where θ is a constant. Note that $\theta > 0$ corresponds to the case of gross substitutes. We assume that the marginal cost is given by $\gamma(z)$. The the flow cost, or minus profits are:

$$
\begin{aligned}
F(x,z,m) &= -\left(x - \gamma(z)\right)\left(D_0(x,z) + \theta X\right) \\
&= -(x - \gamma(z))D_0(x,z) + \gamma(z)\theta \int\int x'm(x',z')dx'dz' + \theta x \int\int x'm(x',z')dx'dz'
\end{aligned}
$$

Again, we can easily check monotonicity in this example. Let $X^k = \int\int x \, m^k(x,z)dxdz$

for $k = a, b$ so that

$$\begin{aligned}
&\int \left(F(x,z,m^a) - F(x,z,m^b)\right)(dm^a - dm^b) \\
&= \theta\left[X^a \int\int x\,(m^a(x,z) - m^b(x,z))dxdz - X^b \int\int x\,(m^a(x,z) - m^b(x,z))\,dxdz\right] \\
&= \theta(X^a - X^b)^2 \geq 0 \iff \theta > 0
\end{aligned}$$

Thus monotonicity requires the goods to be substitutes, i.e. that holding constant the price of the good, its demand increases when the other prices increase.

5.2 Uniqueness of Equilibrium of MFG

We first list the assumptions on $\mu_x, \mu_z, \sigma_x^2, \sigma_z^2$, F and u_T. We then list the regularity of assumption on the equilibrium objects $u, m, \bar{x}$. For the exogenous objects we assume that:

1. Boundedness of μ and σ^2 and its derivatives – to be written

2. We assume that F and u_T satisfies LL weak monotonicity, that that they are continuous and bounded.

The definition of monotonicity is the same, i.e $f : \mathbb{R}^2 \times \mathcal{P} \to \mathbb{R}$, then we say $f$is weakly monotone if:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(f(x,z,m^a) - f(x,z,m^b)\right)\left(m^a(x,z) - m^b(x,z)\right)dxdz \geq 0$$

for any two densities m^a, m^b.

In the definition of a classical regular equilibrium we require $u, m, \bar{x}$ to be smooth and integrable as follows

1. The function $\bar{x}(z,t)$ is continuously differentiable function of time –to be written

2. Boundedness of u and integrability of derivatives for each t – to be written

3. Integrability of derivatives for each m and its derivatives for t – to be written

Now we start with the characterization. The next lemma is a generalization of the analysis of the probability flux from the benchmark case. Note that here it holds for all $z.t$.

LEMMA 4. For any $z \in \mathbb{R}$ and $t \in (0, T)$, if m solves the KFE we have

$$\begin{aligned} &\tfrac{1}{2}\sigma^2(\bar{x}(z,t),z)\left[\partial_x m(\bar{x}_-(z,t),z,t) - \partial_x m(\bar{x}_+(z,t),z,t)\right] \\ &= -\int_{-\infty}^{\infty} m(x,z,t) H_v(v(x,z,t),x,z)\,dx \end{aligned} \tag{41}$$

The next lemma has a simple observation of equality in the key expression for uniqueness of using value functions or relative value functions.

LEMMA 5. For any $z \in \mathbb{R}$ and $t \in (0, T)$. Let $\{u^a, \bar{x}^a, m^a\}$ and $\{u^b, \bar{x}^b, m^b\}$ be two classical regular equilibrium of the MFG. Recall $v^i(x,z,t) \equiv u^i(x,z,t) - u^i(\bar{x}^i(z,t),z),t)$ for $i = a, b$. Then

$$\begin{aligned} K(t,z) &\equiv \int_{-\infty}^{\infty} \left(u^a(x,z,t) - u^b(x,z,t)\right)\left(m^a(x,z,t) - m^b(x,z,t)\right)dx \\ &= \int_{-\infty}^{\infty} \left(v^a(x,z,t) - v^b(x,z,t)\right)\left(m^a(x,z,t) - m^b(x,z,t)\right)dx \end{aligned}$$

The next lemma gives gives a decomposition on the change of K through time. It uses the p.d.e.'s for the HBJ and KF equations.

Lemma 6. Define $\bar{K}(t) = \int K(t,z)dz$. Then for all $t \in [0,T]$:

$$
\begin{aligned}
&\frac{d}{dt}\bar{K}(t) = \rho \bar{K}(t) \\
&+ \int\int \left((v^a(x,z,t) - v^b(x,z,t)\right) \mathcal{L}^*(m^a - m^b)(x,z,t)dxdz \\
&- \int\int \left((m^a(x,z,t) - m^b(x,z,t)\right) \mathcal{L}(v^a - v^b)((x,z,t)dxdz \\
&- \int\int \left(m^a(x,z,t) - m^b(x,z,t)\right) \left(F(x,z,m^a(t)) - F(x,z,m^b(t))\right) dxdz \\
&- \int\int \left((m^a(x,z,t) - m^b(x,z,t)\right) (H(v^a(x,z,t),x,z) - H(v^b(x,z,t),x,z))dxdz \\
&+ \int\int \left((v^a(x,z,t) - v^b(x,z,t)\right) \left(H_v(v^a(x,z,t),x,z)m^a(x,z,t) - H_v(v^b(x,z,t),x,z)m^b(x,z,t)\right) dxdz
\end{aligned}
$$

The next lemma analysis two of the expressions in the previous lemma. Repeated integration by parts, consideration of the limit behavior of different elements imply the following key result:

Lemma 7. Let $\{m^a, u^a, \bar{x}^a\}$ and $\{m^b, u^b, \bar{x}^b\}$ two regular classical equilibrium of the MFG. Then:

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(v^a(x,z,t) - v^b(x,z,t)\right) \mathcal{L}^*(m^a - m^b)(x,z,t)dxdz \\
-&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(m^a(x,z,t) - m^b(x,z,t)\right) \mathcal{L}(v^a - v^b)(x,z,t)dxdz \\
&\leq \bar{H}_v \int_{-\infty}^{\infty} n(z) \left(v^b(\bar{x}^a(z,t),z,t) + v^a(\bar{x}^b(z,t),z,t)\right) dz
\end{aligned}
$$

The concavity properties of the impulse Hamiltonian imply the following result, which bounds two of the terms expressed in Lemma 6.

Lemma 8. For any v^a, m^a, v^b, m^b given the concavity of H we have

$$
\begin{aligned}
&-\left((m^a(x,z,t) - m^b(x,z,t)\right)\left(H(v^a(x,z,t),x,z) - H(v^b(x,z,t),x,z)\right) \\
&+\left((v^a(x,z,t) - v^b(x,z,t)\right)\left(H_v(v^a(x,z,t),x,z)m^a(x,z,t) - H_v(v^b(x,z,t),x,z)m^b(x,z,t)\right) \\
&\leq \bar{H}_{vv}\left(m^a(x,z,t) + m^b(x,z,t)\right)\left(v^a(x,z,t) - v^b(x,z,t)\right)^2
\end{aligned}
$$

The previous lemmas imply the main proposition for this model, which is at the center of the Lasry-Lions style of argument, as we used in the previous section.

Proposition 2. Let $\{u^a, \bar{x}^a, m^a\}$ and $\{u^b, \bar{x}^b, m^b\}$ be two classical regular equilibrium of the MFG. Assume that F is monotone. Using the definition of $\bar{K}$ and the previous result we get:

$$
\begin{aligned}
\bar{K}(t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(u^a(x,z,t) - u^b(x,z,t)\right)\left(m^a(x,z,t) - m^b(x,z,t)\right)dxdz \text{ then} \\
\frac{d}{dt}\bar{K}(t) &= \rho\bar{K}(t) + S(t) \text{ for all } t \in [0,T] \text{ where } S(t) \geq \bar{S}(t) \text{ with} \\
\bar{S}(t) &\leq \bar{H}_v\int_{-\infty}^{\infty} n(z)\,\left(v^b(\bar{x}^a(z,t),z,t) + v^a(\bar{x}^b(z,t),z,t)\right)dz \\
&\quad + \bar{H}_{vv}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(m^a(x,z,t) + m^b(x,z,t)\right)\left(v^a(x,z,t) - v^b(x,z,t)\right)^2 dxdz
\end{aligned}
$$

The uniqueness of equilibrium is obtained following similar steps as in the benchmark case.

6 Multidimensional, noisy optimal decision

In this section we develop a version of the model where we have two generalizations. First we allow that $x \in \mathbb{R}^n$ for $n \geq 1$, i.e. we consider the multidimensional case. Second, when the decision makers decides to adjust the state, instead of jumping to $\bar{x}(t)$, the adjustment is distributed with a density ν_ϵ centered around $\bar{x}(t)$. This distribution is indexed by $\epsilon > 0$, where ϵ measure the dispersion of x around $\bar{x}(t)$. The distribution ν_ϵ is defined as follows:

$$\nu_\epsilon(x - \bar{x}(t)) = \epsilon g\left(\frac{x - \bar{x}(t)}{\epsilon}\right)$$

where $g : \mathbb{R} \to \mathbb{R}_+$ is a smooth density with a maximum at zero, so $g(0) > g'(0) = 0$. We use ν_ϵ to smooth out the choice of optimal return, i.e. when the decision maker selects to move to state from x to $\bar{x}(t)$, the state will will be randomly distributed around $\bar{x}(t)$ according to the density ν_ϵ.

In this case the impulse Hamiltonian is $H : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_-$. The function $H(\cdot, x)$, for a fixed $x \in \mathbb{R}^n$, as the same properties as before, i.e it is negative, monotone, and concave. In this case, the first argument of the jump Hamiltonian is the (negative) of the expected change conditional on an adjustment.

We note that in this section where we let $x \in \mathbb{R}^n$, when the decision maker decides to exercise control, it changes the entire $n-$dimensional state, i.e. it changes all the components of the state. This is similar to the multiproduct pricing model of Midrigan (2011), Alvarez and Lippi (2014), and Bhattarai and Schoenle (2014). An interesting alternative assumption, which we explore separately, is the one in which the decision maker can only control some subset of the state.

Additionally, each of the n coordinates of x when it is uncontrolled follow $dx_i = \mu_i(x)dt + \sigma_i(x)dW_i$ for $i = 1, 2, \ldots, n$, where to simplify the notation we assume that $\{W_i, W_j\}$ are orthogonal for $i \neq j$, i.e. $\mathbb{E}\left[dW_i \, dW_j\right] = \delta_{i,j}$. Thus we have two vectors fields:

$$\mu(x) = \{\mu_1(x), \mu_2(x), \ldots, \mu_n(x)\} \text{ and } \sigma^2(x) = \{\sigma_1^2(x), \sigma_2^2(x), \ldots, \sigma_n^2(x)\}$$

Corresponding to this process, or to μ and σ^2, we have that for any smooth function f :

$\mathbb{R}^n \times [0,T] \to \mathbb{R}$ we define the operators $\mathcal{L}(f)$ and $\mathcal{L}^*(f)$ for all $x \in \mathbb{R}^n$ as follows:

$$\mathcal{L}(f)(x,t) \equiv \sum_{i=1}^{n} \mu_i(x) \frac{\partial}{\partial x_i} f(x,t) + \frac{1}{2} \sum_{i=1}^{n} \sigma_i^2(x) \frac{\partial^2}{\partial x_i \partial x_i} f(x,t)$$
$$\mathcal{L}^*(f)(x,t) \equiv -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(f(x,t)\mu_i(x)\right) + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i \partial x_i} \left(f(x,t)\sigma_i^2(x)\right)$$

We also adapt the notation of $\mathcal{P}$, the set of densities, i.e. $\mathcal{P} = \left\{ f : \mathbb{R}^n \to \mathbb{R}_+ \text{ with } \int f(x)dx = 1 \right\}$. Likewise we adapt the conditions on the vector fields for the drift and volatility. In particular, the function $\mu : \mathbb{R}^n \to \mathbb{R}^n$ is once continuously differentiable, and $\sigma^2 : \mathbb{R}^n \to \mathbb{R}_+^n$ is twice continuously differentiable, with

$$\begin{aligned} &\exists\, B \text{ s.t.} : ||\sigma^2||_\infty \leq B,\ ||\mu||_\infty \leq B,\ ||\partial_x \mu||_\infty,\ ||\partial_x \sigma^2||_\infty \leq B \text{ and} \\ &\exists\, \underline{s} > 0 \text{ s.t.} : \underline{s}^2 \leq \sigma_j^2(x) \text{ all } x,\ \text{ all } j = 1, \ldots, n \end{aligned} \tag{42}$$

Next we define a classical regular equilibrium, for a given ϵ as follows:

Definition 3. Fixing $\epsilon > 0$, a classical regular ϵ-MFG, given (u_T, m_0) is given by a triplet $(u, m, \bar{x})$, where $u : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ and $m : \mathbb{R}^n \times [0,T] \to \mathbb{R}_+$ are once continuously differentiable with respect to t and twice continuously differentiable with respect to x, and where $\bar{x} : [0,T] \to \mathbb{R}^n$ is once continuously differentiable on t. We say a classical regular ϵ-MFG must satisfy:

(a) There is a function $M : [0,T] \to \mathbb{R}_+$ for which

$$||u(\cdot,t)||_\infty \leq M(t), ||\partial_x u(\cdot,t)||_2 \leq M(t), ||\partial_{xx} u(\cdot,t)||_2 \leq M(t) \text{ for all } t \in [0,T]. \tag{43}$$

(b) The density $m(\cdot,t) : \mathbb{R}^n \times [0,T]$ is once continuously differentiable in t and twice continuously differentiable in x with a function $M : [0,T] \to \mathbb{R}_+$ for which:

$$||m(\cdot,t)||_2, ||\partial_x m(\cdot,t)||_2 \leq M(t),\ ||\partial_{xx} m(\cdot,t)||_2 \leq M(t) \tag{44}$$

(c) The HBJ for all $(x,t) \in \mathbb{R}^n \times [0,T]$:

$$\begin{aligned}\rho u(x,t) &= F(x, m(t)) + \mathcal{L}(u)(x,t) + u_t(x,t) \\ &\quad + H\left(u(x,t) - \int u(z,t)\nu_\epsilon(z-\bar{x}(t))dz, x\right)\end{aligned} \tag{45}$$

(d) The KFE for all $(x,t) \in \mathbb{R}^n \times [0,T]$:

$$\begin{aligned}m_t(x,t) &= H_v\left(u(x,t) - \int u(z,t)\nu_\epsilon(z-\bar{x}(t))dz, x\right) m(x,t) + \mathcal{L}^*(m)(x,t) \\ &\quad - \left[\int H_v\left(u(x',t) - \int u(z,t)\nu_\epsilon(z-\bar{x}(t))dz, x'\right) m(x',t)dx'\right] \nu_\epsilon(x-\bar{x}(t))\end{aligned} \tag{46}$$

(e) The Optimal return for all $t \in [0,T] : \bar{x}(t) = \arg\min_x u(x,t)$

(f) The Terminal and Initial condition for all $x \in \mathbb{R}^n$:

$$u(x,T) = u_T(x, m(T)) \text{ and } m(x,0) = m_0(x) \tag{47}$$

Few comments on the Definition 3 are in order:

1. It is convenient to adapt our definition of v to the case of noisy control as follows:

$$v(x,t) = u(x,t) - \int u(z,t)\nu_\epsilon(z-\bar{x}(t))dz \tag{48}$$

2. The term $-H_v\,(v(x,t),x) \geq 0$ gives the probability of an adjustment at (x,t).

3. The term $-\left[\int H_v\,(v(x',t),x')\,m(x',t)dx'\right]\nu_\epsilon(x-\bar{x}(t))$ in the KFE is the product of the fraction of the values which adjust, given by $-\int H_v(v(x',t),x')m(x')dx'$, times the density of those that adjust to the value x, given by $\nu_\epsilon(x-\bar{x}(t))$. This is the probability flow that "enters" at x at time t.

4. The KFE holds for all x, including $x = \bar{x}(t)$. This is because the function ν_ϵ regularizes the problem.

5. As $\epsilon \downarrow 0$, then ν_ϵ becomes a delta function, and then the KFE does not hold at $(x,t) = (\bar{x}(t), t)$.

6. If $\epsilon = 0$ and $n = 1$ we have exactly the baseline case. If $\epsilon = 0$ and $n > 1$ we have the extension of the baseline case to the case where the decision maker controls the entire $n-$ dimensional state.

7. As in the baseline case, $\mathcal{L}$ gives the ($n-$dimensional version of the) linear operator describing the effect on the expected change on u of the drift $\mu(x)$ and volatility $\sigma^2(x)$.

8. As in the baseline case, $\mathcal{L}^*$ gives the ($n-$dimensional version of the) linear operator describing the propagation of the density m due to effect of the drift $\mu(x)$ and volatility $\sigma^2(x)$

We redefine the constants $\bar{H}_v$ and $\bar{H}_{vv}$ analogously:

$$H_v(z,x) \leq \bar{H}_v \leq 0 \text{ for all } z \in \mathbb{R} \text{ and all } x \in \mathbb{R}^n \tag{49}$$

$$H_{vv}(z,x) \leq \bar{H}_{vv} \leq 0 \text{ for all } z \in \mathbb{R} \text{ and all } x \in \mathbb{R}^n \tag{50}$$

and redefine monotonicity in the natural way:

$$\int \big(F(x,m^a) - F(x,m^b)\big)\big(m^a(dx) - m^b(dx)\big) > 0 \text{ if } m^a \neq m^b \in \mathcal{P} \tag{51}$$

$$\int \big(u_T(x,m^a) - u_T(x,m^b)\big)\big(m^a(dx) - m^b(dx)\big) > 0 \text{ if } m^a \neq m^b \in \mathcal{P} \tag{52}$$

where $\mathcal{P} = \big\{ f : \mathbb{R}^n \to \mathbb{R}_+ \text{ with } \int f(x)dx = 1 \big\}$.

We show that a classical ϵ-MFG equilibrium is unique under the analogous assumptions as in the baseline case. We don't write the entire Theorem. Instead, only prove the key proposition for the uniqueness result, establishing the Lasry-Lions type of inequality. In particular we have:

PROPOSITION 3. Assume that $\epsilon > 0$, that F satisfy the monotonicity conditions given by 51, and that H satisfy the conditions given by equation (49) and equation (50). Furthermore, assume that μ and σ^2 are once and twice continuously differentiable in $x \in \mathbb{R}^n$. Suppose that $\{u^a, m^a, \bar{x}^a, \}$ and $\{u^a, m^b, \bar{x}^b\}$ are two classical equilibria of the ϵ-MFG as stated in Definition 3. Let v^a, v^b be defined as in equation (48). Let

$$K(t) \equiv \int \left(u^a(x,t) - u^b(x,t)\right)\left(m^a(x,t) - m^b(x,t)\right) dx \text{ for all } t \in [0,T] \text{ then} \tag{53}$$

$$\frac{d}{dt}K(t) - \rho K(t) \leq \bar{S}(t) \text{ for all } t \in [0,T] \text{ where}$$

$$\bar{S}(t) \equiv \bar{H}_{vv} \int \left[m^a(x,t) + m^b(x,t)\right]\left(v^a(x,t) - v^b(x,t)\right)^2 dx \text{ all } t \in (0,T) \tag{54}$$

Given Lemma 3, the proof of uniqueness of the classical regular equilibria of the ϵ-MFG follows similar steps as in the baseline one dimensional case with $\epsilon = 0$. Comparing the expression for $\bar{S}$ in this case, with the previous case, we note that the effect of $\epsilon > 0$ is to have a differentiable $m(\cdot, t)$, and that the first term $\bar{S}$ no longer applies.

References

Achdou, Yves, Jiequn Han, Jean-Michel Lasry, Pierre-Louis Lions, and Benjamin Moll. 2021. "Income and Wealth Distribution in Macroeconomics: A Continuous-Time Approach." *The Review of Economic Studies* 89 (1):45–86.

Aiyagari, S. Rao. 1994. "Uninsured Idiosyncratic Risk and Aggregate Saving." *The Quarterly Journal of Economics* 109 (3):659–684.

Alvarez, Fernando, Francesco Lippi, and Panagiotis Souganidis. 2023. "Price Setting with Strategic Complementarities as a Mean Field Game." Working paper.

Alvarez, Fernando E. and Francesco Lippi. 2014. "Price setting with menu costs for multi product firms." *Econometrica* 82 (1):89–135.

Alvarez, Fernando E., Francesco Lippi, and Aleksei Oskolkov. 2021. "The Macroeconomics of Sticky Prices with Generalized Hazard Functions." *Quarterly Journal of Economics* forthcoming.

Bertucci, Charles. 2020. "Fokker-Planck equations of jumping particles and mean field games of impulse control." *Annales de l'Institut Henri Poincaré C, Analyse non linéaire* 37 (5):1211–1244.

Bhattarai, Saroj and Raphael Schoenle. 2014. "Multiproduct Firms and Price-Setting: Theory and Evidence from U.S. Producer Prices." *Journal of Monetary Economics* 66:178–192.

Caballero, Ricardo J. and Eduardo M. R. A. Engel. 1993a. "Microeconomic Adjustment Hazards and Aggregate Dynamics." *The Quarterly Journal of Economics* 108 (2):359–383.

———. 1999. "Explaining Investment Dynamics in U.S. Manufacturing: A Generalized (S,s) Approach." *Econometrica* 67 (4):783–826.

Caballero, Ricardo J. and Eduardo M.R.A. Engel. 1991. "Dynamic (S, s) Economies." *Econometrica* 59 (6):1659–86.

———. 1993b. "Heterogeneity and output fluctuations in a dynamic menu-cost economy." *The Review of Economic Studies* 60 (1):95.

———. 1993c. "Microeconomic rigidities and aggregate price dynamics." *European Economic Review* 37 (4):697–711.

———. 2007. "Price stickiness in Ss models: New interpretations of old results." *Journal of Monetary Economics* 54 (Supplement):100–121.

Costain, James and Anton Nakov. 2011. "Price Adjustments in a General Model of State–Dependent Pricing." *Journal of Money, Credit and Banking* 43:385–406.

Dupuis, Paul and Hui Wang. 2002. "Optimal Stopping with Random Intervention Times." *Advances in Applied Probability* 34 (1):141–157.

Lasry, Jean-Michel and Pierre-Louis Lions. 2007. “Mean field games.” *Japanese Journal Mathematics* 1:229–260.

Menaldi, J. L. and M. Robin. 2016. “On Some Optimal Stopping Problems with Constraint.” *SIAM Journal on Control and Optimization* 54 (5):2650–2671.

Midrigan, Virgiliu. 2011. “Menu Costs, Multiproduct Firms, and Aggregate Fluctuations.” *Econometrica* 79 (4):1139–1180.

Wang, Hui. 2001. “Some Control Problems with Random Intervention Times.” *Advances in Applied Probability* 33 (2):404–422.

Woodford, Michael. 2009. “Information-constrained state-dependent pricing.” *Journal of Monetary Economics* 56 (S):100–124.

7 Proofs

Proof. (of Lemma 1) Differentiating equation (6) with respect to time we obtain and replacing 5 :

$$
\begin{aligned}
0 &= \int \partial_t m(x,t)dx \\
&= -\int_{-\infty}^{\infty} \partial_x \left(\mu(x)m(x,t)\right)dx + \int_{-\infty}^{\infty} \tfrac{1}{2}\partial_{xx}\left(\sigma^2(x)m(x,t)\right)dx + \int_{-\infty}^{\infty} H_v(v(x,t),x)m(x,t)dx \\
&= -\mu(x)m(x,t)|_{-\infty}^{\infty} + \tfrac{1}{2}\partial_x\left(\sigma^2(x)m(x,t)\right)|_{\bar{x}_+(t)}^{\infty} + \tfrac{1}{2}\partial_x\left(\sigma^2(x)m(x,t)\right)|_{-\infty}^{\bar{x}_-(t)} \\
&+ \int_{-\infty}^{\infty} H_v(v(x,t),x)m(x,t)dx
\end{aligned}
$$

Using that $\mu(x)$ is bounded and $m(x,t) \to 0$ as $|x| \to \infty$, and that σ^2, σ_x^2, and that $m_x(x,t) \to 0$ as $|x| \to \infty$ then

$$0 = \tfrac{1}{2}\partial_x\left(\sigma^2(x)m(x,t)\right)|_{\bar{x}_+(t)}^{\bar{x}_-(t)} + \int_{-\infty}^{\infty} H_v(v(x,t),x)m(x,t)dx$$

Since $m(\cdot,t), \sigma_x^2(\cdot,t)$ and $\sigma^2(\cdot,t)$ are continuous at $x = \bar{x}(t)$ then

$$\partial_x\left(\sigma^2(x)m(x,t)\right)|_{\bar{x}_+(t)}^{\bar{x}_-(t)} = \sigma^2(\bar{x}(t))\partial_x\left(\partial_x m(x,t)\right)|_{\bar{x}_+(t)}^{\bar{x}_-(t)}$$

obtaining the desired result. □

Proof. (of Proposition 1) By definition of a classical regular equilibrium of the MFG, we have that $||u^i(\cdot,t)|| \leq M(t)$ for $i = a, b$ so that

$$K(t) = \int_{-\infty}^{\infty} \left(u^a(x,t) - u^b(x,t)\right)\left(m^a(x,t) - m^b(x,t)\right)dx$$

is well defined. Since, by definition, $v(x,t) - u(x,t) = -u(\bar{x}(t),t)$ does not depend on x, then

$$\partial_x u(x,t) = \partial_x v(x,t), \text{ and } \partial_{xx}u(x,t) = \partial_{xx}v(x,t)$$

Using that $\int_{-\infty}^{\infty}\left(m^a(x,t) - m^b(x,t)\right)dx = 0$, then:

$$K(t) = \int_{-\infty}^{\infty} \left(v^a(x,t) - v^b(x,t)\right)\left(m^a(x,t) - m^b(x,t)\right)dx$$

We differentiate with K with respect to time to obtain:

$$\begin{aligned}\frac{d}{dt}K(t) &= \frac{d}{dt}\int_{-\infty}^{\infty}\left(v^a(x,t)-v^b(x,t)\right)\left(m^a(x,t)-m^b(x,t)\right)dx\\ &= \int_{-\infty}^{\infty}\left(v^a(x,t)-v^b(x,t)\right)\partial_t\left(m^a(x,t)-m^b(x,t)\right)dx\\ &+ \int_{-\infty}^{\infty}\left(m^a(x,t)-m^b(x,t)\right)\partial_t\left(v^a(x,t)-v^b(x,t)\right)dx\end{aligned}$$

Below we check on sufficient conditions on the different integrals o interchange integrals and time derivatives. Using the properties of $v^i(x,t)-u^i(x,t)=-u^i(\bar{x}^i(t),t)$ for $i=a,b$, and that $\int(m^a-m^b)dx$ and its time derivative are zero:

$$\begin{aligned}\frac{d}{dt}K(t) &= \frac{d}{dt}\int_{-\infty}^{\infty}\left(u^a(x,t)-u^b(x,t)\right)\left(m^a(x,t)-m^b(x,t)\right)dx\\ &= \int_{-\infty}^{\infty}\left(v^a(x,t)-v^b(x,t)\right)\partial_t\left(m^a(x,t)-m^b(x,t)\right)dx\\ &+ \int_{-\infty}^{\infty}\left(m^a(x,t)-m^b(x,t)\right)\partial_t\left(u^a(x,t)-u^b(x,t)\right)dx\end{aligned}$$

Using the p.d.e for the HJB in equation (1), using that $\partial_x v=\partial_x u$ and $\partial_{xx}v=\partial_{xx}u$, and the p.d.e for the KBF in equation (5) to replace in the previous integrals as follows:

$$\begin{aligned}&\frac{d}{dt}\int_{-\infty}^{\infty}\left(u^a(x,t)-u^b(x,t)\right)\left(m^a(x,t)-m^b(x,t)\right)dx\\ &= \int_{-\infty}^{\infty}\left(v^a-v^b\right)\left(\partial_{xx}(\tfrac{\sigma^2}{2}m^a-\tfrac{\sigma^2}{2}m^b)\right)-\left(m^a-m^b\right)\tfrac{\sigma^2}{2}\left(\partial_{xx}(v^a-v^b)\right)dx\\ &+ \int_{-\infty}^{\infty}\left(v^a-v^b\right)\left(-\partial_x(\mu\, m^a-\mu\, m^b)\right)-\left(m^a-m^b\right)\mu\left(\partial_x(v^a-v^b)\right)dx\\ &+ \rho\int_{-\infty}^{\infty}\left(u^a-u^b\right)\left(m^a-m^b\right)dx\\ &+ \int_{-\infty}^{\infty}\left(v^a-v^b\right)\left(m^aH_v(v^a)-m^bH_v(v^b)\right)dx-\int_{-\infty}^{\infty}\left(m^a-m^b\right)\left(H(v^a)-H(v^b)\right)dx\\ &- \int_{-\infty}^{\infty}\left(m^a-m^b\right)\left(F(x,m^a)-F(x,m^b)\right)dx\end{aligned}$$

where we omit the arguments (x,t) or x from the different functions to simplify the notation. Thus we write:

$$\begin{aligned}\frac{d}{dt}K(t) = \frac{d}{dt}\int_{-\infty}^{\infty}\left(u^a-u^b\right)\left(m^a-m^b\right)dx &= \rho\int_{-\infty}^{\infty}\left(u^a-u^b\right)\left(m^a-m^b\right)dx\\ &+ I_V(t)+I_D(t)+I_H(t)+I_F(t)\end{aligned}$$

where

$$
\begin{aligned}
I_V(t) &\equiv \int_{-\infty}^{\infty} \left(v^a - v^b\right)\left(\partial_{xx}(\tfrac{\sigma^2}{2}m^a - \tfrac{\sigma^2}{2}m^b)\right) - \left(m^a - m^b\right)\tfrac{\sigma^2}{2}\left(\partial_{xx}(v^a - v^b)\right)dx \\
I_D(t) &\equiv \int_{-\infty}^{\infty} \left(v^a - v^b\right)\left(-\partial_x(\mu\, m^a - \mu\, m^b)\right) - \left(m^a - m^b\right)\mu\left(\partial_x(v^a - v^b)\right)dx \\
I_H(t) &\equiv \int_{-\infty}^{\infty} \left(v^a - v^b\right)\left(m^a H_v(v^a) - m^b H_v(v^b)\right)dx - \int_{-\infty}^{\infty}\left(m^a - m^b\right)\left(H(v^a) - H(v^b)\right)dx \\
I_F(t) &\equiv -\int_{-\infty}^{\infty}\left(m^a - m^b\right)\left(F(m^a) - F(m^b)\right)dx
\end{aligned}
$$

Note that the integrals in $I_V(t), I_D(t)$, and $I_H(t)$ are all well defined given the integrability assumptions of a classical regular equilibrium, as well as the integrability assumptions on μ and σ^2. Next, we obtain an inequality for each term, i.e. for $I_V(t), I_D(t), I_H(t)$ and $I_F(t)$.

1. We will show that

$$
I_V(t) \leq \left[v^b(\bar{x}^a(t), t) + v^a(\bar{x}^b(t), t)\right]\bar{H}_v
$$

To simplify the notation we let $\hat{v} \equiv v^a - v^b$ and $\hat{m} \equiv m^a - m^b$. With this notation we have

$$
\begin{aligned}
I_V(t)|_L^U &\equiv \int_L^U \left[\left(v^a - v^b\right)\left(\partial_{xx}(\tfrac{\sigma^2}{2}m^a - \tfrac{\sigma^2}{2}m^b)\right) - \left(m^a - m^b\right)\tfrac{\sigma^2}{2}\left(\partial_{xx}(v^a - v^b)\right)\right]dx \\
&= \int_L^U \left[\hat{v}\,\partial_{xx}\left(\tfrac{\sigma^2}{2}\hat{m}\right) - \hat{m}\,\tfrac{\sigma^2}{2}\partial_{xx}\hat{v}\right]dx
\end{aligned}
$$

In an interval $x \in [L, U]$ where $\hat{m}_x$ is twice continuously differentiable we have:

$$
\begin{aligned}
I_V(t)|_L^U &= \int_L^U \left[\hat{v}\,\partial_{xx}\left(\tfrac{\sigma^2}{2}\hat{m}\right) - \hat{m}\,\tfrac{\sigma^2}{2}\partial_{xx}\hat{v}\right]dx \\
&= \int_L^U \hat{v}\,\partial_{xx}\left(\tfrac{\sigma^2}{2}\hat{m}\right)dx + \int_L^U \partial_x\left(\hat{m}\tfrac{\sigma^2}{2}\right)\partial_x\hat{v}dx - \hat{m}\tfrac{\sigma^2}{2}\,\partial_x\hat{v}|_L^U \\
&= \hat{v}\,\partial_x\left(\hat{m}\tfrac{\sigma^2}{2}\right)|_L^U - \hat{m}\tfrac{\sigma^2}{2}\,\partial_x\hat{v}|_L^U
\end{aligned}
$$

where we integrate by parts each term. Assume, without loss of generality that $-\infty < \bar{x}^a(t) < \bar{x}^b(t) < \infty$, so we can write:

$$
I_V(t)|_{-\infty}^{\infty} = I_V(t)|_{-\infty}^{\bar{x}^a} + I_V(t)|_{\bar{x}^a}^{\bar{x}^b} + I_V(t)|_{\bar{x}^b}^{\infty}
$$

We use that, given the assumption on integrability for a classical regular equilibrium, then $|m^i| \to 0$ and $|\partial_x m^i| \to 0$, that $\partial_x v^i \to 0$ as $|x| \to \infty$, and v^i is bounded for $i = a, b$ to obtain:

$$
0 = \lim_{|x|\to\infty} \hat{v}\,\partial_x\left(\hat{m}\tfrac{\sigma^2}{2}\right)(x) - \lim_{|x|\to\infty} \hat{m}\tfrac{\sigma^2}{2}\,\partial_x\hat{v}(x)
$$

Thus can write:

$$I_V(t)|_{-\infty}^{\infty} = I_V(t)|_{\bar{x}_+^a}^{\bar{x}_-^a} + I_V(t)|_{\bar{x}_+^b}^{\bar{x}_-^b}$$

Let concentrate on the first term, $I_V(t)|_{\bar{x}_+^a}^{\bar{x}_-^a}$. We use that $v^a(\bar{x}^a(t), t) = 0$ and $\partial_x v^a(x,t)|_{x=x^a} = 0$ to obtain:

$$\begin{aligned} I_V(t)|_{\bar{x}_+^a}^{\bar{x}_-^a} &= \hat{v}\,\partial_x\left(\hat{m}\tfrac{\sigma^2}{2}\right)|_{\bar{x}_+^a}^{\bar{x}_-^a} - \hat{m}\tfrac{\sigma^2}{2}\,\partial_x\hat{v}|_{\bar{x}_+^a}^{\bar{x}_-^a} \\ &= \hat{v}\,\partial_x\left(\hat{m}\tfrac{\sigma^2}{2}\right)|_{\bar{x}_+^a}^{\bar{x}_-^a} \end{aligned}$$

where the second line uses that m^i and ∂v^i are continuous on x, and hence $0 = \hat{m}\frac{\sigma^2}{2}\,\partial_x\hat{v}|_{\bar{x}_+^a}^{\bar{x}_-^a}$. Then

$$I_V(t)|_{\bar{x}_+^a}^{\bar{x}_-^a} = \hat{v}\,\partial_x\left(\hat{m}\tfrac{\sigma^2}{2}\right)|_{\bar{x}_+^a}^{\bar{x}_-^a} = -v^b\,\tfrac{\sigma^2}{2}\,\partial_x m^a|_{\bar{x}_+^a}^{\bar{x}_-^a} < 0$$

where we use that σ^2 is continuously differentiable, and that $\hat{m} = m^a - m^b$, and that $\partial_x m^b$ is continuous at $x = \bar{x}^a$. Using Lemma 1 we obtain:

$$I_V(t)|_{\bar{x}_+^a}^{\bar{x}_-^a} = v^b(\bar{x}^a(t), t)\int m^a(x,t)H_v(v^a(x,t),x)dx \leq v^b(\bar{x}^a(t),t)\bar{H}_v$$

The argument for $I_V(t)|_{\bar{x}_+^b}^{\bar{x}_-^b}$ is identical, giving:

$$I_V(t)|_{\bar{x}_+^b}^{\bar{x}_-^b} = v^a(\bar{x}^b(t), t)\int m^b(x,t)H_v(v^b(x,t),x)dx \leq v^a(\bar{x}^b(t),t)\bar{H}_v$$

Combining the two results we obtain the desired inequality.

2. We will show that $I_D(t) = 0$. This follows by using integration by parts, since $\hat{v}(\cdot, t)$ is continuously differentiable. The boundaries at $x = \infty$ and $x = -\infty$, the terms vanish given the assumption on the tails. In particular:

$$\begin{aligned} I_D(t) &= \int_{-\infty}^{\infty} \left(v^a - v^b\right)\left(-\partial_x(\mu\, m^a - \mu\, m^b)\right) - \left(m^a - m^b\right)\mu\left(\partial_x(v^a - v^b)\right)dx \\ &= -\left(v^a - v^b\right)\left(\mu\, m^a - \mu\, m^b\right)|_{-\infty}^{\infty} \end{aligned}$$

using that $v^i(\cdot, t)$, and μ are bounded, and that $m^i \to 0$ as $|x| \to \infty$. Thus $I_D(t) = 0$.

3. We will show that

$$I_H(t) \leq \bar{H}_{vv}\int_{-\infty}^{\infty}\left[m^a(x,t) + m^b(x,t)\right]\left(v^a(x,t) - v^b(x,t)\right)^2 dx \leq 0$$

We use the concavity of H and the definition of $I_H(t)$. In particular, we use a first order expansion:

$$H(y_1, x) = H(y_2, x) + H_v(y_2, x)\,(y_1 - y_2) + H_{vv}(\tilde{y}, x)\,(y_1 - y_2)^2$$

for some $\tilde{y} \in [y_1, y_2]$. Letting $y_1 = v^b(x,t)$ and $y_2 = v^a(x,t)$, or reversing it with $y_1 = v^a(x,t)$ and $y_2 = v^b(x,t)$, and omitting arguments:

$$\begin{aligned} H(v^b) - H(v^a) &= H_v(v^a)\left(v^b - v^a\right) + H_{vv}(\tilde{v}^{ba}, x)\left(v^a - v^b\right)^2 \\ H(v^a) - H(v^b) &= H_v(v^b)\left(v^a - v^b\right) + H_{vv}(\tilde{v}^{ba}, x)\left(v^a - v^b\right)^2 \end{aligned}$$

Thus:

$$\begin{aligned} &\left(v^a - v^b\right)\left(m^a H_v(v^a) - m^b H_v(v^b)\right) - \left(m^a - m^b\right)\left(H(v^a) - H(v^b)\right) \\ &= \left(v^a - v^b\right)\left(m^a H_v(v^a) - m^b H_v(v^b)\right) + m^a\left(H(v^b) - H(v^a)\right) + m^b\left(H(v^a) - H(v^b)\right) \\ &= \left[m^a H_{vv}(\tilde{v}^{ba}, x) + m^b H_{vv}(\tilde{v}^{ba}, x)\right]\left(v^a - v^b\right)^2 \\ &\leq \left[m^a + m^b\right]\left(v^a - v^b\right)^2 \bar{H}_{vv} \end{aligned}$$

Integrating across x:

$$\begin{aligned} I_H(t) &\equiv \int \left(v^a - v^b\right)\left(m^a H_v(v^a) - m^b H_v(v^b)\right) - \left(m^a - m^b\right)\left(H(v^a) - H(v^b)\right) dx \\ &\leq \int_{-\infty}^{\infty} \left[m^a + m^b\right]\left(v^a - v^b\right)^2 \bar{H}_{vv}\, dx \end{aligned}$$

4. We will show that $I_F(t) \leq 0$. This follows directly by Assumption (18) on monotonicity for F.

Combining the bounds obtained by $I_V(t)$, $I_D(t)$, $I_H(t)$ and $I_F(t)$ we get the desired result. □

Proof. (of Lemma 2) Proposition 1 implies that

$$K(t) = e^{\rho t} K(0) + \int_0^t e^{\rho(t-\tau)} S(\tau) d\tau \leq e^{\rho t} K(0) + \int_0^t e^{\rho(t-\tau)} \bar{S}(\tau) d\tau \quad \text{all} \quad t \in (0, T)$$

Now we use that, since m^a and m^b are part of the equilibrium, they must satisfy the initial condition $m^a(x,0) = m^b(x,0) = m_0(x)$, so that $K(0) = 0$. Thus $K(t) = \int_0^t e^{\rho(t-\tau)} S(\tau) d\tau$ all $t \in [0,T]$. Note that, given that u^a and u^b, by definition of a MFG, should be equal to the terminal condition u_T then

$$\begin{aligned} K(T) &= \int_{-\infty}^{\infty} \left(u^a(x,T) - u^b(x,T)\right)\left(m^a(x,T) - m^b(x,T)\right) dx \\ &= \int_{-\infty}^{\infty} \left(u_T(x, m^a(T)) - u_T(x, m^b(T))\right)\left(m^a(x,T) - m^b(x,T)\right) dx \geq 0 \end{aligned}$$

where the inequality follows by the monotonicity condition 19 on u_T. Thus

$$0 \leq K(T) = \int_0^T e^{\rho(T-\tau)} S(\tau) d\tau \leq \int_0^T e^{\rho(T-\tau)} \bar{S}(\tau) d\tau \leq 0$$

where we use that $\bar{S}(t) \leq 0$ since $\bar{H}_v \geq 0, \bar{H}_{vv} \geq 0$ and $v^a \geq 0$ and $v^b \geq 0$. Hence, if $S(\tau) \geq 0$ for all $\tau \in [0,T]$, then it must be that $S(\tau) = 0$ for all $\tau \in [0,T]$. In this case $K(t) = 0$ for all $t \in [0,T]$ too.

□

Proof. (of Lemma 3) Assume, by contradiction that $\bar{x}^a \neq \bar{x}^b$. Given the assumed continuity, then without loss of generality we can assume that $\bar{x}^a(t) < \bar{x}^b(t)$ for $t \in (t_0, t_1)$. Since $S(t) = 0$ for $t \in [t_0, t_1]$ and $\bar{H}_v < 0$ then $v^a(\bar{x}^b(t), t) = v^b(\bar{x}^a(t), t) = 0$ for $t \in [t_0, t_1]$.

This implies that $u^i(x,t) = u^i(\bar{x}(t), t)$ for all for $x \in (\bar{x}^a(t), \bar{x}^b(t))$ and $t \in (t_1, t_0)$, and $i = a, b$. Hence, for each $t \in (t_1, t_0)$, then $\partial_x u^i(x,t) = \partial_{xx} u^i(x,t) = \partial_{xt} u^i(x,t) = 0$ for all $x \in (\bar{x}^a(t), \bar{x}^b(t))$. Replacing this into the p.d.e. for u^i we have:

$$\rho u^i(\bar{x}(t), t) = \rho u^i(x,t) = F(x, m(t)) + H(0, x) + \partial_t u^i(x,t) = F(x, m(t)) + \partial_t u^i(x,t)$$

Differentiating again with respect to x we get:

$$0 = \partial_x F(x, m(t)) + \partial_{xt} u^i(x,t) = \partial_x F(x, m(t))$$

which is a contradiction with the assumption that $F(\cdot, m)$ is strictly single peaked. Hence $\bar{x}^a = \bar{x}^b$.

□

Proof. (of Theorem 1) Under the stated assumptions we can verify the conditions for Proposition 1 and Lemma 2 and hence $\bar{S}(t) = 0$. Furthermore, we assume that $\bar{H}_{vv} < 0$, and hence

$$0 = \int_{-\infty}^{\infty} \left(m^a(x,t) + m^b(x,t)\right) \left(v^a(x,t) - v^b(x,t)\right)^2 dx$$

implies $v^a(x,t) = v^b(x,t)$ for all x, since m^a and m^b are supported on the entire real line. Thus, using the definition of v^a, v^b and the p.d.e. for u^a, u^b we have:

$$\begin{aligned}
\rho v^a(x,t) &\equiv \rho u^a(x,t) - \rho u^a(\bar{x}^a(t), t) \\
&= F(x, m^a(t)) - \rho u^a(\bar{x}^a(t), t) + \mathcal{L}(v^a)(x,t) + H(v^a(x,t), x) - \partial_t u^a(x,t) \\
\rho v^b(x,t) &\equiv \rho u^b(x,t) - \rho u^b(\bar{x}^b(t), t) \\
&= F(x, m^b(t)) - \rho u^b(\bar{x}^b(t), t) + \mathcal{L}(v^b)(x,t) + H(v^b(x,t), x) - \partial_t u^b(x,t)
\end{aligned}$$

for all x, t. But since $v^a(x,t) = v^b(x,t)$ we have:

$$F(x, m^a(t)) - \rho u^a(\bar{x}^a(t), t) + \partial_t u^a(x,t) = F(x, m^b(t)) - \rho u^b(\bar{x}^b(t), t) + \partial_t u^b(x,t) \text{ for all } x, t$$

Differentiating $v^i(x,t) = u^i(x,t) - u^i(\bar{x}^i(t), t)$ with respect to time and using $\partial_x u^i(\bar{x}^i(t), t) = 0$

for $i = a, b$ we get:

$$\partial_t v^i(x,t) = \partial_t u^i(x,t) - \partial_t u^i(\bar{x}^i(t),t)$$

Then replacing this into the previous equality

$$\begin{aligned} & F(x, m^a(t)) - \rho u^a(\bar{x}^a(t),t) + \partial_t v^a(x,t) + \partial_t u^a(\bar{x}^a(t),t) \\ =& F(x, m^b(t)) - \rho u^b(\bar{x}^b(t),t) + \partial_t v^b(x,t) + \partial_t u^b(\bar{x}^a(t),t) \text{ for all } x, t \end{aligned}$$

Differentiating $v^a(x,t) = v^b(x,t)$ with respect to time we have $\partial_t v^a(x,t) = \partial_t v^b(x,t)$, so we can write:

$$\begin{aligned} & F(x, m^a(t)) - \rho u^a(\bar{x}^a(t),t) + \partial_t u^a(\bar{x}^a(t),t) \\ =& F(x, m^b(t)) - \rho u^b(\bar{x}^b(t),t) + \partial_t u^b(\bar{x}^a(t),t) \text{ for all } x, t \end{aligned}$$

Defining $g^i(t) \equiv -\rho u^a(\bar{x}^i(t),t) + \partial_t u^i(\bar{x}^i(t),t)$ for $i = a, b$ and all t, we can write:

$$F(x, m^a(t)) + g^a(t) = F(x, m^b(t)) + g^b(t) \text{ for all } x, t$$

Multiplying by $m^a(x,t) - m^b(x,t)$ and integrating with respect to x we get:

$$\begin{aligned} & \int \left[F(x, m^a(t)) - F(x, m^b(t))\right] \left(m^a(x,t) - m^b(x,t)\right) dx \\ =& \int \left(g^b(t) - g^a(t)\right) \left(m^a(x,t) - m^b(x,t)\right) dx \\ =& \left(g^b(t) - g^a(t)\right) \int \left(m^a(x,t) - m^b(x,t)\right) dx = 0 \end{aligned}$$

which, if $\int (m^a(x,t) - m^b(x,t))^2 dx > 0$, gives a contradiction with the (strong) monotonicity of F as stated in condition 18. Thus $m^a = m^b$.

Once $m^a = m^b$, then it must be the case that not only $v^a = v^b$, but also $u^a = u^b$.

Finally, Lemma 3 implies that $\bar{x}^a = \bar{x}^b$, and hence the equilibrium is unique.

□

Proof. (of Lemma 4) Since $\int_{-\infty}^{\infty} m(x,z,t)dx = n(z)$ with does not depend on time:

$$
\begin{aligned}
0 &= \int_{-\infty}^{\infty} \partial_t m(x,z,t)dx \\
&= -\int_{-\infty}^{\infty} \partial_x(\mu_x(x,z,t)m(x,z,t))dx - \int_{-\infty}^{\infty} \partial_z(\mu_z(z)n(z)m(x,t|z))dx \\
&+ \int_{-\infty}^{\infty} \partial_{xx}(\tfrac{1}{2}\sigma_x^2(x,z,t)m(x,z,t))dx + \int_{-\infty}^{\infty} \partial_{zz}(\tfrac{1}{2}\sigma_z^2(z,t)n(z)m(x,t|z))dx \\
&+ \int_{-\infty}^{\infty} H_v(v(x,z,t),x,z)m(x,z,t)dx \\
&= -\int_{-\infty}^{\infty} \partial_x(\mu_x(x,z,t)m(x,z,t))dx - \partial_z(\mu_z(z)n(z)) \int_{-\infty}^{\infty} m(x,t|z)dx \\
&- \mu_z(z)n(z) \int_{-\infty}^{\infty} \partial_z(m(x,t|z))dx \\
&+ \int_{-\infty}^{\infty} \partial_{xx}(\tfrac{1}{2}\sigma_x^2(x,z,t)m(x,z,t))dx + \partial_{zz}(\tfrac{1}{2}\sigma_z^2(z,t)n(z)) \int_{-\infty}^{\infty} m(x,t|z)dx \\
&+ 2\partial_z(\tfrac{1}{2}\sigma_z^2(z,t)n(z)) \int_{-\infty}^{\infty} \partial_z m(x,t|z)dx + \tfrac{1}{2}\sigma_z^2(z,t)n(z) \int_{-\infty}^{\infty} \partial_{zz} m(x,t|z)dx \\
&+ \int_{-\infty}^{\infty} H_v(v(x,z,t),x,z)m(x,z,t)dx
\end{aligned}
$$

Using the p.d.e for n:

$$
\begin{aligned}
0 = &-\int_{-\infty}^{\infty} \partial_x(\mu_x(x,z,t)m(x,z,t))dx \\
&- \mu_z(z)n(z) \int_{-\infty}^{\infty} \partial_z m(x,t|z)dx + \int_{-\infty}^{\infty} \partial_{xx}(\tfrac{1}{2}\sigma_x^2(x,z,t)m(x,z,t))dx \\
&+ 2\partial_z(\tfrac{1}{2}\sigma_z^2(z,t)n(z)) \int_{-\infty}^{\infty} \partial_z m(x,t|z)dx + \tfrac{1}{2}\sigma_z^2(z,t)n(z) \int_{-\infty}^{\infty} \partial_{zz} m(x,t|z)dx \\
&+ \int_{-\infty}^{\infty} H_v(v(x,z,t),x,z)m(x,z,t)dx
\end{aligned}
$$

Exchanging the integrals with derivatives:

$$
\begin{aligned}
0 = &-\int_{-\infty}^{\infty} \partial_x(\mu_x(x,z,t)m(x,z,t))dx \\
&- \mu_z(z)n(z)\partial_z \int_{-\infty}^{\infty} m(x,t|z)dx + \int_{-\infty}^{\infty} \partial_{xx}(\tfrac{1}{2}\sigma_x^2(x,z,t)m(x,z,t))dx \\
&+ 2\partial_z(\tfrac{1}{2}\sigma_z^2(z,t)n(z))\partial_z \int_{-\infty}^{\infty} m(x,t|z)dx \\
&+ \tfrac{1}{2}\sigma_z^2(z,t)n(z)\partial_{zz} \int_{-\infty}^{\infty} m(x,t|z)dx + \int_{-\infty}^{\infty} H_v(v(x,z,t),x,z)m(x,z,t)dx
\end{aligned}
$$

But since for all z:

$$1 = \int_{-\infty}^{\infty} m(x,t|z)dx \text{ then } 0 = \partial_z \int_{-\infty}^{\infty} m(x,t|z)dx = \partial_{zz} \int_{-\infty}^{\infty} m(x,t|z)dx$$

then

$$\begin{aligned}0 = &- \int_{-\infty}^{\infty} \partial_x(\mu_x(x,z,t)m(x,z,t))dx + \int_{-\infty}^{\infty} \partial_{xx}(\tfrac{1}{2}\sigma_x^2(x,z,t)m(x,z,t))dx \\ &+ \int_{-\infty}^{\infty} H_v(v(x,z,t),x,z)m(x,z,t)dx\end{aligned}$$

Using the boundedness of μ and integrability of m:

$$\int \partial_x(\mu_x(x,z,t)m(x,z,t))dx = \mu_x(x,z,t)m(x,z,t)|_{x=-\infty}^{x=\infty} = 0$$

Finally, using the boundedness and integrability of m at infinity, we get:

$$\begin{aligned}&\int_{-\infty}^{\infty} \partial_{xx}(\tfrac{1}{2}\sigma_x^2(x,z,t)m(x,z,t))dx = \partial_x(\tfrac{1}{2}\sigma_x^2(x,z,t)m(x,z,t))|_{x=-\infty}^{x=\infty} \\ &+ \partial_x(\tfrac{1}{2}\sigma_x^2(x,z,t)m(x,z,t))|_{x=\bar{x}_+(z,t)}^{x=\bar{x}_-(z,t)} = \partial_x(\tfrac{1}{2}\sigma_x^2(x,z,t)m(x,z,t))|_{x=\bar{x}_+(z,t)}^{x=\bar{x}_-(z,t)} \\ &= \tfrac{1}{2}\sigma_x^2(\bar{x}(z),z,t)\partial_x(m(x,z,t))|_{x=\bar{x}_+(z,t)}^{x=\bar{x}_-(z,t)}\end{aligned}$$

Replacing these expression we obtain the desired result. $\square$

Proof. (of Lemma 5). Let $g(z,t) = u^a(\bar{x}^a(z,t),z),t) - u^b(\bar{x}^b(z,t),z),t)$ then $\int(u^a - u^b)(m^a - m^b)dx = \int(v^a - v^b)(m^a - m^b)dx + \int g(m^a - m^b)dx$. But $\int m^a(x,z,t)dx = \int m^b(x,z,t)dx = n(z)$ so that $\int g(z,t)(m^a(x,z,t) - m^b(x,z,t))dx = g(z,t)[n(z) - n(z)] = 0$.
$\square$

Proof. (of Lemma 6) Start with

$$\begin{aligned}\frac{d}{dt}\bar{K}(t) &= \int \frac{d}{dt}K(t,z)dz = \int\int \frac{d}{dt}(u^a - u^b)(m^a - m^b)dxdz \\ &= \int\int (m^a - m^b)\partial_t(u^a - u^b)dxdz + \int\int (u^a - u^b)\partial_t(m^a - m^b)dxdz\end{aligned}$$

Using p.d.e. obtained from the HJB equation:

$$\begin{aligned}&\int\int (m^a - m^b)\partial_t(u^a - u^b)dxdz \\ &= \int\int (m^a - m^b)\rho(u^a - u^b)dxdz - \int\int (m^a - m^b)(F(m^a) - F(m^b))dxdz \\ &- \int\int (m^a - m^b)(H(v^a) - H(v^b))dxdz - \int\int (m^a - m^b)\mathcal{L}(u^a - u^b)dxdz\end{aligned}$$

and using the p.d.e. obtained from the KF equation:

$$\begin{aligned}
&\int\int (u^a - u^b)\partial_t(m^a - m^b)dxdz \\
&= \int\int (u^a - u^b)\mathcal{L}^*(m^a - m^b)dxdz + \int\int (u^a - u^b)(H_v(v^a)m^a - H_v(v^b)m^b)dxdz
\end{aligned}$$

Use that $[u^a(x,z,t) - u^b(x,z,t)] - [v^a(x,z,t) - v^b(x,z,t)] = g(z,t)$ where the function g does not depend on x, then

$$\begin{aligned}
&\int\int (u^a - u^b)\partial_t(m^a - m^b)dxdz - \int\int (v^a - u^v)\partial_t(m^a - m^b)dxdz \\
&= \int\int g\partial_t(m^a - m^b)dxdz = \int g\left[\int \partial_t(m^a - m^b)dx\right]dz = 0
\end{aligned}$$

since $\int \partial_t m^a dx = \int \partial_t m^b dx = 0$ for any z. Thus:

$$\begin{aligned}
&\int\int (u^a - u^b)\partial_t(m^a - m^b)dxdz = \int\int (v^a - v^b)\partial_t(m^a - m^b)dxdz \\
&= \int\int (v^a - v^b)\mathcal{L}^*(m^a - m^b)dxdz + \int\int (v^a - v^b)(H_v(v^a)m^a - H_v(v^b)m^b)dxdz
\end{aligned}$$

Finally, using the linearity of $\mathcal{L}$ and the definition of g above:

$$\begin{aligned}
&\int\int (m^a - m^b)\mathcal{L}(u^a - u^b)dxdz - \int\int (m^a - m^b)\mathcal{L}(v^a - v^b)dxdz \\
&= \int\int (m^a - m^b)\mathcal{L}(g)dxdz = \int \mathcal{L}(g)\left[\int (m^a - m^b)dx\right]dz = 0
\end{aligned}$$

since $\int m^a dx = \int m^b dx = n$ for any z.

Thus

$$\begin{aligned}
\frac{d}{dt}\bar{K}(t) = &-\int\int (m^a - m^b)(F(m^a) - F(m^b))dxdz - \int\int (m^a - m^b)(H(v^a) - H(v^b))dxdz \\
&+ \rho K(t) + \int\int (v^a - v^b)\mathcal{L}^*(m^a - m^b)dxdz - \int\int (m^a - m^b)\mathcal{L}(v^a - v^b)dxdz \\
&+ \int\int (v^a - v^b)(H_v(v^a)m^a - H_v(v^b)m^b)dxdz
\end{aligned}$$

□

Proof. (of Lemma 7)

Let $\hat{v} = v^a - v^b$ and $\hat{m} = m^a - m^b$. Use the linearity of the operator to get:

$$\begin{aligned}
&\int\int (v^a - v^b)\mathcal{L}^*(m^a - m^b)\, dxdz = \int\int \hat{v}\, \mathcal{L}^*(\hat{m})\, dxdz \\
&= -\int\int \hat{v}\, \partial_x(\mu_x \hat{m})\, dxdz - \int\int \hat{v}\, \partial_z(\mu_z \hat{m})\, dzdx \\
&+ \int\int \hat{v}\, \partial_{xx}(\tfrac{1}{2}\sigma_x^2 \hat{m})\, dxdz + \int\int \hat{v}\, \partial_{zz}(\tfrac{1}{2}\sigma_z^2 \hat{m})\, dzdx
\end{aligned}$$

Likewise:

$$\begin{aligned}
&\int\int (m^a - m^b)\mathcal{L}(v^a - v^b)\, dxdz = \int\int \hat{m}\mathcal{L}(\hat{v})\, dxdz \\
&= \int\int \hat{m}\mu_x\, \partial_x \hat{v}\, dxdz + \int\int \hat{m}\mu_z\, \partial_z \hat{v}\, dzdx \\
&+ \int\int \hat{m}\tfrac{1}{2}\sigma_x^2\, \partial_{xx}\hat{v}\, dxdz + \int\int \hat{m}\tfrac{1}{2}\sigma_z^2\, \partial_{zz}\hat{v}\, dzdx
\end{aligned}$$

Notice we have changed the order of integration across terms.

To compute $\int\int \hat{v}\mathcal{L}^*(\hat{m}) dxdz$ we use that for a fixed z we integrate by parts with respect to x to obtain:

$$\begin{aligned}
&-\int \hat{v}\, \partial_x(\mu_x \hat{m})\, dx = \int \mu_x \hat{m}\partial_x(\hat{v})\, dx - \hat{v}\mu_x\hat{m}|_{-\infty}^{\infty} \\
&\int \hat{v}\, \partial_{xx}(\tfrac{1}{2}\sigma_x^2 \hat{m})\, dx = -\int \partial_x \hat{v}\, \partial_x(\tfrac{1}{2}\sigma_x^2\hat{m})\, dx \\
&+ \hat{v}\, \partial_x(\tfrac{1}{2}\sigma_x^2\hat{m})|_{-\infty}^{\bar{x}^a_-} + \hat{v}\, \partial_x(\tfrac{1}{2}\sigma_x^2\hat{m})|_{\bar{x}^a_+}^{\bar{x}^b_-} + \hat{v}\, \partial_x(\tfrac{1}{2}\sigma_x^2\hat{m})|_{\bar{x}^b_+}^{\infty}
\end{aligned}$$

and for a fixed x we integrate by parts with respect to z to obtain:

$$\begin{aligned}
&-\int \hat{v}\, \partial_z(\mu_z \hat{m})\, dz = \int \mu_z \hat{m}\, \partial_z \hat{v}\, dz - \hat{v}\mu_z\hat{m}|_{-\infty}^{\infty} \\
&\int \hat{m}\tfrac{1}{2}\sigma_z^2\, \partial_{zz}\hat{v}\, dz = -\int \partial_z(\hat{m}\tfrac{1}{2}\sigma_z^2)\, \partial_z\hat{v}\, dz + \hat{m}\tfrac{1}{2}\sigma_z^2\, \partial_z\hat{v}|_{-\infty}^{\infty}
\end{aligned}$$

Likewise to compute $-\int\int \hat{m}\, \mathcal{L}(\hat{v})\, dxdz$ we use that for a fixed z we integrate by parts with respect to x to obtain:

$$\begin{aligned}
&-\int \hat{m}\mu_x\, \partial_x \hat{v}\, dx = \int \hat{v}\, \partial_x \hat{m}\mu_x\, dx - \hat{m}\mu_x\hat{v}|_{-\infty}^{\infty} \\
&-\int \hat{m}\tfrac{1}{2}\sigma_x^2\, \partial_{xx}\hat{v}\, dx = \int \partial_x(\hat{m}\tfrac{1}{2}\sigma_x^2)\, \partial_x\hat{v}\, dx - \hat{m}\tfrac{1}{2}\sigma_x^2\, \partial_x\hat{v}|_{-\infty}^{\infty}
\end{aligned}$$

and for a fixed x we integrate by parts with respect to x to obtain:

$$
\begin{aligned}
-\int \hat{m}\mu_z\, \partial_z \hat{v}\, dz &= \int \hat{v}\, \partial_z(\hat{m}\mu_z)\, dz - \hat{m}\mu_z\, \partial_z \hat{v}|_{-\infty}^{\infty} \\
-\int \hat{m}\tfrac{1}{2}\sigma_z^2\, \partial_{zz}\hat{v}\, dz &= \int \partial_z(\hat{m}\tfrac{1}{2}\sigma_z^2)\, \partial_z \hat{v}\, dz - \hat{m}\tfrac{1}{2}\sigma_z^2\, \partial_z \hat{v}|_{-\infty}^{\infty}
\end{aligned}
$$

Using the properties assumed for μ, σ^2 and the integrability assumptions for v^i and m^i at a regular classical equilibrium we get that for a fixed z:

$$
0 = \hat{v}\mu_x \hat{m}|_{-\infty}^{\infty} = \hat{v}\, \partial_x(\tfrac{1}{2}\sigma_x^2 \hat{m})|_{-\infty}^{\infty} = \hat{m}\mu_x \hat{v}|_{-\infty}^{\infty} = \hat{m}\tfrac{1}{2}\sigma_x^2\, \partial_x \hat{v}|_{-\infty}^{\infty}
$$

and likewise for a fixed x we get:

$$
0 = \hat{v}\mu_z \hat{m}|_{-\infty}^{\infty} = \hat{m}\tfrac{1}{2}\sigma_z^2\, \partial_z \hat{v}|_{-\infty}^{\infty} = \hat{m}\mu_z\, \partial_z \hat{v}|_{-\infty}^{\infty} = \hat{m}\tfrac{1}{2}\sigma_z^2\, \partial_z \hat{v}|_{-\infty}^{\infty}
$$

Note that

$$
\begin{aligned}
A &= \hat{v}\, \partial_x(\tfrac{1}{2}\sigma_x^2 \hat{m})|_{-\infty}^{\bar{x}_-^a} + \hat{v}\, \partial_x(\tfrac{1}{2}\sigma_x^2 \hat{m})|_{\bar{x}_+^a}^{\bar{x}_-^b} + \hat{v}\, \partial_x(\tfrac{1}{2}\sigma_x^2 \hat{m})|_{\bar{x}_+^b}^{\infty} \\
&= \hat{v}\, \partial_x(\tfrac{1}{2}\sigma_x^2 \hat{m})|_{\bar{x}_+^a}^{\bar{x}_-^a} + \hat{v}\, \partial_x(\tfrac{1}{2}\sigma_x^2 \hat{m})|_{\bar{x}_+^b}^{\bar{x}_-^b}
\end{aligned}
$$

Using that $\partial_x \sigma_x^2(x, z)$ and $m(x, z, t)$ are continuous on x everywhere we obtain:

$$
A \equiv \hat{v}\, \partial_x(\tfrac{1}{2}\sigma_x^2 \hat{m})|_{\bar{x}_+^a}^{\bar{x}_-^a} + \hat{v}\, \partial_x(\tfrac{1}{2}\sigma_x^2 \hat{m})|_{\bar{x}_+^b}^{\bar{x}_-^b} = \hat{v}\tfrac{1}{2}\sigma_x^2\, \partial_x \hat{m})|_{\bar{x}_+^a}^{\bar{x}_-^a} + \hat{v}\tfrac{1}{2}\sigma_x^2\, \partial_x \hat{m}|_{\bar{x}_+^b}^{\bar{x}_-^b}
$$

At this point it may be more clear to write all the arguments. In particular

$$
\begin{aligned}
A(z,t) = &\left(v^a(\bar{x}^a(z,t), z, t) - v^b(\bar{x}^a(z,t), z, t)\right) \tfrac{1}{2}\sigma_x^2(\bar{x}^a(z,t), z) \\
&\times \left(\partial_x m^a(x,z,t) - \partial_x m^b(x,z,t)\right)|_{x=\bar{x}_+^a(z,t)}^{x=\bar{x}_-^a(z,t)} \\
&+ \left(v^a(\bar{x}^b(z,t), z, t) - v^b(\bar{x}^b(z,t), z, t)\right) \tfrac{1}{2}\sigma_x^2(\bar{x}^b(z,t), z) \\
&\times \left(\partial_x m^a(x,z,t) - \partial_x m^b(x,z,t)\right)|_{x=\bar{x}_+^b(z,t)}^{x=\bar{x}_-^b(z,t)}
\end{aligned}
$$

Using that $\partial_x m^a(x,z,t)$ is continuous at $\bar{x}^b(z,t)$ and that $\partial_x m^b(x,z,t)$ is continuous at $\bar{x}^a(z,t)$ whenever $\bar{x}^a(z,t) \neq \bar{x}^b(z,t)$. Also we use that $v^a(\bar{x}^a(z,t),z,t) = v^b(\bar{x}^b(z,t),z,t) = 0$, then

$$
\begin{aligned}
A(z,t) = &- v^b(\bar{x}^a(z,t), z, t)\tfrac{1}{2}\sigma_x^2(\bar{x}^a(z,t), z)\partial_x m^a(x,z,t)|_{x=\bar{x}_+^a(z,t)}^{x=\bar{x}_-^a(z,t)} \\
&- v^a(\bar{x}^b(z,t), z, t)\tfrac{1}{2}\sigma_x^2(\bar{x}^b(z,t), z)\partial_x m^b(x,z,t)|_{x=\bar{x}_+^b(z,t)}^{x=\bar{x}_-^b(z,t)}
\end{aligned}
$$

And using Lemma 4, we have:

$$
\begin{aligned}
\tfrac{1}{2}\sigma_x^2(\bar{x}^a(z,t),z)\partial_x m^a(x,z,t)\big|_{x=\bar{x}^a_+(z,t)}^{x=\bar{x}^a_-(z,t)} &= -\int_{-\infty}^{\infty} m^a(x,z,t)H_v(v^a(x,z,t),x,z)\,dx \\
&\geq -\bar{H}_v \int_{-\infty}^{\infty} m^a(x,z,t)dx = -\bar{H}_v\, n(z) \\
\tfrac{1}{2}\sigma_x^2(\bar{x}^b(z,t),z)\partial_x m^b(x,z,t)\big|_{x=\bar{x}^b_+(z,t)}^{x=\bar{x}^b_-(z,t)} &= -\int_{-\infty}^{\infty} m^b(x,z,t)H_v(v^a(x,z,t),x,z)\,dx \\
&\geq -\bar{H}_v \int_{-\infty}^{\infty} m^b(x,z,t)dx = -\bar{H}_v\, n(z)
\end{aligned}
$$

Hence we have that:

$$
\begin{aligned}
&\int\int \hat{v}\mathcal{L}^*(\hat{m})dxdz - \int\int \hat{m}\,\mathcal{L}(\hat{v})\,dxdz \\
&\equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(v^a(x,z,t) - v^b(x,z,t)\right)\mathcal{L}^*(m^a - m^b)(x,z,t)dxdz \\
-&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(m^a(x,z,t) - m^b(x,z,t)\right)\mathcal{L}(v^a - v^b)(x,z,t)dxdz \\
&\leq \bar{H}_v \int_{-\infty}^{\infty} n(z)\, v^b(\bar{x}^a(z,t),z,t)dz + \bar{H}_v \int_{-\infty}^{\infty} n(z)\, v^a(\bar{x}^b(z,t),z,t)dz
\end{aligned}
$$

□

Proof. (of Lemma 8) Fix a (x,z,t) an omit them to simplify the notation.

$$
\begin{aligned}
&-\left(m^a - m^b\right)\left(H(v^a) - H(v^b)\right) + \left(v^a - v^b\right)\left(H_v(v^a)m^a - H_v(v^b)m^b\right) \\
=&m^a\left(H(v^b) - H(v^a) + H_v(v^a)\left(v^a - v^b\right)\right) + m^b\left(H(v^a) - H(v^b) + H_v(v^b)\left(v^b - v^a\right)\right)
\end{aligned}
$$

The concavity of H and the upper bound on the second derivative

$$
\begin{aligned}
H(v^b) - H(v^a) + H_v(v^a)(v^a - v^b) &\leq \bar{H}_{vv}(v^a - v^b)^2 \\
H(v^a) - H(v^b) + H_v(v^b)(v^b - v^a) &\leq \bar{H}_{vv}(v^a - v^b)^2
\end{aligned}
$$

Thus

$$
\begin{aligned}
&-\left(m^a - m^b\right)\left(H(v^a) - H(v^b)\right) + \left(v^a - v^b\right)\left(H_v(v^a)m^a - H_v(v^b)m^b\right) \\
&\leq \bar{H}_{vv}(m^a + m^b)(v^a - v^b)^2
\end{aligned}
$$

□

Proof. (of Proposition 3) Since $v(x,t) = u(x,t) - \int u(z,t)\nu_\epsilon(z - \bar{x}(t))dz$ does not depend on x, then

$$
\partial_x u(x,t) = \partial_x v(x,t), \text{ and } \partial_{xx} u(x,t) = \partial_{xx} v(x,t)
$$

Using that $\int \big(m^a(x,t) - m^b(x,t)\big)\, dx = 0$, then:

$$\begin{aligned}
&\int \big(v^a(x,t) - v^b(x,t)\big) \big(m^a(x,t) - m^b(x,t)\big)\, dx \\
=&\int \big(u^a(x,t) - u^b(x,t)\big) \big(m^a(x,t) - m^b(x,t)\big)\, dx
\end{aligned}$$

We start with:

$$\begin{aligned}
&\frac{d}{dt}\int \big(u^a(x,t) - u^b(x,t)\big) \big(m^a(x,t) - m^b(x,t)\big)\, dx \\
=&\int \big(u^a(x,t) - u^b(x,t)\big)\, \partial_t \big(m^a(x,t) - m^b(x,t)\big)\, dx \\
+&\int \big(m^a(x,t) - m^b(x,t)\big)\, \partial_t \big(u^a(x,t) - u^b(x,t)\big)\, dx
\end{aligned}$$

Using the properties of $v^i(x,t) - u^i(x,t) = -u^i(\bar{x}^i(t), t)$ for $i = a, b$, and that $\int (m^a - m^b) dx$ and its time derivative are zero:

$$\begin{aligned}
&\frac{d}{dt}\int \big(u^a(x,t) - u^b(x,t)\big) \big(m^a(x,t) - m^b(x,t)\big)\, dx \\
=&\int \big(u^a(x,t) - u^b(x,t)\big)\, \partial_t \big(m^a(x,t) - m^b(x,t)\big)\, dx \\
+&\int \big(m^a(x,t) - m^b(x,t)\big)\, \partial_t \big(u^a(x,t) - u^b(x,t)\big)\, dx \\
=&\int \big(v^a(x,t) - v^b(x,t)\big)\, \partial_t \big(m^a(x,t) - m^b(x,t)\big)\, dx \\
+&\int \big(m^a(x,t) - m^b(x,t)\big)\, \partial_t \big(u^a(x,t) - u^b(x,t)\big)\, dx
\end{aligned}$$

Using the p.d.e for the HJB in equation (45) and the one in KBF equation (46) to replace in the previous integrals as follows:

$$
\begin{aligned}
&\frac{d}{dt}\int \left(u^a(x,t)-u^b(x,t)\right)\left(m^a(x,t)-m^b(x,t)\right)dx \\
&= \int \left(v^a-v^b\right)\left(\mathcal{L}^*(m^a-m^b)\right)-\left(m^a-m^b\right)\left(\mathcal{L}(u^a-u^b)\right) \\
&+\rho\int \left(u^a-u^b\right)\left(m^a-m^b\right)dx \\
&+\int \left(v^a-v^b\right)\left(m^a H_v\left(v^a\right)-m^b H_v\left(v^b\right)\right)dx \\
&-\int \left(m^a-m^b\right)\left(H\left(v^a\right)-H\left(v^b\right)\right)dx \\
&-\int \left(m^a-m^b\right)\left(F(x,m^a)-F(x,m^b)\right)dx \\
&-\int \left(v^a-v^b\right)\left(\left(\int H_v\left(v^a\right)m^a\right)\nu^a-\left(\int H_v\left(v^b\right)m^b\right)\nu^b\right)dx
\end{aligned}
$$

where we omit the arguments (x,t) or x from the different functions to simplify the notation, and where we use the notation:

$$
\begin{aligned}
H\left(v^i\right) &\text{ as } H\left(u(x,t)-\int u^i(z,t)\nu_\epsilon(z-\bar{x}^i(t))dz, x\right) \\
H_v\left(v^i\right) &\text{ as } H_v\left(u(x,t)-\int u^i(z,t)\nu_\epsilon(z-\bar{x}^i(t))dz, x\right) \\
v^i &\text{ as } u(x,t)-\int u^i(z,t)\nu_\epsilon(z-\bar{x}^i(t))dz \\
\nu^i &\text{ as } \nu_\epsilon(z-\bar{x}^i(t))
\end{aligned}
$$

Using that $\partial_x v=\partial_x u$ and $\partial_{xx} v=\partial_{xx} u$ we can write:

$$
\begin{aligned}
&\frac{d}{dt}\int \left(u^a(x,t)-u^b(x,t)\right)\left(m^a(x,t)-m^b(x,t)\right)dx \\
&= \int \left(v^a-v^b\right)\left(\mathcal{L}^*(m^a-m^b)\right)-\left(m^a-m^b\right)\left(\mathcal{L}(v^a-v^b)\right)dx \\
&+\rho\int \left(u^a-u^b\right)\left(m^a-m^b\right)dx \\
&+\int \left(v^a-v^b\right)\left(m^a H_v\left(v^a\right)-m^b H_v\left(v^b\right)\right)dx \\
&-\int \left(m^a-m^b\right)\left(H\left(v^a\right)-H\left(v^b\right)\right)dx \\
&-\int \left(m^a-m^b\right)\left(F(x,m^a)-F(x,m^b)\right)dx \\
&-\int \left(v^a-v^b\right)\left(\left(\int H_v\left(v^a\right)m^a\right)\nu^a-\left(\int H_v\left(v^b\right)m^b\right)\nu^b\right)dx
\end{aligned}
$$

Thus we write:

$$\frac{d}{dt}\int \left(u^a - u^b\right)\left(m^a - m^b\right) dx = \rho \int \left(u^a - u^b\right)\left(m^a - m^b\right) dx \\ + I_L(t) + I_H(t) + I_F(t)$$

where

$$\begin{aligned} I_L(t) &\equiv \int \left(v^a - v^b\right)\left(\mathcal{L}^*(m^a - m^b)\right) - \left(m^a - m^b\right)\left(\mathcal{L}(v^a - v^b)\right) dx \\ I_F(t) &\equiv -\int \left(m^a - m^b\right)\left(F(x, m^a) - F(x, m^b)\right) dx \\ I_H(t) &\equiv \int \left(v^a - v^b\right)\left(m^a H_v\left(v^a\right) - m^b H_v\left(v^b\right)\right) dx \\ &\quad - \int \left(m^a - m^b\right)\left(H\left(v^a\right) - H\left(v^b\right)\right) dx \\ &\quad - \int \left(v^a - v^b\right)\left(\left(\int H_v\left(v^a\right) m^a\right)\nu^a - \left(\int H_v\left(v^b\right) m^b\right)\nu^b\right) dx \end{aligned}$$

Next, we obtain an inequality from each of the following terms.

1. We have that
$$I_L(t) = 0$$
since $\mathcal{L}$ and $\mathcal{L}^*$ are adjoints. In particular by integrating by parts twice, and using the boundary conditions in equation (43) and equation (44).

2. $I_F(t) \le 0$: holds directly by Assumption on monotonicity of F, i.e. that
$$\int \left(m^a - m^b\right)\left(F(x, m^a) - F(x, m^b)\right) dx \ge 0.$$

3. $I_H(t) \le \int \left[m^a + m^b\right]\left(v^a - v^b\right)^2 \bar{H}_{vv} dx$. This follows from expanding each term, i.e
$$\begin{aligned} I_H(t) \equiv &\int \left(v^a - v^b\right)\left(m^a H_v\left(v^a\right) - m^b H_v\left(v^b\right)\right) dx \\ &- \int \left(m^a - m^b\right)\left(H\left(v^a\right) - H\left(v^b\right)\right) dx \\ &- \int \left(v^a - v^b\right)\left(\left(\int H_v\left(v^a\right) m^a\right)\nu^a - \left(\int H_v\left(v^b\right) m^b\right)\nu^b\right) dx \end{aligned}$$
We use a first order expansion:
$$H(y_1, x) = H(y_2, x) + H_v(y_2, x)\left(y_1 - y_2\right) + H_{vv}(\tilde{y}, x)\left(y_1 - y_2\right)^2$$
for some $\tilde{y} \in [y_1, y_2]$. Letting $y_1 = v^b(x, t)$ and $y_2 = v^a(x, t)$, or reversing it with

$y_1 = v^a(x,t)$ and $y_2 = v^b(x,t)$, and omitting arguments:

$$H(v^b) - H(v^a) = H_v(v^a)\left(v^b - v^a\right) + H_{vv}(\tilde{v}^{ba}, x)\left(v^a - v^b\right)^2$$
$$H(v^a) - H(v^b) = H_v(v^b)\left(v^a - v^b\right) + H_{vv}(\tilde{v}^{ba}, x)\left(v^a - v^b\right)^2$$

Thus:

$$\begin{aligned}
&\left(v^a - v^b\right)\left(m^a H_v(v^a) - m^b H_v(v^b)\right) - \left(m^a - m^b\right)\left(H(v^a) - H(v^b)\right) \\
&= \left(v^a - v^b\right)\left(m^a H_v(v^a) - m^b H_v(v^b)\right) + m^a\left(H(v^b) - H(v^a)\right) + m^b\left(H(v^a) - H(v^b)\right) \\
&= \left[m^a H_{vv}(\tilde{v}^{ba}, x) + m^b H_{vv}(\tilde{v}^{ba}, x)\right]\left(v^a - v^b\right)^2 \\
&\leq \left[m^a + m^b\right]\left(v^a - v^b\right)^2 \bar{H}_{vv}
\end{aligned}$$

Integrating across x:

$$\begin{aligned}
I_H(t) = &\int \left[m^a + m^b\right]\left(v^a - v^b\right)^2 \bar{H}_{vv} dx \\
&- \int \left(v^a - v^b\right)\left(\left(\int H_v\left(v^a\right) m^a\right)\nu^a - \left(\int H_v\left(v^b\right) m^b\right)\nu^b\right) dx
\end{aligned}$$

We can rewrite the last line to get:

$$\begin{aligned}
I_H(t) = &\int \left[m^a + m^b\right]\left(v^a - v^b\right)^2 \bar{H}_{vv} dx \\
&- \left(\int H_v\left(v^a\right) m^a dx\right)\int \left(v^a - v^b\right)\nu^a dx \\
&- \left(\int H_v\left(v^b\right) m^b dx\right)\int \left(v^b - v^a\right)\nu^b dx
\end{aligned}$$

Using the definition we have:

$$\begin{aligned}
\int v^a \nu^a dx &= \int \left[u^a(x,t) - \int u^a(z,t)\nu_\epsilon(z - \bar{x}^a(t))dz\right]\nu_\epsilon(x - \bar{x}^a(t))dx \\
&= \int u^a(x,t)\nu_\epsilon(x - \bar{x}^a(t))dx - \int u^a(z,t)\nu_\epsilon(z - \bar{x}^a(t))dz = 0
\end{aligned}$$

and

$$\int v^b \nu^a dx = \int \left[v^b(x,t)\right]\nu_\epsilon(x - \bar{x}^a(t))dx \geq 0$$

Thus:

$$\int \left(v^a - v^b\right)\nu^a dx \leq 0 \text{ and } \int \left(v^b - v^a\right)\nu^b dx \leq 0$$

and using that

$$-\left(\int H_v\left(v^a\right) m^a dx\right) \geq 0 \text{ and } -\left(\int H_v\left(v^b\right) m^b dx\right) \geq 0$$

Hence we obtain the desired result:

$$I_H(t) \leq \bar{H}_{vv} \int \left[m^a + m^b\right] \left(v^a - v^b\right)^2 dx$$

Combining the expressions for $I_L(t)$, $I_H(t)$, $I_F(t)$ we obtain the desired results.
□

8 Discrete Time Version: Costly Probabilities

In this section we write a discrete time version of the problem in Section 3.1, and show that its continuous time limit is given by equation (9).

We let Δ be the length of the time period. We assume that when x is uncontrolled it evolves as:

$$x_{t+\Delta} = x_t + \mu(x_t)\Delta + \sigma(x_t)\sqrt{\Delta}\, e_{t+\Delta}$$

where

$$e_{t+\Delta} = \begin{cases} +1 & \text{with probability} = \frac{1}{2} \\ -1 & \text{with probability} = \frac{1}{2} \end{cases}$$

and where $e_{t+\Delta}$ is independent of e_τ for any τ.

To simplify notation, we let $f(x,t) = F(x,(m(t))$. We use a discount factor given by $1/(1+\Delta\rho)$.

We denote the value function by $u(x,t)$. We measure the value function at the end of the period t, after the decision of adjustment has been made. At this time the agent has a value x and the time is t. So the agent gets $f(x,t)\Delta$ flow cost during the discrete time period t to $t+\Delta$. At the end of period t the agent decides the probability λ per unit of time for an adjustment opportunity. The agent pays a cost $c(x,\lambda)\Delta$ if it chooses λ. After λ is chosen, at the beginning of next period, the binomial random variable $a_{t+\Delta}$ is realized. If $a_{t+\Delta} = 1$, then the agent can set the state at any desired value. We assume that the agent decides the new value of x, before seeing the realization of $e_{t+\Delta}$.

Before writing down the discrete time Bellman equation, we introduce the following notation for the conditional expectation of the next period value function:

$$E_x(u_{t+\Delta}) \equiv \frac{1}{2} u\left(x + \mu(x)\Delta + \sigma(x)\sqrt{\Delta}, t+\Delta\right) + \frac{1}{2} u\left(x + \mu(x)\Delta - \sigma(x)\sqrt{\Delta}, t+\Delta\right)$$

We can now write the discrete time Bellman equation for this problem:

$$
\begin{aligned}
u(x,t) &= f(x,t)\Delta + \min_{\{0\le\lambda\le 1/\Delta\}} c(x,\lambda)\Delta \\
&\quad + \frac{1}{1+\rho\Delta}(1-\lambda\Delta)E_x(u_{t+\Delta}) + \frac{1}{1+\rho\Delta}\lambda\Delta u(\bar{x}_{t+\Delta}, t+\Delta) \\
\text{where } \bar{x}_{t+\Delta} &= \arg\min_z u(z, t+\Delta)
\end{aligned}
$$

The first term of the right hand side is the period cost. The second term has the cost of selecting λ. The third and fourth terms correspond to the continuation. The third term corresponds to the case where the agent does not get the opportunity to adjust. The last one contains the case where the agent has a possibility of adjusting the state.

Now we derive, heuristically, the continuous time limit HJB equation. Multiplying both sides by $(1+\rho\Delta)$

$$
\begin{aligned}
(1+\rho\Delta)u(x,t) &= f(x,t)(1+\rho\Delta)\Delta + \min_{\{0\le\lambda\le 1/\Delta\}} c(x,\lambda)(1+\rho\Delta)\Delta \\
&\quad + (1-\lambda\Delta)E_x(u_{t+\Delta}) + \lambda\Delta u(\bar{x}_{t+\Delta}, t+\Delta)
\end{aligned}
$$

rearranging

$$
\begin{aligned}
\rho\Delta u(x,t) &= f(x,t)(1+\rho\Delta)\Delta + (1+\rho\Delta)\min_{\{0\le\lambda\le 1/\Delta\}} c(x,\lambda)(1+\rho\Delta)\Delta \\
&\quad + E_x(u_{t+\Delta}) - u(x,t) + \lambda\Delta\left[u(\bar{x}_{t+\Delta}, t+\Delta) - E_x(u_{t+\Delta})\right]
\end{aligned}
$$

Dividing by Δ

$$
\begin{aligned}
\rho u(x,t) &= f(x,t)(1+\rho\Delta) + (1+\rho\Delta)\min_{\{0\le\lambda\le 1/\Delta\}} c(x,\lambda)(1+\rho\Delta) \\
&\quad + \frac{E_x(u_{t+\Delta}) - u(x,t)}{\Delta} + \lambda\left[u(\bar{x}_{t+\Delta}, t+\Delta) - E_x(u_{t+\Delta})\right]
\end{aligned}
$$

Taking $\Delta \downarrow 0$ and using that :

$$
\begin{aligned}
\frac{E_x(u_{t+\Delta}) - u(x,t)}{\Delta} &\to u_x(x,t)\mu(x) + \frac{\sigma^2(x)}{2}u_{xx}(x,t) + u_t(x,t) \\
E_x(u_{t+\Delta}) &\to u(x,t) \\
u(\bar{x}_{t+\Delta}, t+\Delta) &\to u(\bar{x}(t), t) \\
f(x,t)(1+\rho\Delta) &\to f(x,t)
\end{aligned}
$$

We obtain the desired result:

$$
\begin{aligned}
\rho u(x,t) &= f(x,t) + \min_{\{0\le\lambda\}} c(x,\lambda) + \lambda\left[u(\bar{x}(t), t) - u(x,t)\right] \\
&\quad + u_x(x,t)\mu(x) + \frac{\sigma^2(x)}{2}u_{xx}(x,t) + u_t(x,t)
\end{aligned}
$$

9 Discrete Time Version: Random Fixed Cost Model

In this section we write a discrete time version of the problem in Section 3.2, and show that its continuous time limit is given by equation (14).

We let Δ be the length of the time period. We assume that when x is uncontrolled it evolves as:

$$x_{t+\Delta} = x_t + \mu(x_t)\Delta + \sigma(x_t)\sqrt{\Delta}\, e_{t+\Delta}$$

where

$$e_{t+\Delta} = \begin{cases} +1 & \text{with probability} = \frac{1}{2} \\ -1 & \text{with probability} = \frac{1}{2} \end{cases}$$

and where $e_{t+\Delta}$ is independent of e_τ for any τ. We consider another random variable, binomially distributed, which we denote by a which has the interpretation than when $a = 1$ the firm has an opportunity to adjust, and when $a = 0$ it does not.

$$a_{t+\Delta} = \begin{cases} 1 & \text{with probability} = \kappa(x_t)\Delta \\ 0 & \text{with probability} = 1 - \kappa(x_t)\Delta \end{cases}$$

for some function κ. Conditionally on x_t, we assume that the realization of $a_{t+\Delta}$ are independently distributed of e_τ for any τ.

To simplify notation, we let $f(x,t) = F(x,(m(t))$. We use a discount factor given by $1/(1 + \Delta\rho)$.

We denote the value function by $u(x,t)$. We measure the value function at the end of the period t, after the decision of adjustment has been made. At this time the agent has a value x and the time is t. So the agent gets $f(x,t)\Delta$ flow cost during the discrete time period t to $t + \Delta$. At the beginning of next period, two random variables are realized, independently of each other, are realized: $a_{t+\Delta}$ and $e_{t+\Delta}$. If $a_{t+\Delta} = 1$, which occurs with probability $\kappa(x)\Delta$, the firm can adjust its state. In particular, if $a_{t+\Delta} = 1$, the agent draws a fixed cost ψ from a distribution with CDF given by G. The realization of ψ is independent of the realizations of e_τ and a'_τ for all τ, τ'. We assume that the agent has to decide whether to adjust x or not, before seeing the realization of $e_{t+\Delta}$.

Before writing down the discrete time Bellman equation, we introduce the following notation for the conditional expectation of the next period value function:

$$E_x(u_{t+\Delta}) \equiv \frac{1}{2}u\left(x + \mu(x)\Delta + \sigma(x)\sqrt{\Delta}, t+\Delta\right) + \frac{1}{2}u\left(x + \mu(x)\Delta - \sigma(x)\sqrt{\Delta}, t+\Delta\right)$$

We can now write the discrete time Bellman equation for this problem:

$$\begin{aligned} u(x,t) &= f(x,t)\Delta + \frac{1}{1+\rho\Delta}(1 - \kappa(x)\Delta)E_x(u_{t+\Delta}) \\ &\quad + \frac{1}{1+\rho\Delta}\kappa(x)\Delta \int \min\left\{E_x(u_{t+\Delta})\,,\, \psi + u(\bar{x}_{t+\Delta}, t+\Delta)\right\} dG(\psi) \\ \text{where } \bar{x}_{t+\Delta} &= \arg\min_z u(z, t+\Delta) \end{aligned}$$

The first term of the right hand side is the period cost. The second and third term correspond to the continuation. The second term corresponds to the case where the agent does not get the opportunity to adjust. The last one contains the case where the agent has a possibility of adjusting the state. This last term contains the only decision of the problem.

Now we derive, heuristically, the continuous time limit HJB equation. Multiplying both sides by $(1+\rho\Delta)$

$$\begin{aligned}(1+\rho\Delta)u(x,t) &= f(x,t)(1+\rho\Delta)\Delta + (1-\kappa(x)\Delta)E_x(u_{t+\Delta}) \\ &\quad + \kappa(x)\Delta \int \min\left\{E_x(u_{t+\Delta})\,,\ \psi + u(\bar{x}_{t+\Delta}, t+\Delta)\right\} dG(\psi)\end{aligned}$$

rearranging

$$\begin{aligned}u(x,t)\rho\Delta &= f(x,t)(1+\rho\Delta)\Delta + E_x(u_{t+\Delta}) - u(x,t) \\ &\quad + \kappa(x)\Delta\left[\int \min\left\{E_x(u_{t+\Delta})\,,\ \psi + u(\bar{x}_{t+\Delta}, t+\Delta)\right\} dG(\psi) - E_x(u_{t+\Delta})\right]\end{aligned}$$

collecting terms in the minimum:

$$\begin{aligned}u(x,t)\rho\Delta &= f(x,t)(1+\rho\Delta)\Delta + E_x(u_{t+\Delta}) - u(x,t) \\ &\quad + \kappa(x)\Delta \int \min\left\{0\,,\ \psi + u(\bar{x}_{t+\Delta}, t+\Delta) - E_x(u_{t+\Delta})\right\} dG(\psi)\end{aligned}$$

Dividing by Δ

$$\begin{aligned}u(x,t)\rho &= f(x,t)(1+\rho\Delta) + \frac{E_x(u_{t+\Delta}) - u(x,t)}{\Delta} \\ &\quad + \kappa(x) \int \min\left\{0\,,\ \psi + u(\bar{x}_{t+\Delta}, t+\Delta) - E_x(u_{t+\Delta})\right\} dG(\psi)\end{aligned}$$

Taking $\Delta \downarrow 0$ and using that :

$$\begin{aligned}\frac{E_x(u_{t+\Delta}) - u(x,t)}{\Delta} &\to u_x(x,t)\mu(x) + \frac{\sigma^2(x)}{2}u_{xx}(x,t) + u_t(x,t) \\ E_x(u_{t+\Delta}) &\to u(x,t) \\ u(\bar{x}_{t+\Delta}, t+\Delta) &\to u(\bar{x}(t), t) \\ f(x,t)(1+\rho\Delta) &\to f(x,t)\end{aligned}$$

We obtain the desired result:

$$\begin{aligned}\rho u(x,t) &= f(x,t) + u_x(x,t)\mu(x) + \frac{\sigma^2(x)}{2}u_{xx}(x,t) + u_t(x,t) \\ &\quad + \kappa(x) \int \min\left\{0\,,\ \psi + u(\bar{x}(t), t) - u(x,t)\right\} dG(\psi)\end{aligned}$$