A Side Effect of Financial Innovation

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Abstract

Financial innovation often involves the creation of new instruments that bundle existing securities such as mutual funds or exchange traded funds. In this paper we argue that while the creation of such assets has advantages such as lowering transaction costs, there can also be negative side effects. Investment in the new instrument may reveal information about future demand for the underlying securities, as the fund managing the instrument often needs to mechanically rebalance its position. Informed arbitrageurs may trade, exploiting their knowledge of this rebalancing. In general equilibrium, this trading activity can make the price of the underlying securities less informative and more sensitive to rebalancing shocks. We show that this loss of information may outweigh the direct gains from lower transaction costs that uninformed investors achieve. So welfare of uninformed investors may be lower. We argue that the potential effect on the market for the underlying securities may be substantial. To illustrate our mechanism, we analyze two episodes of financial innovations in the oil and volatility markets. The magnitude of the effect can be gauged by the fact that introduction of the new instrument in the oil market led to a temporary violation of a no-physical arbitrage condition.

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1 Introduction

Financial innovation often involves the creation of new instruments that bundle existing securities. The creation of such assets has advantages such as lowering trading costs. Possible ways to lower costs include more efficient margin management, netting of orders, or avoidance of frequent trading of illiquid stocks. One example are mutual funds that provide access to diversified portfolios of equities, including illiquid stocks. Another example are exchange traded funds (ETF) that replicate a trading strategy on the underlying market, issue shares that give the holder a claim on the revenue, and allow the shares to be traded on a liquid secondary market. Thus all agents can trade shares of the ETF instead of trading the underlying asset, obtaining the same exposure but at a lower costs. Our paper aims to show a potential side effect of such financial innovations.

Our first observation is that trading replicating instruments is not informationally equivalent to trading the underlying assets. Trading strategies involve rebalancing that may be partially predictable. Observing the aggregate demand for the new instrument, an arbitrageur can extract information about future transactions and trade based on that information. This type of trading is usually called “front-running”. Not surprisingly, front-running will affect both the market for the underlying asset and for the new instrument. However, it is not a priori clear that the effect should be negative. Intuitively, if the rebalancing induces a temporary price impact, the front-runner may provide the liquidity needed during the rebalancing process, and help to spread out the price impact into the preceding periods, trading in advance. Thus investors could lose less during the rebalancing. This is the usual way in which we think about the stabilizing role of speculation. However, that logic abstracts from general equilibrium effects. In other words, the degree to which prices are sensitive to shocks may depend on the presence of front-running. Front-running may potentially amplify temporary mispricings, implying larger losses for investors. To understand these forces we build a general equilibrium model with partially revealing prices.

Throughout the paper, we will use the following example as a motivation for our analysis. The usual way to invest in oil is to trade oil futures. Given that West Texas Intermediate (WTI) is the grade of crude oil used as a benchmark in oil pricing, investors prefer to trade the New York Mercantile Exchange’s oil futures contracts that have WTI as the underlying commodity. The futures contracts have unique feature - they expire, and when they do so physical oil should be delivered. In order to maintain long exposure and avoid taking the delivery, one has to regularly replace soon to expire futures with more distant ones, an operation known as rolling. Historically, investors would perform rolling independently, choosing random days and times. The situation changed with the introduction of the United States Oil Fund(USO). USO is an exchange traded fund that offers low cost exposure to oil prices. The Fund invests in liquid short-term oil futures and issues shares that are traded as usual stocks on the New York Stock Exchange. The supply of shares is flexible - if demand for USO shares increases, the price of its shares on the secondary market goes up, then more shares are issued and distributed. New shares are backed with additional nearby futures contracts acquired by the Fund. The Fund became popular in 2008 and attracted $4 bln of assets by February, 2009, equivalent to 30% of the open position in the nearby contract[1]. The problem is that the Fund has to regularly roll its entire portfolio. Moreover, the

[1]When several futures contracts are considered, the contract with the closest settlement date is called the nearby futures contract.

regulators require that the Fund publish a rolling schedule in advance and reveal the number of contracts on its balance every day. Therefore, the rolling operation represents a predictable and large demand shock unrelated to fundamentals.

The question arises whether the market is able to absorb the rolling shock. If arbitrage is limited, the potential pressure on prices of two futures contracts may be large given the large number of contracts to be rolled. In particular, the price of the soon to expire contract goes down and the price of the next out contract increases. Thus, the price difference, called contango, spikes on rolling days, and the investors in the Fund experience losses as they sell underpriced nearby contracts and buy overpriced second month contracts. However, given that the rolling is predictable, any arbitrageur could in principle front-run - sell nearby contracts and buy second month contracts in advance. When the rolling comes and the Fund itself sells nearby contracts and buys second month contracts, the arbitrageur could step in and take the opposite side of the deal: at the same time offsetting his own position and potentially decreasing the price impact of the rolling. However, the oil futures market experienced large and prolonged contango in the winter 2008-2009. The price of the second month futures contract exceeded the price of the nearby contract, or spot price, by so much that physical arbitrage was possible: anyone could sell the second month futures contract, buy oil on the spot market, store it for about a month, and deliver using the second futures contract[2]. So what can explain such unprecedented events, such a violation of a no-arbitrage condition? We argue that USO rolling and front-running added to the large and prolonged price divergence. The rolling makes the futures market vulnerable to large demand shocks and associated front-running. During normal times, there is enough arbitrage capital and we might not observe large misprisings, and the impact of the new instrument on the underlying market is hard to detect. However, during extreme times, such as the winter 2008-2009, when the arbitrage capital was particularly scarce, the new instrument played a particularly detrimental role, thus motivating our analysis.

In this paper we ask how the introduction of a new instrument that bundles existing demand affects the informativeness of prices and market efficiency, and whether agents lose more from trading with front-running arbitrageur than they gain from the lower transaction costs that the instrument offers.

The framework for our analysis is a two period Kyle model that answers how informed traders trade in order to maximize the value of their private information. There is one risky asset with unknown fundamental value that can be traded during two periods. There are four types of agents: fundamental traders, uninformed traders, market makers, and one large arbitrageur. Informed agents, namely fundamental traders and arbitrageur, get private signals about fundamental value and trade based on that information. The arbitrageur is large, and thus accounts for the price impact of his trades. Informed agents trade with uninformed market makers, who set prices and clear the market. Uninformed traders' demand is exogenous and precludes market makers from perfectly observing informed agents' demand and learning the fundamental value θ. Thus, equilibrium prices end up being partially revealing.

[2]Assessment of associated costs, and analysis of storage capacity is done in Selezneva (2010).

One major difference between our model and the canonical Kyle model is the introduction of trading costs. We assume that both fundamental and uninformed traders must pay transaction costs in order to trade the risky asset. However, the transaction costs may be reduced. Indeed, fundamental traders get noisy private signals, thus, demand of each individual trader has an idiosyncratic component. Similarly, uninformed traders have idiosyncratic hedging needs. Therefore, there is scope for intermediation: an intermediary could collect orders of the clients, net them out, and bring to the market only the net demand.

We model such intermediation as the introduction of a new instrument that replicates the trading of the underlying asset and that can be traded through a monopolistic broker that serves as an intermediary. Agents self-select to trade the new instrument, and the market for the new instrument coexists with the market for the underlying risky asset. However, investment in the new instrument reveals information about future demand for the underlying asset, as required by replication. The arbitrageur observes the investment in the new instrument and extracts information about future transactions. The main questions that we ask are: i) how does arbitrageur use that extra information?; and ii) how does it change market efficiency and welfare of investors?

We start with the case with no informed traders, which can be solved analytically. The new instrument reveals future uninformed traders' demand to the arbitrageur. We show that in equilibrium the arbitrageur partially offsets the uninformed demand shock in period 2, but trades in the direction of the demand shock in period 1. In other words, the arbitrageur front-runs. For given price functions, front-running in period 1 hurts uninformed traders as it pushes the price in the direction of the shock. In contrast, offsetting of the shock in period 2 benefits investors, as it reduces the price impact of the uninformed demand shock. The reallocation of uninformed demand shock from period 2 to period 1 as a result of front-running, in turn, induces the arbitrageur to change his fundamental trading. Price setting functions adjust for the new behavior of the arbitrageur. The first main result of the paper states that introduction of the new instrument is beneficial, in particular, prices become more informative and uninformed investors profit increases.

The welfare implications changes if informed traders trade in period 1. The arbitrageur faces the tradeoff between fundamental trading and front-running. Front-running, as it involves substantial offset of the uninformed demand shock in period 2, makes fundamental trading less profitable and also forces the arbitrageur to reallocate fundamental trading to period 1. Fundamental trading becomes less attractive, and more so when the arbitrageur has to face competition from informed traders also active in period 1. The main result of the paper is that the introduction of the new instrument can be detrimental. The new instrument triggers aggressive front-running and implies less informative prices and larger loses of investors, large enough to exceed direct gains from lower transaction costs. Finally, we allow informed traders to trade in period 2 and invest in the new instrument. In that case investment in the new instrument represents an endogenous mix of fundamental signals and signals about future uninformed demand shock. In general, we show that front-run is associated with larger price responses to uninformed demand shocks and larger losses of uninformed investors.

The analysis in this paper captures the implications of front-running triggered by the new instrument for the efficiency of the financial market in isolation. More generally, there may be additional social value attached to the informativeness of market prices. The price of the nearby futures contract on WTI is often considered as a benchmark in oil pricing, thus anything that pushes it away from the fundamental value potentially has real effects. That question is studied in more detail in Selezneva (2015).

To provide empirical evidence for our mechanism, we focus on two replicating instruments that require regular rebalancing. Our empirical exercise aims to document a significant price pressure, consistent with rebalancing transactions, that these instruments impose on the underlying market. Observed price impact would suggest limited arbitrage on the underlying market. Given limited arbitrage, we may expect to observe front-running, as front-running arbitrageurs intent to only partially offset the rebalancing shock. The front-running affects the underlying market in the periods preceding the rebalancing period. Thus, we intent to relate the flow of investment in the replicating instrument to observable changes in the underlying market in the period before the rebalancing. Additional investment in the new instrument inevitably increases the rebalancing demand, and thus makes front-running more attractive. We expect to observe the effect of extra front-running transactions on the prices of the underlying assets.

We consider two instruments: i) the United States Oil Fund (USO) in the oil market; and ii) exchange traded notes issued by Barclays and to linked to the S&P VIX Short-Term Futures index (VXX) in the volatility market. Both entities require regular rebalancing. In the oil market monthly rebalancing is required to replace expiring futures contracts and avoid physical delivery of oil. In the volatility market daily rebalancing is necessary to achieve a constant maturity volatility index. Namely, a particular fraction of the nearby futures contracts must be sold and the next futures contracts must be bought every day. In both cases rebalancing is predictable. USO specifies rolling days in its prospectus a year in advance and publishes the holdings every day on the official website. Barclays, the issuer of VXX, is expected to replicate the underlying volatility index to hedge its own exposure. Observing total investment in VXX on its official website, Barclay's daily rebalancing can also be calculated. Thereby, any sophisticated agent may predict future rebalancing demand and use that information to front-run. In both cases front-running takes the form of sale of the nearby futures and purchase of the next out futures. That transaction tends to increase the difference between the prices of the second month and the nearby futures, sometimes called contango. We aim to relate changes in the investment in the new instruments with changes in contango. We work with daily data, thus in the case of VXX we can only document the combined price impact of both front-running and rolling, as rebalancing frequency is also daily. In the case of USO we can document front-running, analyzing the days that precede the rolling.

We use a number of sources to assess the inflows of investment in USO and VXX. Shares outstanding data and share prices can be found in Compustat. However, shares outstanding data are often out of date. Thus we cross-check the Compustat data with inflows data from etf.com[3], with our own data on USO holdings downloaded from the USO website, and with monthly 8-k forms from SEC filings that report shares outstanding. We calculate contango using daily futures prices. In the VXX case we document that current changes in the investment in VXX predict future changes in contango. Thus, we document a significant price impact of daily rebalancing

[3]Data provider is FactSet Research Systems, Inc.

and associated front-running. In the USO case we also account for market specific factors, and show that changes in investment predict future changes in contango. However, as we intentionally restrict our sample to the period before the rebalancing, the observed pattern is consistent with front-running behavior of arbitrageurs. Front-running was particularly strong in the winter 2008-2009 and in 2015, when USO attracted substantial investment.

The paper is organized as follows. We discuss the main motivating example in more detail in section 2. Section 3 presents a theoretical model without the instrument, shows the existence of a linear equilibrium, and studies general features of arbitrageur's behavior. Then a model with the new instrument is presented, and a new linear equilibrium is characterized in a general form. Section 4 studies a special case with no fundamental traders, and discusses welfare and market efficiency implications of front-running associated with the introduction of the new instrument. Section 5 examines the second special case with fundamental traders trading in period 1. Section 6 uses a numerical example to illustrate the difference between the two cases and considers an extended version of the model with fundamental traders trading in both periods and investing in the new instrument. Section7 provides empirical evidence for our mechanism, studying two instruments in the oil and volatility markets. Section 8 discusses the related literature. Finally, section 9 concludes.

2 Motivating example - the Unites States Oil Fund

The usual way to trade oil is to trade oil futures, offered at the NYMEX. The universe of oil traders comprises of two broad groups, depending on the horizon of strategies used. First, traders looking for short-term exposure to oil prices, for example, in pursuit of diversification of an equity portfolio. Second, traders who would like to hedge their long-term exposure, including real producers and consumers of oil, but financial traders as well. The problem that the second group faces is the fact that futures have expiration dates, thus any open position in futures on delivery date implies actual delivery of physical oil, and not many traders are eager to receive that. Futures with long maturities are available, but they are illiquid. Thus most traders choose to buy the nearby futures and roll it over, when delivery date approaches. Rolling means an instantaneous selling of the contract soon to expire and purchase of the next contract.

The situation changes with the introduction of the United States Oil Fund (USO). USO advertises itself as an investment vehicle that tracks oil prices. Being an exchange traded fund USO issues shares and invests in the nearby futures contracts (crude light oil futures, WTI, NYMEX). Liquid secondary market for USO shares exists, shares are traded as any other stocks at NYSE, and thereby USO can be viewed as an ideal instrument by first group of oil investors, mentioned above. Moreover, orders of that first group are likely to be balanced, thus giving any broker an opportunity to profit from netting of order flows and share that profit with their clients.

USO attracts the second group as well, as they also view it as a low costs instrument that gives exposure to oil. The problem lies in the Fund's rolling procedure. First, the Fund rolls its entire portfolio in a short time period - one day until March 2009, and four days since then. Second, USO is required by regulators to specify its rolling days in a publicly available prospectus in advance. Third, any one is able to see the amount of money invested in the Fund at the end of each trading day and thus may estimate the open futures position it implies. Altogether that creates a predictable rolling demand. Thus, the Fund adds up positions of long-term investors, coordinates their rolling, and makes it public.

We argue that sophisticated arbitrageurs front-run on that information. The rolling transaction, selling of the nearby contract and purchase of the next one, tends to increase the price difference between the two contracts, $F_{2,t} - F_{1,t}$, if positive also called contango. One might imagine that the observed effect of rolling transactions on the market would depend on the liquidity in futures market available at the moment of rolling and on the level of contango just before the rolling happens. But both liquidity and pre-trade contango are endogenous, and both may be affected by previous actions of other market participants, namely arbitrageurs. Actions of arbitrageurs depend on whether they are aware of rolling demand. If they are aware, arbitrageurs may trade in the same direction as Fund's rolling before the Fund itself (front-run) while other market participants are uninformed about the Fund and contango is low; and then close position with the Fund, thus providing liquidity but not enough to fully absorb the Fund's rolling demand and therefore allowing the contango to shoot up and thus profiting at the expense of Fund's long-term investors[4].

The effect on the dynamics of contango was tremendous at times. Figure 19 presents on one graph USO position and contango level. One can notice that two noticeable episodes of huge contango coincide with a sharp growth of USO assets. Moreover, figure 20 confirms that although contango may have different explanations, only WTI oil traded at NYMEX and being tracked by USO experienced contango. In particular, futures based on European Brent oil do not display the same patterns, suggesting that the presence of the USO rather than global conditions in the market for oil account for the anomalous movements. Section 7 analyzes the empirical data in more details, and confirms that contango may be attributed to USO positions and is not likely to be explained by the lack of storage facilities, or high borrowing costs, or high volatility[5].

[4]WSJ 'CFTC Fines Morgan Stanley, UBS Over Oil Block Trade' Apr 2010: "On Feb. 6, 2009, Morgan Stanley and UBS, on behalf of its customer, conducted the trade. Morgan Stanley allegedly bought 33,110 March 2009 light sweet crude oil contracts on the New York Mercantile Exchange and sold 33,110 April 2009 crude oil contracts. UBS, according to the CFTC, didn't report this trade until 2:37 p.m., which was after the market had closed. The CFTC said that both Morgan Stanley and UBS concealed the block trade from NYMEX, a violation of NYMEX rules, which require a block trade to be reported to the exchange within five minutes of the time it is executed. "

[5]However, the effect of USO may not be limited to the disturbance of the futures market, the real market could have been affected as well as shown in Selezneva (2015). First, financial market is used by real producers to hedge their exposure to the oil price and insure the cash flow. If anything affects the futures prices that automatically transmits to the production decisions. Thus a new instrument may have an effect on the real variables by disturbing the risk shifting role of the financial market. Second, agents tend to look at financial markets for information. Thus price discovery role of financial market can be affected as well. Finally, contango tends to trigger a buildup in inventories. Inventories also perform a role of a source of information for market participants, even more important source than financial market itself, as inventories are perceived to be "closer to the real things". However, the rolling shock would propagate to the level of inventories as well, large inventories would be considered as a sign of a weak conditions on the current oil market. But inventories are large only as a result of the financialization and rolling shock, therefore price discovery role of inventories diminishes as well.

3 Model

We consider a two period Kyle model, which aims to answer how informed traders would choose to trade in order to maximize the value of their private information. There are two periods, 1 and 2. There is one risky asset with fundamental value $\theta \sim N(0, \sigma_\theta^2)$, that is realized at the beginning of period 1 and is revealed to everyone at the end of period 2 after all trading is completed. Any agent that holds a unit of the asset gets the payoff θ at date 2. The asset is in zero net supply. There are four types of agents: financial traders, uninformed noise traders, market makers, and one large arbitrageur. Later on we will add a new financial instrument, that can be traded by agents, and introduce a broker that trades this instrument, but for now only the risky asset is available.

The timeline is as follows. First, private signals are observed. Then agents trade the asset in periods one and two, and finally payoffs are realized. Trading in periods 1 and 2 follows the same protocol. First, traders simultaneously and anonymously submit market orders - only quantities are specified. Uninformed risk neutral market makers must clear the market, which means that they set up the execution price and absorb the net demand for the risky asset.

3.1 Model without the instrument

Informed financial traders Financial traders get a signal about the fundamental value $s_i = \theta + \nu_i$ and trade based on that signal, buying/selling units of the risky asset and keeping it until liquidation to get the payoff θ. We assume that there is a measure one of financial traders, where fraction μ of them trades only in the first period and fraction $1-\mu$ trades only in the second period. All financial traders are risk neutral and all of them are price takers. A financial trader forms an expectation about the asset price p_t, and submits the market order z_i - how much he wants to trade, subject to transaction costs. If a trader demands z_i units of the asset, he has to pay $c_i z_i^2$ units of numeraire to the exchange, where $c_i \sim U[\underline{c}, \bar{c}]$ is independent from both period of activity and signal s_i. Heterogeneity in transaction costs will be used later to obtain the coexistence of different instruments replicating same strategies. In sum, a financial trader active in period t maximizes expected profit given by $\pi_{fin,t}(s_i, c_i) = \underset{z_i(s_i)}{max} \left\{ E\left[(\theta - p_t)|s_i\right] z_i - c_i z_i^2 \right\}$.

For simplicity, we assume that market orders by financial traders at time 2 are not conditional on the price p_1.

Arbitrageur A single arbitrageur gets a signal $s = \theta + \varepsilon$ about the fundamental value and trades in both periods; his trades are denoted by $x_1$and x_2. Similarly to financial traders we assume that market orders by the arbitrageur at time 2 are not conditional on the price p_1, although new information may be revealed by first period price[6]. In contrast to financial traders, the arbitrageur is strategic and large. He realizes his influence on the market and thus trades accounting for the impact of his orders on the prices, given market makers' price setting strategy. The arbitrageur maximizes $\pi_{arbitrageur} = \underset{x_1(s), x_2(s)}{max} E\left[(\theta - p_1)x_1 + (\theta - p_2)x_2 | s\right]$.

[6]In the appendix we solve the model where the arbitrageur observes first period price before deciding how much to trade in the second period and show that the main results continue to hold(see appendix, section 12.5).

Uninformed noise traders The demand of uninformed noise traders is exogenous. There is a measure one of uninformed noise traders that trade in period 1, and a measure one in period 2. We assume that each individual trader active in period t trades $u_t + \xi_{j,t}$, where $\xi_{j,t} \sim N(0, \sigma_\xi^2)$ defines idiosyncratic motive for trading, and $u_t \sim N(0, \sigma_{ut}^2)$ is an aggregate shock affecting all noise traders, possibly corresponding to a systematic or correlated judgment bias. Because uninformed noise traders trade with informed traders, on average they will make losses. Their expected profits are given by $E\pi_{noise,t}(c_j) = E\left[(\theta - p_t)(u_t + \xi_{j,t}) - c_j(u_t + \xi_{j,t})^2\right]$, where heterogeneous costs are given by $c_j \sim U[\underline{c}, \bar{c}]$ as in the case of informed traders.

Order flow Combined order flow includes orders submitted by arbitrageur, financial traders, and uninformed traders. We need to integrate orders of informed traders. Fraction μ of informed agents is active in the first period, and we label them by $i \in [0, \mu]$ from lowest to largest costs c_i, fraction $1 - \mu$ are active in second and we similarly label them by $i \in [\mu, 1 - \mu]$. Using that we can write informed traders combined order flow in period 1 as $\int_0^\mu z_i di$ and in period 2 as $\int_\mu^1 z_i di$. Aggregation of orders submitted by uninformed eliminates idiosyncratic demand $\xi_{j,t}$, and gives u_1 in period 1 and u_2 in period 2. Hence, combined order flow can be written as

$$
\begin{aligned}
y_1 &= \int_0^\mu z_i di + u_1 + x_1, \\
y_2 &= \int_\mu^1 z_i di + u_2 + x_2.
\end{aligned}
$$

Market makers Risk neutral uninformed competitive market makers collect market orders submitted by other agents, and take the opposite side of the net demand. Thus if agents altogether want to by a unit of asset, market maker sells them that unit, absorbing net demand. Hence, if $y_1$is combined order flow in the first period and y_2- in the second, then utility of market maker in each period is given by

$$
\begin{aligned}
U_1 &= E\left[(p_1 - \theta) y_1 | y_1\right], \\
U_2 &= E\left[(p_2 - \theta) y_2 | y_1, y_2\right].
\end{aligned}
$$

Competition forces bring the price back to the efficient level: $p_1 = E[\theta|y_1]$ and $p_2 = E[\theta|y_1, y_2]$.

Definition. An equilibrium is a combination of price functions $p_1(y_1)$, $p_2(y_1, y_2)$, trading strategy of the arbitrageur $x_1(s)$ and $x_2(s)$, trading strategy of informed traders $z_i(s_i, c_i)$ such that

- $x_1(s)$ and $x_2(s)$ maximize arbitrageur's profit, given price functions and given trading strategy of informed traders;

- $z_i(s_i, c_i)$ maximizes profit of informed trader i with costs c_i, given price functions and given trading strategy of arbitrageur;

- $p_1(y_1)$ and $p_2(y_1, y_2)$ are such that $p_1 = E[\theta|y_1]$ and $p_2 = E[\theta|y_1, y_2]$ for given trading strategies of the arbitrageur and informed;

where $y_1 = \int\limits_0^\mu z_i di + u_1 + x_1$ and $y_2 = \int\limits_\mu^1 z_i di + u_2 + x_2$ are combined order flows.

Given normal distributions of all random variables, we will be looking for a linear equilibrium. Thus we conjecture that

-prices set by market makers are linear functions of order flows: $p_1 = \lambda_{11} y_1$ and $p_2 = \lambda_{12} y_1 + \lambda_{22} y_2$;

-arbitrageur trading function is linear in his signal: $x_1 = as$ and $x_2 = bs$;

-trading strategy of informed traders active in period one is a linear function of signal s_i: $z_i = \frac{1}{c_i} d_1 s_i$;

-trading strategy of informed traders active in period two is a linear function of signal s_i: $z_i = \frac{1}{c_i} d_2 s_i$.

The coefficients $\{\lambda_{11}, \lambda_{12}, \lambda_{22}, a, b, d_1, d_2\}$ in the linear expressions above need to be consistent with optimality and market clearing conditions. We will now check that this is the case and characterize a linear equilibrium.

Notice that in a linear equilibrium combined informed traders demand is proportional to fundamental value θ. Consider first period informed traders. A fraction μ of informed agents is active in the first period, we label them by $i \in [0, \mu]$ from lowest to largest costs c_i,which we can parametrize by $c_i = \frac{\bar{c} - \underline{c}}{\mu} i + \underline{c}$. The noise in private signal ν_i is independent from costs shock c_i. Total informed order flow in the first period is given by

$$\int\limits_0^\mu z_i di = \int\limits_0^\mu \frac{1}{c_i} d_1 s_i di = d_1 \int\limits_0^\mu \frac{1}{c_i} (\theta + \nu_i) di = d_1 \theta \frac{\mu}{\bar{c} - \underline{c}} \int\limits_{\underline{c}}^{\bar{c}} \frac{1}{c_i} dc_i = \frac{1}{\bar{c} - \underline{c}} ln \left(\frac{\bar{c}}{\underline{c}} \right) \mu d_1 \theta \equiv \mu g_1 \theta. \qquad (1)$$

Thus total informed order flow is proportional to fundamental value θ with coefficient of proportionality g_1. Similarly, informed traders active in the second period trade $\int\limits_\mu^1 z_i di = (1 - \mu) g_2 \theta$ where $g_2 = \frac{1}{\bar{c} - \underline{c}} ln \left(\frac{\bar{c}}{\underline{c}} \right) d_2$. Given that results and also arbitrageur's strategy, combined order flow in a linear equilibrium is given by

$$\begin{aligned} y_1 &= (\mu g_1 + a) \theta + u_1 + a\varepsilon, \\ y_2 &= ((1 - \mu) g_2 + b) \theta + u_2 + b\varepsilon. \end{aligned}$$

Notice that order flow represents a noisy signal about fundamental value θ. Define ρ as coefficient of projection of second period demand on first period demand, thus $E[y_2 | y_1] = \rho y_1$.

First we show that a linear equilibrium indeed exists and present a system of equations that defines it.

Proposition 1. *A unique linear equilibrium exists and is parametrized by* $\{\lambda_{11}, \lambda_{12}, \lambda_{22}, \rho, a, b, g_1, g_2\}$ *that solve the following system of equations*

$$\lambda_{11} = \frac{(\mu g_1 + a)\,\sigma_\theta^2}{(\mu g_1 + a)^2\,\sigma_\theta^2 + \sigma_{u1}^2 + a^2\sigma_\varepsilon^2},$$

$$\rho = \frac{(\mu g_1 + a)\,((1-\mu)g_2 + b)\,\sigma_\theta^2 + ab\sigma_\varepsilon^2}{(\mu g_1 + a)^2\,\sigma_\theta^2 + \sigma_{u1}^2 + a^2\sigma_\varepsilon^2},$$

$$\lambda_{22} = \frac{((1-\mu)g_2 + b - \rho\,(\mu g_1 + a))\,\sigma_\theta^2}{((1-\mu)g_2 + b - \rho\,(\mu g_1 + a))^2\,\sigma_\theta^2 + \sigma_{u2}^2 + \rho^2\sigma_{u1}^2 + (b-\rho a)^2\sigma_\varepsilon^2},$$

$$\lambda_{12} = \lambda_{11} - \rho\lambda_{22},$$

$$a = \frac{2\lambda_{22}\,[1-\lambda_{11}\mu g_1] - \lambda_{12}\,[1-\lambda_{12}\mu g_1 - \lambda_{22}(1-\mu)g_2]}{4\lambda_{22}\lambda_{11} - \lambda_{12}^2}\frac{\sigma_\theta^2}{\sigma_\theta^2 + \sigma_\varepsilon^2},$$

$$b = \frac{2\lambda_{11}\,[1-\lambda_{12}\mu g_1 - \lambda_{22}(1-\mu)g_2] - \lambda_{12}\,[1-\lambda_{11}\mu g_1]}{4\lambda_{22}\lambda_{11} - \lambda_{12}^2}\frac{\sigma_\theta^2}{\sigma_\theta^2 + \sigma_\varepsilon^2},$$

$$g_1 = \frac{1}{2}\frac{1}{\bar{c} - \underline{c}} ln\left(\frac{\bar{c}}{\underline{c}}\right)\frac{[1-\lambda_{11}a]}{\left(1 + \frac{1}{2}\lambda_{11}\mu\frac{1}{\bar{c}-\underline{c}} ln\left(\frac{\bar{c}}{\underline{c}}\right)\frac{\sigma_\theta^2}{\sigma_\theta^2+\sigma_\nu^2}\right)}\frac{\sigma_\theta^2}{\sigma_\theta^2 + \sigma_\nu^2},$$

$$g_2 = \frac{1}{2}\frac{1}{\bar{c} - \underline{c}} ln\left(\frac{\bar{c}}{\underline{c}}\right)\frac{[1-\lambda_{12}\,(\mu g_1 + a) - \lambda_{22}b]\,\frac{\sigma_\theta^2}{\sigma_\theta^2+\sigma_\nu^2}}{1 + \frac{1}{2}\lambda_{22}(1-\mu)\frac{1}{\bar{c}-\underline{c}} ln\left(\frac{\bar{c}}{\underline{c}}\right)\frac{\sigma_\theta^2}{\sigma_\theta^2+\sigma_\nu^2}}.$$

Proof. The full proof is in the appendix (section 12.1) but let me sketch the main steps. In the first period, market makers extract information about θ from y_1, which using the standard formula gives the expression for λ_{11}. In the second period, a market maker observes second period order flow y_2 but does not forget what he learned in period 1. Following Bernhardt and Taub (2008) the projection theorem may be used to find p_2:

$$\begin{aligned} p_2 &= E[\theta|y_1, y_2] \\ &= E[\theta|y_1] + E\left[\theta - E[\theta|y_1]\middle| y_2 - E[y_2|y_1]\right] \\ &= \lambda_{11}y_1 + E\left[\theta\middle| y_2 - E[y_2|y_1]\right]. \end{aligned}$$

Notice that because $E[\theta|y_1]$ is proportional to y_1, and projection error $y_2 - E[y_2|y_1]$ is orthogonal to y_1, the last equality follows. Using ρ defined as coefficient of projection of second period demand on first period demand,$E[y_2|y_1] = \rho y_1$, second period price can be written as $p_2 = \lambda_{11}y_1 + \lambda_{22}\left(y_2 - E[y_2|y_1]\right) = \lambda_{12}y_1 + \lambda_{22}y_2$, where $\lambda_{12} = \lambda_{11} - \rho\lambda_{22}$.

Arbitrageur maximizes

$$E\left[(\theta - p_1)x_1 + (\theta - p_2)x_2|s\right],$$

taking as given the pricing strategy of market makers

$$\begin{aligned} p_1 &= \lambda_{11}y_1 = \lambda_{11}\left(\mu g_1\theta + u_1 + x_1\right), \\ p_2 &= \lambda_{12}y_1 + \lambda_{22}y_2 = \lambda_{12}\left(\mu g_1\theta + u_1 + x_1\right) + \lambda_{22}\left((1-\mu)g_2\theta + u_2 + x_2\right). \end{aligned}$$

Moreover, the arbitrageur knows that his trading x_1 and x_2 has an impact on prices and accounts for that. Therefore, even though the arbitrageur is risk neutral and possesses private information, he limits his trade to limit price impact.

Arbitrageur has a signal about fundamental value θ, therefore his estimate of θ is $\hat{\theta} = E[\theta|s] = \frac{\sigma_\theta^2}{\sigma_\theta^2 + \sigma_\varepsilon^2}s$. Thus the arbitrageur is not only able to assess the fundamental value, but can also estimate the magnitude of informed demand. But the arbitrageur has no information about uninformed demand either in period 1 or in period 2. Therefore, expected profit simplifies to

$$\pi_{arbitrageur} = \left(\left[1 - \lambda_{11}\mu g_1\right]\hat{\theta} - \lambda_{11}x_1\right)x_1 + \left(\left[1 - \lambda_{12}\mu g_1 - \lambda_{22}(1-\mu)g_2\right]\hat{\theta} - \lambda_{12}x_1 - \lambda_{22}x_2\right)x_2.$$

First order conditions imply

$$\begin{aligned} x_1 &= \frac{\left[1 - \lambda_{11}\mu g_1\right]\hat{\theta} - \lambda_{12}x_2}{2\lambda_{11}}, \\ x_2 &= \frac{\left[1 - \lambda_{12}\mu g_1 - \lambda_{22}(1-\mu)g_2\right]\hat{\theta} - \lambda_{12}x_1}{2\lambda_{22}}, \end{aligned}$$

which can be solved to get x_1 and x_2 as functions of $\hat{\theta}$ and thus as functions of s, which gives equilibrium values of a and b.

Assume that signals are such that $\hat{\theta} > 0$. Notice that the arbitrageur buys more if he has a better estimate of the fundamental value, $\hat{\theta}$ is larger; if fundamental traders do not trade much,

g_1 and g_2 are smaller; prices are not sensitive to order flow, λ_{11} and λ_{22} are smaller. On top of that dynamic setup creates an intertemporal tradeoff. If the arbitrageur trades more in the first period, x_1 is larger, then his trade has impact not only on the first period price, but on the second period price as well. □

In order to fully characterize the equilibrium we need to solve that system of equations. However, any linear equilibrium implies particular features of price functions and arbitrageur's strategy. But before that we need to introduce the notion of price informativeness.

Market makers price setting strategy defines prices as functions of order flows. We can rewrite the prices as functions of exogenous shocks, and separate fundamental part from the rest:

$$\begin{aligned} p_1 &= \lambda_{\theta,1}\theta + shocks_1, \\ p_2 &= \lambda_{\theta,2}\theta + shocks_2, \end{aligned}$$

where $\lambda_{\theta,1} = \lambda_{11}(\mu g_1 + a)$ and $\lambda_{\theta,2} = \lambda_{12}(\mu g_1 + a) + \lambda_{22}((1-\mu)g_2 + b)$ denote sensitivity of prices to fundamental value θ.

The question is how well prices reflect fundamental value θ, a measure of informativeness of prices is needed. Imagine that an outsider observes the prices and extracts information about θ. Given normal distributions, conditional expectation of fundamental value θ and conditional variance are given by

$$E[\theta|p_t] = \frac{\lambda_{\theta,t}\sigma_\theta^2}{Var(p_t)},$$

$$Var[\theta|p_t] = \sigma_\theta^2 - \frac{\lambda_{\theta,t}^2\sigma_\theta^4}{Var(p_t)} = \sigma_\theta^2\left(1 - \frac{\lambda_{\theta,t}^2\sigma_\theta^2}{Var(p_t)}\right).$$

If the price does not contain much noise relative to fundamental signal, than the price reveals a lot of information about θ and conditional variance is low, ideally zero. Therefore, one can define informativeness of prices as a ratio of 'fundamental variance' to the total variance, formally

$$I(p_t) = \frac{\lambda_{\theta,t}^2\sigma_\theta^2}{Var(p_t)}.$$

$I(p_t)$ is always greater than zero and smaller than 1. We will say that the price is more informative, if informativeness is larger. But one should not forget that price functions result from market makers extracting information about θ from order flows, thus $\lambda_{\theta,t}$ are endogenous objects. Next proposition shows that actually $I(p_t) = \lambda_{\theta,t}$ and compares price informativeness across periods.

Proposition 2. *In any linear equilibrium*

1. Prices become more informative over time and underreact to fundamental value θ, in particular $I(p_t) = \lambda_{\theta,t}$ and $0 < \lambda_{\theta,1} \leq \lambda_{\theta,2} \leq 1$.

2. The arbitrageur trades in the direction of fundamentals in period 2, or $b > 0$.

3. The arbitrageur and informed traders profit at the expense of uninformed traders.

Arbitrageur may trade against fundamentals in period 1, if g_1 is sufficiently large.

Proof. Full proof can be found in the appendix (section 12.1.1) , but a few things are worth mentioning. Consider first period. Let's find $I(p_1)$, given that $p_1 = \lambda_{11} (\mu g_1 + a) \theta + \lambda_{11} (u_1 + a\varepsilon)$. Indeed, λ_{11}^2 cancels out and

$$I(p_1) \;=\; \frac{(\mu g_1 + a)^2 \sigma_\theta^2}{(\mu g_1 + a)^2 \sigma_\theta^2 + \sigma_{u1}^2 + a^2 \sigma_\varepsilon^2} = \lambda_{11} (\mu g_1 + a),$$

where we use the formula for λ_{11} from proposition 1. Therefore, $I(p_1) = \lambda_{\theta,1} = \lambda_{11} (\mu g_1 + a) < 1$. As long as there is some uninformed trading, market makers would be afraid to mix up uninformed demand with informed trading, and thus would never fully react to order flow. Underreaction is a standard result in any kind of signal extraction problems. Next, intuitively, as market makers do not forget information, the second period price which is $E[\theta|y_1, y_2]$, ends up being more informative than the first period price.

The intuition for the arbitrageur's trading behavior is the following. Consider the case in which the arbitrageur gets a perfect private signal about fundamental value θ. And assume that the arbitrageur trades $a^*$and b^* in equilibrium. Given that in any equilibrium $0 < \lambda_{\theta,1} \leq \lambda_{\theta,2} \leq 1$, if the arbitrageur trades against fundamentals in any period, if $a^* < 0$ or $b^* < 0$, then he must experience losses from trading in that period. Imagine that fundamentals are good, $\theta > 0$, and the arbitrageur knows it, as his signal is perfect. Given that $\lambda_{\theta,t} < 1$, the price in period t is such that asset on average is underpriced. Hence, if the arbitrageur sells underpriced asset in accordance with trading against fundamentals strategy, then he will experience losses when θ is revealed. Thus the question arises if we can observe trading against fundamentals in any period.

Assume that $b^* < 0$. But trading in the second period does not affect profit earned in the first period. Thus for the arbitrageur does not make any sense to trade against fundamentals in the second period and earn negative profit. Instead he may stop trading in period two, make $b = 0$, and increase his total profit. Thus we cannot observe $b^* < 0$ in equilibrium.

However, we may observe $a^* < 0$. If g_1 is sufficiently large, than the arbitrageur may trade against fundamentals in the first period. If there are a lot of informed traders in the first period, they may trade too intensely based on fundamental information and push the first period price, and hence, second period price, too much in the direction of fundamentals. Therefore, the arbitrageur may actually find it optimal to experience losses in the first period, trading against fundamentals, but achieve positive profit in the second period. If the arbitrageur does not have good information about θ, the results are still true.

Finally, apart from transaction costs, trading is a zero sum game. Market makers by construction earn zero expected profit. Therefore, any positive trading profit that the arbitrageur and informed traders obtain, should come from trading losses experienced by uninformed traders. □

So far we explored some features of the solution that would be true in any equilibrium. Next we will introduce the new instrument to show how that changes the system of equations that defines equilibrium, and the we will solve both and assess how introduction of observable instrument changes the equilibrium.

3.2 Model with the new instrument

Both informed and uninformed traders pay transaction costs, and both have idiosyncratic components in their demand. Idiosyncrasy in fundamental traders demand originates from noise in private signals, whereas idiosyncrasy in uninformed demand originates from idiosyncratic trading needs. However, when market maker collects submitted orders, that idiosyncratic components get washed out. Thus there is scope for intermediation: intermediary may internalize order flow to eliminate idiosyncratic parts, before routing that order flow to market makers. However, we assume that internalization is forbidden in the risky asset market. To overcome that a new instrument is introduced to the market that replicates the same trading strategy, but is traded on a different exchange that allows for internalization.

The new asset A new asset is introduced and is traded on a separate market. We assume that a new asset is designed to replicate a strategy of purchase of one unit of original risky asset in the second period. The new asset offers expected value of the cash flow, and thus simply equivalent to a forward contract on a risky asset. A purchase of one unit of the new instrument at specified price f in period 1, gives its owner one unit of the original asset in period 2. The new asset is in zero supply, as in the case of any derivative. The trading procedure is the same as in the market of θ-asset. Mainly, competitive uninformed risk neutral market makers collect market orders for the new derivative, and set execution price f efficiently given information they observe, which is total demand for a new asset, y_f, and absorb the net demand for the new asset. Thus if $y_f > 0$ then market maker sells y_f units of new asset to his customers and collects fy_f. In the second period market maker goes to the market of the original asset, buys y_f units of the original asset and delivers to his customers, while doing this he pays $p_2 y_f$. Hence his profit is the difference between the two: $(f - p_2)y_f$. As before we can either directly assume price efficiency so that $f = E[p_2|y_f]$, or we can say that market makers maximize expected profit by choosing the price strategy $f(y_f)$,$U = \underset{f(y_f)}{max} E[(f - p_2)y_f|y_f]$. and assume that Bertran competition drives competitive price f to expected value of second period spot price.

Broker as an intermediary Now we can describe the role of an intermediary represented by a single monopolistic broker. Contrary to the original risky asset, all traders that want to invest in the new instrument, may use the services of the broker. The broker collects orders and cross-matches them, and sends to the market (submits orders at the market for the new instrument) only the net demand, netting all idiosyncratic parts out. For his services broker sets a transaction price w for a unit of squared order flow to maximize his profit, defined by the difference between what other agents pay him and what he pays in transaction costs when he uploads the net demand to the market[7]. We assume that broker faces costs c_b. Both because broker may be more efficient that some of the traders, $c_b < \bar{c}$, and because he may internalize the order flow and thus avoid paying extra transaction costs, broker will be able and willing to offer a price that would be smaller than what agents can achieve if they trade the risky asset, $w < \bar{c}$.

Now all agents active in the second period can choose whether to trade a new instrument or continue trading the risky asset as before. We assume that the decision has to made before the signals are realized. The solution will have the threshold form, where agents with largest costs will trade the new instrument. Consider again all informed agents active in the second period, that are labeled by $i \in [\mu, 1]$, where larger i means larger costs c_i. Then all agents with largest

[7]Thus we are not looking for an optimal arbitrary payments schedule that a broker may possibly offer to his clients, but simply assume that he offers a transaction costs w similar to the ones agents face themselves on the market of the risky asset.

costs, starting from i^*, would prefer to trade a new instrument. Similarly let's label by $j \in [0,1]$ uninformed agents active in the second period. Again, all agents with largest costs trade the new instrument, starting from j^*. Broker optimally chooses w solving

$$E\pi_{broker} = \max_{w} \left\{ wE\left[\left(\int_{i^*}^{1} z_i^2 di + \int_{j^*}^{1} (u_t + \xi_{j,t})^2 dj\right)\right] - c_b E\left[\left(\int_{i^*}^{1} z_i di + \int_{j^*}^{1} (u_t + \xi_{j,t}) dj\right)^2\right]\right\}.$$

First term represents the combined payment made by informed and uninformed traders. For example, all informed traders that have larger costs than agent i^*has, choose the new instrument, trade z_i and pay wz_i^2 to the broker. When we integrate over them we get $w\int_{i^*}^{1} z_i^2 di$ in payments to the broker. Similarly for uninformed. Second term represents the transaction costs that the broker itself will have to pay in order to trade the imbalance on the market for the new asset. Both orders from fundamental traders and uninformed traders have correlated component and thus do not net out, and a broker is left with an imbalanced position that he has to bring back to the market, suffering costs proportional to c_b and magnitude of imbalanced. Combined order flow that is not matched is whatever left after integration: $\int_{i^*}^{1} z_i di + \int_{j^*}^{1} (u_t + \xi_{j,t}) dj$, thus he pays c_b for that order squared. So what is left?

First, idiosyncratic part of the demand is internalized $\int_{j^*}^{1} \xi_{j,t} dj = 0$, and also a part of $\int_{i^*}^{1} z_i di$ proportional to ε_i will be cross-matched as well. Assume that fraction $\varkappa_\theta = \dfrac{1-i^*}{1-\mu}$ of informed hedgers and fraction $\varkappa_{u2} = 1-j^*$ of uninformed traders chooses new instrument. Then, aggregating all idiosyncratic parts, using independence of uninformed demand from fundamentals and getting population means we get

$$E\pi_{broker}(w) = \max_{w} \left\{ wE\left[\int_{i^*}^{1} z_i^2 di\right] + w\varkappa_{u2}\left(\sigma_{u2}^2 + \sigma_\xi^2\right) - c_b\left(E\left(\int_{i^*}^{1} z_i di\right)^2 + \varkappa_{u2}^2\sigma_{u2}^2\right)\right\}.$$

Fractions of agents choosing the new instrument Let's now find the volumes traded on each exchange for a given value of w. If an agent with costs c_i and signal s_i trades the risky asset, he trades $z_i(s_i)$ that solves

$$\pi_{asset}^f(s_i, c_i) = \max_{z_i(s_i)} \left\{E[(\theta - p_2)|s_i]z_i - c_i z_i^2\right\},$$

if instead he trades the instrument

$$\pi_{instrument}^f(s_i, c_i) = \max_{z_i(s_i)} \left\{E[(\theta - f)|s_i]z_i - w z_i^2\right\}.$$

Solving both problems, we get $z_i = \dfrac{E[(\theta - p_2)|s_i]}{2c_i}$ if asset is traded and $z_i = \dfrac{E[(\theta - f)|s_i]}{2w}$ if instrument is traded. That strategy assures the following profit: $\pi^f_{asset}(s_i, c_i) = \dfrac{(E[(\theta - p_2)|s_i])^2}{4c_i}$ and $\pi^f_{instrument}(s_i, c_i) = \dfrac{(E[(\theta - f)|s_i])^2}{4w}$.

A fundamental trader has to choose the venue before the signal s_i is realized, thus he has to compare expected profit. Therefore, if equilibrium is interior, there exists a threshold value of costs

$$c_{i^*} = w\frac{E\left[(E[(\theta - p_2)|s_i])^2\right]}{E\left[(E[(\theta - f)|s_i])^2\right]} \equiv wA_1,$$

such that all agents with $c_i \geq c_{i^*}$ choose the instrument, constituting $\varkappa_\theta = \dfrac{\bar{c} - wA_1}{\bar{c} - \underline{c}}$ fraction of informed traders, whereas the rest prefer to trade the risky asset.

Uninformed traders that are active in the second period, cannot choose how much to trade, but can minimize expected losses to informed traders and the arbitrageur by choosing the better trading venue. If a trader trades the asset he expects to loose

$$E\pi^n_{asset}(c_j) = E\left[(\theta - p_2)(u_2 + \xi_j) - c_j(u_2 + \xi_j)^2\right],$$

if instead new instrument is preferred, than

$$E\pi^n_{instrument}(, c_j) = E\left[(\theta - f)(u_2 + \xi_j) - c_j(u_2 + \xi_j)^2\right].$$

Monotonicity implies that in interior equilibrium all agents with costs larger than c_{j^*}defined by $c_{j^*} = w + \dfrac{E[(f - p_2)u_2]}{\sigma^2_{u2} + \sigma^2_\xi} \equiv w + B_0$ prefer to trade the new instrument, and that adds up to $\varkappa_{u2} = \dfrac{\bar{c} - w - B_0}{\bar{c} - \underline{c}}$ fraction of uninformed agents active in second period that switch to the new instrument. Given that hedgers do not choose how much to trade it is easy to find order flow for the new instrument, $\varkappa_{u2}u_2$, and for the asset in the second period $(1 - \varkappa_{u2})u_2$, as usual idiosyncratic components are "washed out" in the aggregate.

Arbitrageur Final and the most important change is that now the arbitrageur is able to observe the investment in the new instrument, y_f, in addition to his private signal y_f, thus he now solves

$$\pi_{arbitrageur} = \max_{x_1(s,y_f), x_2(s,y_f)} E\left[(\theta - p_1)x_1 + (\theta - p_2)x_2 | s, y_f\right].$$

Timeline 1. Broker chooses w and agents choose whether to trade the new instrument, $\varkappa_\theta$ and $\varkappa_{u2}$ are defined.

2. Private signals are received.
3. Instrument is traded, y_f and f are defined.
4. The arbitrageur observes y_f.
5. First round of trading of risky asset, y_1 and p_1 are defined.
6. Second round of trading of risky asset, y_2 and p_2 are defined.
7. Payoff $\theta$is realized.

Definition. An equilibrium is a combination of price functions $p_1(y_1)$, $p_2(y_1, y_2)$, $f(y_f)$, trading strategy of the arbitrageur $x_1(s, y_f)$ and $x_2(s, y_f)$, trading strategy of informed traders $z_i(s_i, c_i)$ that choose the risky asset, trading strategy of informed traders $z_i(s_i, w)$ that choose the new instrument, choice of trading venue done by each agent, and price of the broker w such that

- $x_1(s, y_f)$ and $x_2(s, y_f)$ maximize arbitrageur's profit;
- $z_i(s_i, c_i)$ maximizes profit of informed trader i with costs c_i that trades original asset;
- $z_i(s_i, w)$ maximizes profit of informed trader i that trades the new asset;
- $p_1(y_1)$ and $p_2(y_1, y_2)$ are such that $p_1 = E[\theta|y_1]$ and $p_2 = E[\theta|y_1, y_2]$;
-$f(y_f)$ is such that $f = E[p_2|y_f]$
- $y_1 = \int\limits_0^\mu z_i di + u_1 + x_1$ and $y_2 = \int\limits_{i^*}^1 z_i di + u_2 + x_2 + y_f$ are combined order flow for the risky asset;
- $y_f = \int\limits_{i^*}^1 z_i di + \varkappa_{u2} u_2$ order flow for the new instrument;
- $\forall i \in [\mu, 1]$ and $\forall j \in [0, 1]$ choice of the new instrument is optimal;
- w maximizes broker's expected profit.

Given normal distributions of all random variables, we will be looking for a linear equilibrium. Thus we conjecture that

-prices set by market makers are linear functions of order flows: $p_1 = \lambda_{11} y_1$ and $p_2 = \lambda_{12} y_1 + \lambda_{22} y_2$, and $f = \lambda_f y_f$;

-arbitrageur trading function is linear in his signal: $x_1 = a_1 \frac{1}{\varkappa} y_f + a_2 s$ and $x_2 = b_1 \frac{1}{\varkappa} y_f + b_2 s$;

-trading strategy of informed traders active in period one is a linear function of signal s_i: $z_i = \frac{1}{c_i} d_1 s_i$;

-trading strategy of informed traders active in period two is a linear function of signal s_i: $z_i = \frac{1}{c_i} d_2 s_i$;

-trading strategy of informed traders active in period two who trades the new instrument is a linear function of signal s_i: $z_i = \frac{1}{w} d_f s_i$.

The coefficients $\{\lambda_{11}, \lambda_{12}, \lambda_{22}, \lambda_f, a_1, a_2, b_1, b_2, d_1, d_2, d_f\}$ in the linear expressions above need to be consistent with optimality and market clearing conditions.

Investment in the new instrument y_f Now we can combine the informed agents with uninformed agents and get the total demand for the new instrument.

Thus we denote $d_f = \dfrac{E[(\theta - f)|s_i]}{2s_i}$ - and the order flow from informed traders for the instrument will be

$$\int_{i^*}^{1} z_i di = \int_{i^*}^{1} \frac{1}{w} d_f s_i di = \frac{1-\mu}{\bar{c}-\underline{c}} \frac{1}{w} d_f \int_{wA_1}^{\bar{c}} (\theta + \nu_i)\, dc_i = \frac{1}{w} d_f \frac{\bar{c} - wA_1}{\bar{c}-\underline{c}} (1-\mu)\theta = \varkappa_\theta \frac{1}{w} d_f (1-\mu)\theta,$$

and therefore

$$y_f = \int_{i^*}^{1} z_i di + \varkappa_{u2} u_2 = \varkappa_\theta \frac{1}{w} d_f (1-\mu)\theta + \varkappa_{u2} u_2,$$

which for the sake of simplification can be rewritten as $y_f = \varkappa\left((1-\mu) g_f \theta + u_2\right)$, where $g_f = \dfrac{\varkappa_\theta}{\varkappa_{u2}} \dfrac{1}{w} d_f$ and $\varkappa_{u2} = \varkappa$.

Order flow for the risky asset in the second period We also need to recalculate order flow for the risky asset in the second period. Orders submitted will include arbitrageurs demand, x_2; combined demand of informed and uninformed traders that decided to trade the risky asset, $\int_{\mu}^{i^*} z_i di$ and $(1-\varkappa_{u2})u_2$, and also the demand that market makers from the market for the new instrument, y_f. First, using $d_2 = \dfrac{E[(\theta - p_2)|s_i]}{2s_i}$, order flow from informed traders for the asset in the second period is

$$\int_{\mu}^{i^*} z_i di = \int_{\mu}^{i^*} \frac{1}{c_i} d_2 s_i di = \frac{1-\mu}{\bar{c}-\underline{c}} d_2 \int_{\underline{c}}^{wA_1} \frac{1}{c_i} (\theta + \nu_i)\, dc_i = \frac{1}{\bar{c}-\underline{c}} d_2 ln \left(\frac{wA_1}{\underline{c}}\right) (1-\mu)\theta.$$

Thus

$$\begin{aligned} y_2 &= y_f + \frac{1}{\bar{c}-\underline{c}} d_2 ln \left(\frac{wA_1}{\underline{c}}\right) (1-\mu)\theta + (1-\varkappa_{u2})u_2 + x_2 \\ &= \left(\varkappa_\theta \frac{1}{w} d_f + \frac{1}{\bar{c}-\underline{c}} d_2 ln \left(\frac{wA_1}{\underline{c}}\right)\right) (1-\mu)\theta + u_2 + x_2. \end{aligned}$$

Simplifying again the order flow can be written as before

$$y_2 = (1-\mu) g_2 \theta + u_2 + x_2.$$

Thus order flow for the asset equals

$$\begin{aligned} y_1 &= \left(\mu g_1 + a_1(1-\mu) g_f + a_2\right)\theta + u_1 + a_1 u_2 + a_2 \varepsilon, \\ y_2 &= \left((1-\mu) g_2 + b_1(1-\mu) g_f + b_2\right)\theta + (1+b_1) u_2 + b_2 \varepsilon. \end{aligned}$$

Information extraction y_f Notice, that investment in the new instrument, $y_f = \varkappa\left((1-\mu)g_f\theta + u_2\right)$, and thus represents a mixed signal about both fundamental value θ and future uninformed demand u_2. Therefore, observing y_f the arbitrageur may estimate both:

$$
\begin{aligned}
E[\theta|y_f, s] &= i_{\theta y}y_f + i_{\theta s}s, \\
E[u_2|y_f, s] &= i_{uy}y_f + i_{us}s,
\end{aligned}
$$

where $i_{\theta y}, i_{\theta s}, i_{uy}, i_{sy}$ can be found using standard formulas [8].

Proposition 3. *For a given w a unique linear equilibrium exists.*

Proof. The proof can be found in the appendix (section 12.2) . □

The system of equations is similar to the one in proposition 1. But one also needs to find an equilibrium value of w. The particular value of w leads to the investment in the fund to be a particular mix of signals about fundamental value and future demand shock and defines an equilibrium.

Given the complexity of the equilibrium we will study the effect of the introduction of the new instrument in three cases, depending on whether we allow for informed traders or not.

Case I: no informed traders

Case II: informed traders only in the first period

Case III: informed trader in both periods, but only uninformed may invest in the new instrument

Case IV: general case

Absence of informed traders in the pool of possible investors in the instrument separates brokers problem from the rest of the model. Absence of informed traders in the first period simplifies finding fundamental trading strategy of arbitrageur.

4 Results when there are no informed traders (case I)

Let's assume that there are no informed traders, which means that $g_1 = g_2 = g_f = 0$. In that case the arbitrageur is the only one who possesses information about θ, and when the new instrument is introduced he perfectly observes u_2.

In that case, both system of equations corresponding to the model with and without the new instrument can be solved. The solution method implies writing everything as a function of ρ, where ρ is defined as $E[y_2|y_1] = \rho y_1$, and finding the equation that relates ρ with the ratio $\dfrac{\sigma_{u1}^2}{\sigma_{u2}^2}$.

Proposition 4. *Introduction of the new instrument changes arbitrageur's behavior.*

1. The arbitrageur front-runs, $a_1 > 0$ and $b_1 < 0$, and

a) $\dfrac{1}{2} < |b_1| < 4 - 2\sqrt{3}$, the arbitrageur never fully offsets the shock;

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[8] $i_\theta = \dfrac{1}{\varkappa}\dfrac{(1-\mu)g_f\sigma_\theta^2}{(1-\mu)^2 g_f^2\sigma_\theta^2 + \sigma_{u2}^2}$; $i_{\theta s} = \dfrac{(1 - i_\theta\varkappa(1-\mu)g_f)\,\sigma_\theta^2}{(1 - i_\theta\varkappa(1-\mu)g_f)^2\,\sigma_\theta^2 + i_\theta^2\varkappa^2\sigma_{u2}^2 + \sigma_\varepsilon^2}$; $i_{\theta y} = i_\theta\,(1 - i_{\theta s})$

$i_{u2} = \dfrac{1}{\varkappa}\dfrac{\sigma_{u2}^2}{(1-\mu)^2 g_f^2\sigma_\theta^2 + \sigma_{u2}^2}$, $i_{us} = \dfrac{-\,i_\theta\varkappa\sigma_{u2}^2}{(1 - i_\theta\varkappa(1-\mu)g_f)^2\,\sigma_\theta^2 + i_\theta^2\varkappa^2\sigma_{u2}^2 + \sigma_\varepsilon^2}$; $i_{uy} = i_{u2} - i_{us}i_\theta$

b) $a_1 < \dfrac{7-\sqrt{45}}{2}$, *the arbitrageur limits the noise induced in the first period order flow. Therefore, front-running is limited.*

2. The arbitrageur increases fundamental trading in period 1 and decreases period 2, $a < a_2$ *and* $b > b_2$*. The arbitrageur continues to trade in the direction of fundamentals in both periods*

Proof. The full proof can be found in the appendix(section 12.3), it involves solving for equilibrium in both cases with and without the new instrument(sections 12.3.1 and 12.3.2) and comparing the solutions(section 12.3.3). We would like to highlight the main points.

Without the new instrument ρ is defined by

$$\frac{\sigma_{u1}^2}{\sigma_{u2}^2} = \frac{3\sqrt{9+8\rho}-7-4\rho}{8\rho^2}. \tag{2}$$

and arbitrageur's strategy can be written as

$$\begin{aligned} a &= \frac{1-\sqrt{1-4\rho^2\dfrac{\sigma_{u1}^2}{\sigma_{u2}^2}}}{2\rho}\frac{1}{\sqrt{1+\sigma_\varepsilon^2}}\sigma_{u2}, \\ b &= \frac{1}{\sqrt{1+\sigma_\varepsilon^2}}\sigma_{u2}. \end{aligned}$$

Notice that the arbitrageur indeed trades in the direction of fundamentals in both periods: $b > 0$, and it can also be shown that always $a > 0$. [9]

With the new instrument ρ can be found from

$$\frac{\sigma_{u1}^2}{\sigma_{u2}^2} = \frac{(2\rho+1)^2(2\rho+3)}{16\rho^2(\rho+1)^2(4\rho+3)}. \tag{3}$$

Arbitrageur's strategy is then given by $a_2 = \dfrac{1}{2\rho\bar{C}_a}\dfrac{1}{\sqrt{1+\sigma_\varepsilon^2}}\sigma_{u2}$, $b_2 = \dfrac{1}{\bar{C}_a}\dfrac{1}{\sqrt{1+\sigma_\varepsilon^2}}\sigma_{u2}$, $a_1 = \dfrac{(1+2\rho)}{2(4\rho+3)(\rho+1)}$, $b_1 = -\dfrac{(1+2\rho)(3+2\rho)}{2(4\rho+3)(\rho+1)}$, where $\bar{C}_a \approx 2$.[10]

Arbitrageur front-runs on knowledge of future u_2: $a_1 > 0$ and $-1 < b_1 < 0$. Intuition behind front-running is the following. Let's abstract for now from fundamental trading, namely let's assume that realized $\theta = 0$ and realized $u_1 = 0$. For definiteness also assume that $u_2 > 0$, uninformed traders will buy risky asset in the second period. When the demand shock comes to the market in period 2, market makers partially mix it up with fundamental trading and thus respond by increasing p_2, thus if the arbitrageur does not interfere the price will be set up at $p_2 = \lambda_{22}u_2$. But if the arbitrageur knows u_2 in advance he may actually decide to sell a

[9] $1-\sqrt{1-4\rho^2\dfrac{\sigma_{u1}^2}{\sigma_{u2}^2}} = \dfrac{5-\sqrt{9+8\rho}}{2} > 0$ because $\rho < 2$, which follows from $\dfrac{\sigma_{u1}^2}{\sigma_{u2}^2} = \dfrac{3\sqrt{9+8\rho}-7-4\rho}{8\rho^2} > 0$

[10] More precisely $\bar{C}_a$ solves $\bar{C}_a^2 = \dfrac{4(\rho+1)^2(4\rho+3)^2}{(4\rho^2+6\rho+3)(2\rho+3)(2\rho+1)}$ but it can be shown that it is almost constant and equals ~ 4.

few units to uninformed traders in period two but less than a full amount, and profit from the remaining liquidity driven push up in price, which now is smaller because the arbitrageur reduced the demand for the asset $p_2 = \lambda_{22}(1+b_1)u_2 < \lambda_{22}u_2$. That gives the arbitrageur the profit of $E(\theta - p_2|s, u_2) = -\lambda_{22}(1+b_1)b_1u_2^2 > 0$. The arbitrageur offsets the shock u_2 in the second period but only partially.

But that explains why $-1 < b_1 < 0$. Why does he trade in period one? Notice, that if the arbitrageur trades a little bit in period on 1, buys a few units, he will push up both prices in period 1 and 2, as market makers seeing larger order flow partially attribute it to better fundamentals. Hence, given fixed b_1, if in addition the arbitrageur buys $a_1u_2$units in period one, $p_1 = \lambda_{11}a_1u_2$ and $p_2 = \lambda_{12}a_1u_2 + \lambda_{22}(1+b_1)u_2$. Push up in first period price is unfortunate as it brings extra losses, however, push up in second period price allows to sell contracts even at larger price. Indeed, combined profit $E(\theta - p_1|s, u_2) + E(\theta - p_2|s, u_2) = -\lambda_{11}a_1^2u_2^2 - \lambda_{12}a_1b_1u_2^2 - \lambda_{22}(1+b_1)u_2b_1u_2$, where $-\lambda_{11}a_1^2u_2^2 < 0$ but $-\lambda_{12}a_1b_1u_2^2 > 0$. Notice, that extra losses are proportional to a_1^2, whereas extra benefits are proportional to a_1. Thus optimal $a_1^* > 0$ - the arbitrageur manipulates the first period price to get a better deal in the second period.

It should also be noted that front-run is limited. Although the arbitrageur offsets more than half of the shock, which would be the case in a one period setup[11], he never offsets more than $4 - 2\sqrt{3} \approx 0.54$. Similarly the arbitrageur never introduces more than $\dfrac{7-\sqrt{45}}{2} \approx 0.15$ noise in the first period. Thus the arbitrageur limits front-running activity.

Finally, the arbitrageur continues to trade in the direction of fundamentals in both periods $a_2 > 0$ and $b_2 > 0$ when new instrument is introduced. But, $b_2 < b$, thus the arbitrageur decreases fundamental trading in the second period roughly by one half. That is because as we have seen, the arbitrageur decreases the noise in the second period also roughly by one half. Moreover, as $a_1 > 0$ the arbitrageur introduces additional noise in the first period. As a result it can be shown that this additional noise allows the arbitrageur to stronger trade based on fundamentals in the first period, $a_2 > a$. □

We would like to assess whether introduction of the fund is socially beneficial. There are two criteria: i) informativeness of prices; ii) trading losses and transaction costs paid by uninformed. We have seen that introduction of the new instrument changes the behavior of arbitrageur: he front-runs on his knowledge of future uninformed trade, increases fundamental trading in the first period, and decreases fundamental trading in the second period. How does that affect price informativeness? Does it increase trading losses of uninformed? And if yes, does instrument allow to economize on costs enough to cover trading losses? The following proposition states the main positive result of the paper:

Proposition 5. *Introduction of the new instrument is considered beneficial.*

1. Prices become more informative in both periods.

2. New instrument allows uninformed investors economize on costs more than they lose while trading with more informed arbitrageur, if $\bar{c}\sigma_{u2}$ is sufficiently large.

Proof. The full proof can be found in the appendix (section 12.3.3).

[11] In a one period setup if the arbitrageur knows liquidity shock, he offsets exactly half of it. In response market makers set λ that is twice as large, also forcing the arbitrageur to decrease fundamental trading by one half. At the end although trading strategies are different, informativeness and losses are exactly the same.

1. In the appendix we derive the expressions for price informativeness. In the case without the new instrument price informativeness is given by

$$\begin{aligned} I(p_1) &= \frac{5-\sqrt{9+8\rho}}{4}\frac{1}{1+\sigma_\varepsilon^2}, \\ I(p_2) &= \frac{4}{3+\sqrt{9+8\rho}}\frac{1}{1+\sigma_\varepsilon^2}, \end{aligned}$$

and when instrument is available,

$$\begin{aligned} I(p_1) &= \frac{1}{2}\frac{(2\rho+3)}{(4\rho+3)}\frac{1}{1+\sigma_\varepsilon^2}, \\ I(p_2) &= 2\frac{\rho+1}{(4\rho+3)}\frac{1}{1+\sigma_\varepsilon^2}. \end{aligned}$$

Regardless of the presence of the new instrument, the following is true. First, prices are more informative when the arbitrageur gets a better private signal, because as we have seen before ρ does not depend on σ_ε^2. Being more informed, the arbitrageur trades more aggressively based on his private information about θ. Second, prices become more informative over time, $I(p_2) > I(p_1)$, partially because market makers do not forget and learn more information over time. Third, if $\frac{\sigma_{u1}^2}{\sigma_{u2}^2}$ increase, or equivalently if ρ decreases, prices in both periods become more informative. If there is more uninformed demand in period 1, the arbitrageur reallocates fundamental trading towards period 1. As a result information is revealed early on and both current and future prices tend to be more informative. The observation that extra fundamental trading in period 1 adds to price informativeness, also helps to understand why price informativeness increases when instrument is present.

Consider the limiting case and $\sigma_\varepsilon^2 = 0$. Equation 2shows that when $\frac{\sigma_{u1}^2}{\sigma_{u2}^2} \to 0$, $\rho \to 2$. As ratio of noises goes to zero, $I(p_1) \to 0$, because the arbitrageur ceases trading based on fundamentals in period 1, $a \to 0$, and first period price become completely uninformative[12]. In contrast, equation 3 shows that in the case with the new instrument as the ratio of noises goes to zero, $\rho \to \infty$, which implies that $I(p_1) \to \frac{1}{2}$. When the new instrument is present, informativeness of the first period price is bounded below. Intuitively, additional noise that the arbitrageur brings by front-running, allows him to trade based on fundamentals, even if there are no uninformed traders in period 1 to camouflage his orders. And hence price informativeness in period 1 is larger when instrument exists. More informative first period price adds to informativeness of the second period price, even though the arbitrageur substantially cuts fundamental trading in period 2 due to front-running. Figure 1 shows informativeness of prices as a function of $\frac{\sigma_{u1}^2}{\sigma_{u2}^2}$ with and without the new instrument

[12]we show in the appendix that when $\frac{\sigma_{u1}^2}{\sigma_{u2}^2} \to 0$, $\rho \to 2$ when there is no instrument, whereas $\rho \to \infty$ when there is instrument

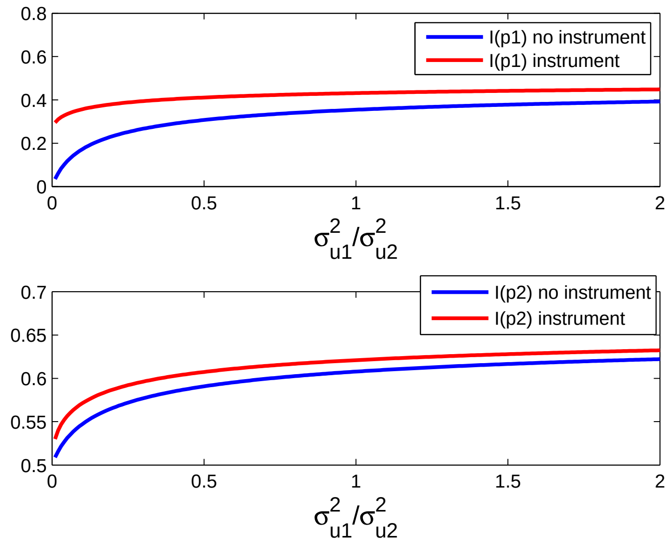

Figure 1: Informativeness of prices in each period as a function of $\frac{\sigma_{u1}^2}{\sigma_{u2}^2}$, given $\sigma_\varepsilon^2 = 0$. Blue line corresponds to the case without the new instrument, red line - to the case with the new instrument.

for $\sigma_\varepsilon^2 = 0$. It illustrates the fact that both prices become more informative when instrument is available.

2. The order flow for the new instruments perfectly reveals u_2 to market makers on the market for the new instrument. Hence, trading losses of uninformed are the same no matter whether they invest in the new instrument or in the underlying risky asset. Market makers are able to perfectly predict the contribution of u_2 to the second period price. Indeed, as $f = E[p_2|y_f = u_2]$, then expected trading losses of the uninformed investors in the new instrument equal $E[-fu_2] = E[-E[p_2|u_2]u_2] = E[-p_2u_2]$.

Expected trading losses of the uninformed in the case without the new instrument are given by

$$E[-p_2u_2] = -\lambda_{22}\sigma_{u2}^2 \quad = \quad -\frac{2(1+\rho)}{3+2\rho+\sqrt{9+8\rho}}\frac{1}{\sqrt{1+\sigma_\varepsilon^2}}\sigma_{u2},$$

with instrument

$$E[-p_2u_2] = E[-fu_2] = -\left[\lambda_{12}a_1 + \lambda_{22}\left(1+b_1\right)\right]\sigma_{u2}^2 = -\bar{C}_a\frac{(2\rho+1)^2(2\rho+3)}{2(4\rho+3)^2(\rho+1)}\frac{1}{\sqrt{1+\sigma_\varepsilon^2}}\sigma_{u2}. \quad (4)$$

Notice three channels through which information revelation associated with the introduction of the new instrument affects the welfare of investors. First, as $b_1 < 0$, arbitrageur provides liquidity or partially offsets the large demand shock and associated price impact and thus increases welfare of investors. Second, $a_1 > 0$, arbitrageur front-runs, market interprets that as fundamental trading and prices in both periods tend to move in the direction of the demand shocks. That decreases welfare of investors. Finally, through general equilibrium effects λ_{12} and λ_{22} change and that also affects welfare of investors. Notice also, that introduction of the fund affects not only investors in the new financial instrument, but traders that keep using the underlying asset.

Of course, we also need to account for the transaction costs. When the new instrument is introduced all agents with relatively large costs are now able to trade the new instrument and thus save. At the same time the arbitrageur can now front run on knowledge of future u_2, and that changes the equilibrium and as a result uninformed traders experience larger trading losses. In the appendix we derive the expression that shows how much all uninformed agents taken together save on costs, Δ^{+}_{costs}, and how much they have to lose from trading Δ^{-}_{trade} when new instrument is introduced. We show that no matter what parameters are, the losses from trading cannot exceed a particular amount: $\Delta^{-}_{trade} \leq \bar{R}\sigma_{u2}$, where $\bar{R} = 0.01$. Thus although present, trading losses are relatively small. Savings on costs, however, are larger if there is more idiosyncrasy in uninformed demand; if broker has smaller own costs and if costs of uninformed are larger. Denote $\sigma^2_\xi \equiv \alpha\sigma^2_{u2}$, $c_b \equiv \beta\bar{c}$, and assume that $\underline{c} = 0$. Then savings can be written as $\Delta^{+}_{costs} = (1-\omega)^2 \frac{\bar{c}}{2}(1+\alpha)\,\sigma^2_{u2}$. Therefore, if $\bar{c}\sigma_{u2}$ is larger than some threshold value, than instrument is beneficial for uninformed investors. Notice, that we normalized $\sigma^2_\theta = 1$, and thus extremely low value of $\bar{c}$ is enough to make the instrument beneficial. □

5 Results when there are informed traders in period 1 only (case II)

Let us now assume that there are fundamental traders active in period 1, which means that $g_1 > 0$. As before $g_2 = g_f = 0$ and when the new instrument is introduced, the arbitrageur is able to perfectly observe u_2. We start with analyzing how the presence of informed traders shapes arbitrageur's fundamental trading in the case with no instrument. We consider a particular case in which all informed agents get perfect signals and analytical solution can be found. Then we show some features of the equilibrium that continue to be true for arbitrary precision of arbitrageur's signal. In particular, we study what happens with intensity of fundamental trading as $\sigma^2_\varepsilon \to \infty$. Then we show the solution in the case with the instrument, still assuming $\sigma^2_\varepsilon \to \infty$. The limiting case is insightful for the following reason: it is the case with the strongest front-running. When instrument is present, the arbitrageur has a tradeoff between fundamental trading and front-running. The front-running as it involves substantial cut of uninformed demand shocks in period 2, forces the arbitrageur to cut fundamental trading in period 2, and only partially reallocate it to period 1. Thus front-running makes fundamental trading less attractive, which is especially true if the arbitrageur has to compete with fundamental traders. As $\sigma^2_\varepsilon \to \infty$, the arbitrageur cares less and less about fundamental trading, not being able to do both, predict fundamental value of the asset and predict total demand of fundamental traders. And thus, we expect to see the arbitrageur switching away from fundamental trading to extensive front-running. The presence of fundamental traders plays a crucial role, as it both, allows and forces the arbitrageur to front-run strongly. We will show the consequences of front-running in the limiting case, that will allow us to claim that the introduction of the new instrument may be detrimental when front-running is strong.

Assume that $\sigma^2_\varepsilon = 0$ and $\sigma^2_\nu = 0$, so that the arbitrageur and informed traders are able to get perfect signals about fundamentals. As before, we express all variables as functions of ρ and then find an equation that relates ρ with the parameters. Define $C_1 = \frac{1}{\bar{c} - \underline{c}}(ln\bar{c} - ln\underline{c})$, as given by equation 1. Notice, if $\underline{c}$ goes to zero, which means that some informed traders are able to trade with extremely small costs, then C_1increases. Thus C_1 represents a measure of intensity of trading by informed traders. Proposition 6 shows how the intensity of trading by informed traders, parametrized by C_1, shapes arbitrageur's behavior.

Proposition 6. *When there is no instrument and when the arbitrageur and informed traders get perfect signals*

1. *The arbitrageur always trades in the direction of fundamentals in period 2 and* $b = \sigma_{u2}$
2. *The arbitrageur may trade against fundamentals in period 1 if* C_1 *is large enough, and*

a) if $C_1 = \frac{8}{3}\sigma_{u2}$*, then* $Q = 0$*,* $g_1 + a = \frac{1}{2\rho}\sigma_{u2} = \sigma_{u1}$*;* $g_1 = \frac{2}{3}\sigma_{u2}$*;* $a = \sigma_{u1} - \frac{2}{3}\sigma_{u2}$*;*

b) if $C_1 > \frac{8}{3}\sigma_{u2}$*, then* $g_1 + a = \frac{1+Q}{2\rho}\sigma_{u2}$*,* $a = \frac{1+Q}{2\rho}\sigma_{u2} - \frac{1}{2}C_1\left[1 - \frac{1+Q}{2}\right]$*,* $g_1 = \frac{1}{2}C_1\left[1 - \frac{1+Q}{2}\right]$

and ρ *solves* $\frac{\rho}{\left[\frac{1}{2}(1-Q)+1\right]} = \frac{1}{4}C_1\left[1-Q\right]\frac{\rho}{\sigma_{u2}} - Q$*;*

c) if $C_1 < \frac{8}{3}\sigma_{u2}$*, then* $g_1 + a = \frac{1-Q}{2\rho}\sigma_{u2}$*,* $a = \frac{1-Q}{2\rho}\sigma_{u2} - \frac{1}{2}C_1\left[1 - \frac{1-Q}{2}\right]$*,* $g_1 = \frac{1}{2}C_1\left[1 - \frac{1-Q}{2}\right]$

and ρ *solves* $\frac{\rho}{\left[\frac{1}{2}(1+Q)+1\right]} = \frac{1}{4}C_1\left[1+Q\right]\frac{\rho}{\sigma_{u2}} + Q$*.*

where $Q = \sqrt{1 - 4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}}$ *.*

Proof. The full proof can be found in the appendix (section 12.4.1). When second period comes the arbitrageur is the only informed trader thus he trades optimal amount given by $b = \sigma_{u2}$ in our case. However, the arbitrageur has to account for other informed traders when he trades in period one. Let's consider the case, when informed traders do not trade much, $C_1 < \frac{8}{3}\sigma_{u2}$.

When the arbitrageur is the only informed agent, $C_1 = 0$, he trades $a = \frac{1-Q}{2\rho}\sigma_{u2}$. Whereas when informed traders also trade in the first period, $g_1 + a = \frac{1-Q}{2\rho}\sigma_{u2}$. Hence, combined informed demand is always the same, meaning that it depends on ρ in exactly the same way. Why is that? The arbitrageur is a large and strategic trader, he realizes how his trade affects prices, and thus return on private information about θ. Whereas informed traders do not account for that effect when they trade, and their trading is limited only by costs. When the arbitrageur has perfect information about θ, he is able to perfectly predict the total demand of informed traders. Thus it is in his power to offset their demand, and bring the combined fundamental trading back to the optimal level for a given ρ. Therefore $g_1 + a$ equals what would the arbitrageur trade if he was the only informed trader given equilibrium ρ.

Moreover, if informed traders have relatively small costs, if they trade a lot and C_1 is large, then g_1 may be larger than optimality requires. Then the arbitrageur may find it optimal even to trade against fundamentals in the first period $a < 0$, to still bring back amount of fundamental trading to the optimum. That can easily be seen for the case when $C_1 = \frac{8}{3}\sigma_{u2}$, as $a = \sigma_{u1} - \frac{2}{3}\sigma_{u2}$ which is negative if σ_{u1} is small relative to σ_{u2}.

Proposition 6 helps to identify the role of informed traders - they "crowd out" the arbitrageur from fundamental trading. Moreover, in the presence of informed traders the arbitrageur has to not only predict fundamental value, but predict informed traders demand as well, thus σ_ε^2 plays more complicated role. □

We solved for the case with $\sigma_\varepsilon^2 = 0$. It can be shown that for any σ_ε^2 and any C_1 the arbitrageur trades optimally in period 2 and $b = \frac{1}{\sqrt{1+\sigma_\varepsilon^2}}\sigma_{u2}$. Thus as $\sigma_\varepsilon^2 \to \infty$, the arbitrageur trades less based on fundamentals and $b \to 0$ at the rate $\frac{1}{\sigma_\varepsilon}$. Moreover, we have seen in Case I with no informed traders, that when instrument is introduced $b_2 = \frac{1}{\bar{C}_a}\frac{1}{\sqrt{1+\sigma_\varepsilon^2}}\sigma_{u2}$and thus also goes to zero at the rate$\frac{1}{\sigma_\varepsilon}$. Introduction of the new instrument changes that, the arbitrageur ceases fundamental trading faster, because the arbitrageur is able to completely switch to front-run, and prices will still respond to order flow, because informed trades continue to trade.

Proposition 7. *When there is a new instrument and $\sigma_\varepsilon^2 \to \infty$, front-run is not limited in the presence of informed traders and the arbitrageur ceases fundamental trading faster than when there are no informed traders. In particular,*

a) $|b_1| \to 1$, the arbitrageur offsets the shock entirely;

b) $a_1 \to \frac{1}{2}$, the arbitrageur front-runs extensively;

c) $b_2 \approx \frac{1-\lambda_{11}g_1}{\lambda_{11}}\frac{1}{\sigma_\varepsilon^2}$ and hence goes to zero at rate $\frac{1}{\sigma_\varepsilon^2}$.

Proof. One can use formulas derived in proposition 3(see appendix, section 12.4.2). When the arbitrageur does not have good private information, he cannot add much to price informativeness, which depends solely on how much informed traders trade. Thus market makers in period 2 do not get much more information relative to what they've learned from informed demand g_1 in period 1, and $p_2 = E[\theta|y_1, y_2] \approx E[\theta|y_1] \approx \lambda_{11}y_1$. Hence, $\lambda_{12} \approx \lambda_{11}$ and $\rho\lambda_{22}$ goes to zero.

Similar to proposition 4, expected profit of the arbitrageur from front running, can be written as $E(\theta - p_1|s, u_2) + E(\theta - p_2|s, u_2) = -\lambda_{11}a_1^2u_2^2 - \lambda_{12}a_1b_1u_2^2 - \lambda_{22}(1+b_1)b_1u_2^2 = [-\lambda_{11}a_1(a_1+b_1) - \lambda_{22}(1+b_1)b_1]\, u$ which means that $a_1 = -\frac{b_1}{2}$ and $b_1 = -\frac{2\lambda_{22}}{4\lambda_{22}-\lambda_{11}}$. The magnitude of front run depends on the ratio of λ_{22} to λ_{11}.

Let's conjecture that in equilibrium $\lambda_{22} = \frac{1}{2}\lambda_{11}$ and thus $b_1 = -1$ and $a_1 = -\frac{1}{2}$. Also assume that $a_2 = 0$, so that it goes to zero faster than $1/\sigma_\varepsilon^2$. Then presence of informed guys means that

λ_{11} can still be greater than zero, in particular informed traders submit demand $g_1 = \dfrac{\frac{1}{2}C_1}{1+\lambda_{11}\frac{1}{2}C_1}$ and λ_{11} then solves

$$\lambda_{11}\left(1+\lambda_{11}\frac{1}{2}C_1\right)^2 = \frac{\frac{1}{2}C_1}{\sigma_{u1}^2+\frac{1}{4}\sigma_{u2}^2}. \tag{5}$$

Thus $\lambda_{11} > 0$ and $g_1 > 0$ and does not depend on σ_ε^2 when it is sufficiently large. Combined order flow equals

$$\begin{aligned} y_1 &= (g_1+a_2)\,\theta + u_1 + a_1 u_2 + a_2\varepsilon = g_1\theta + u_1 + \frac{1}{2}u_2, \\ y_2 &= b_2\theta + (1+b_1)u_2 + b_2\varepsilon = b_2\theta + b_2\varepsilon. \end{aligned}$$

First order flow is informative about second order flow only if $b_2 > 0$, in that case $E[y_2|y_1] = \rho y_1$ where $\rho = \dfrac{b_2 g_1}{g_1^2+\sigma_{u1}^2+\frac{1}{4}\sigma_{u2}^2}$. Thus ρ and b_2 should decline at the same rate when $\sigma_\varepsilon^2 \to \infty$. Then using $E[\theta|y_2-\rho y_1] = \lambda_{22}(y_2-\rho y_1)$ we can find λ_{22}

$$\lambda_{22} = \frac{(b_2-\rho g_1)}{(b_2-\rho g_1)^2 + (-\rho a_1)^2\,\sigma_{u2}^2 + \rho^2\sigma_{u1}^2 + b_2^2\sigma_\varepsilon^2} \approx \frac{(b_2-\rho g_1)}{b_2^2\sigma_\varepsilon^2}.$$

where all terms that have ρ^2 or b_2^2, or ρb_2 can be neglected relative to $b_2^2\sigma_\varepsilon^2$. First period $\lambda_{11} = \dfrac{g_1}{g_1^2+\sigma_{u1}^2+\frac{1}{4}\sigma_{u2}^2} = \dfrac{\rho}{b_2}$ should equal $2\lambda_{22}$ according to our conjecture. Using that one can then solve that for ρ and b_2 to get $\rho \approx \dfrac{1}{2}\dfrac{1-\lambda_{11}^2 g_1^2}{\lambda_{11}g_1}\dfrac{1}{\sigma_\varepsilon^2}$ and $b_2 \approx \dfrac{1-\lambda_{11}g_1}{\lambda_{11}}\dfrac{1}{\sigma_\varepsilon^2}$. Finally, one can use formula for $a_2$from proposition 3 and check that $a_2 = \dfrac{[2\lambda_{22}(1-\lambda_{11}g_1) - (1-\lambda_{12}g_1)\lambda_{12}]}{4\lambda_{22}\lambda_{11}-\lambda_{12}^2}\dfrac{1}{1+\sigma_\varepsilon^2} = 0$ confirming our conjecture. Thus we have found the solution when σ_ε^2 is large. □

Corollary. *In the limit informativeness of prices is determined solely by informed traders demand, and converge to $I(p_1) \approx I(p_2) \approx \lambda_{11}g_1$, where λ_{11} and g_1 are determined by equation 5.*

Corollary. *Losses of uninformed do not go to zero $E[-p_2u_2] > 0$ as $\sigma_\varepsilon^2 \to \infty$.*

Proof. Indeed, $E[-p_2u_2] = -[\lambda_{12}a_1 + \lambda_{22}(1+b_1)]\,\sigma_{u2}^2 = -\lambda_{11}\frac{1}{2}\sigma_{u2}^2 > 0.$ □

Corollary. *If σ_ε^2 is large enough, then when new instrument is introduced the arbitrageur trades less on fundamentals, $b_2 < b$.*

Proof. Follows from the fact that $b = \frac{1}{\sqrt{1+\sigma_\varepsilon^2}}\sigma_{u2}$ for any σ_ε^2, and when σ_ε^2 is large enough, $b_2 \approx \frac{1-\lambda_{11}g_1}{\lambda_{11}}\frac{1}{\sigma_\varepsilon^2} = const\frac{1}{\sigma_\varepsilon^2}$. □

Because of the complexity of the problem with instrument when informed traders are present, it is difficult to establish further analytical results of introduction of the new instrument for an arbitrary σ_ε^2. We therefore start with stating the hypotheses based on already proven limiting results. We conducted a series of numerical experiments and did not find contradiction with the hypotheses stated.

Degree of front-running The limiting cases give intuition of what to expect when a new instrument is introduced and informed traders are present. If $\sigma_\varepsilon^2 > 0$, the arbitrageur cannot ideally predict fundamental value and informed demand. Thus as uncertainty increases, trading based on fundamentals becomes less attractive and less certain relative to front-running, as the arbitrageur gets perfect information about future u_2 and may exploit it. We also have seen that front-run is limited when there are no informed traders and the degree of front-running did not depend on σ_ε^2. Whereas the arbitrageur fully offsets the shock and front-runs half of it when informed traders are active and $\sigma_\varepsilon^2 \to \infty$. Thus presence of informed traders actually is likely to push the arbitrageur towards front run, amplifying front-running.

Conjecture. *As C_1 or σ_ε^2 increases the arbitrageur front-runs more, namely a_1 and $|b_1|$ increase.*

Price informativeness In the previous section we have shown that if $C_1 = 0$, both prices are always more informative when instrument is available, as the arbitrageur increases fundamental trading in the first period. Thus we may expect that for small C_1 and σ_ε^2 introduction of new instrument still increases informativeness of both prices. However, there is discontinuity when we switch from $C_1 = 0$, in which case as $\sigma_\varepsilon^2 \to \infty$ both $I(p_1)$ and $I(p_2)$ go to zero, and $C_1 > 0$, in which case $I(p_1)$ and $I(p_2)$ are limited from below as a result of fundamental trading by informed traders no matter how small. Thus we have seen that when $C_1 > 0$ as $\sigma_\varepsilon^2 \to \infty$ informativeness of both prices converge to the same constant(corollary 7.1). When the arbitrageur is not the only informed trader, even if he ceases fundamental trading, someone else still trades based on fundamental and thus forces market makers to react to order flow, allowing the arbitrageur to front-run. Taken together that allows us to expect the following:

Conjecture. *For any $C_1 > 0$, $\exists\sigma_\varepsilon^2$ large enough, such that price in period 2 becomes less informative when new instrument is introduced.*

Losses of uninformed investors Usually, as "insider" gets worse information, uninformed traders who have to trade with that insider, lose less. We have seen that when $C_1 = 0$ so that there are no other informed traders, losses of uninformed go to zero as $\sigma_\varepsilon^2 \to \infty$ (see equation 4), no matter if instrument is present or not, rate is the same. The arbitrageur trades less on fundamentals and uninformed lose less, front-run is irrelevant for convergence.

Consider the case with informed traders but without instrument, and with σ_ε^2 large enough. As the arbitrageur further gets worse information, he is also likely to cease his fundamental trading, both because he cannot predict fundamentals and other agents demand. Thus as before we may expect to see the losses of uninformed to decrease as $\sigma_\varepsilon^2 \to \infty$, however they would now be bounded away from zero, as uninformed still continue to trade against other "insiders".

But when instrument is introduced, the arbitrageur may also front-run and thus move the noise to the first period. That actually increases total fundamental trading and makes market maker more sensitive to order flow, and hence increases losses of uninformed. As $\sigma_\varepsilon^2 \to \infty$ front-run increases, and losses of uninformed increase as well.

Conjecture. *If σ_ε^2 is larger enough, further increase in σ_ε^2 makes losses of uninformed smaller when there is no instrument, and larger when there is instrument. Thus relative increase in losses in response to introduction of a new instrument is larger for larger σ_ε^2.*

More generally, three conjectures may be summarized in the following proposition that constitutes the main result of the paper:

Proposition 8. *Strong front-running is associated with*

- larger losses of uninformed;

- smaller price informativeness;

- and larger price response of uninformed demand shock.

Introduction of the new instrument may be detrimental when informed traders trade in period 1, as it induces strong front-running behavior of arbitrageur.

6 Numerical results

We use a numerical example to illustrate the front-running and its consequences. Namely, we show that introduction of the new instrument is beneficial in the case without fundamental traders, as it makes prices more informative and increases welfare of investors. In contrast, we argue that introduction of the new instrument may be detrimental, when fundamental traders trade in period 1. Finally, we extend the analysis, allowing fundamental traders to trade in period 2 and invest in the new instrument, and we again show detrimental effects of the new instrument. More generally, we show that strong front-running is associated with larger losses of uninformed traders, smaller price informativeness, and larger response of prices to the uninformed demand shock.

6.1 Numerical results for cases I and II

Consider $\sigma_\theta^2 = \sigma_{u1}^2 = \sigma_{u2}^2 = 1$. We assume costs of uninformed are distributed as $U[0, 0.5]$. We will compare two cases, studied above, with and without fundamental traders. When present, informed traders have the following distribution of costs $U[0.01, 0.1]$, and $\sigma_\nu^2 = \sigma_\xi^2 = 0$, $c_b = 0.2$.

Figures 2 and 3 display arbitrageur's strategy in both cases with and without the new instrument and for different values of σ_ε^2. If there is no instrument, front-run is impossible. As was shown before, when there are no informed traders, front run is limited, and not sensitive to σ_ε^2 - the arbitrageur offset roughly half of the shock in period 2 and trades in the direction of the shock in period one. In contrast when informed traders are active, the arbitrageur front-runs more as his

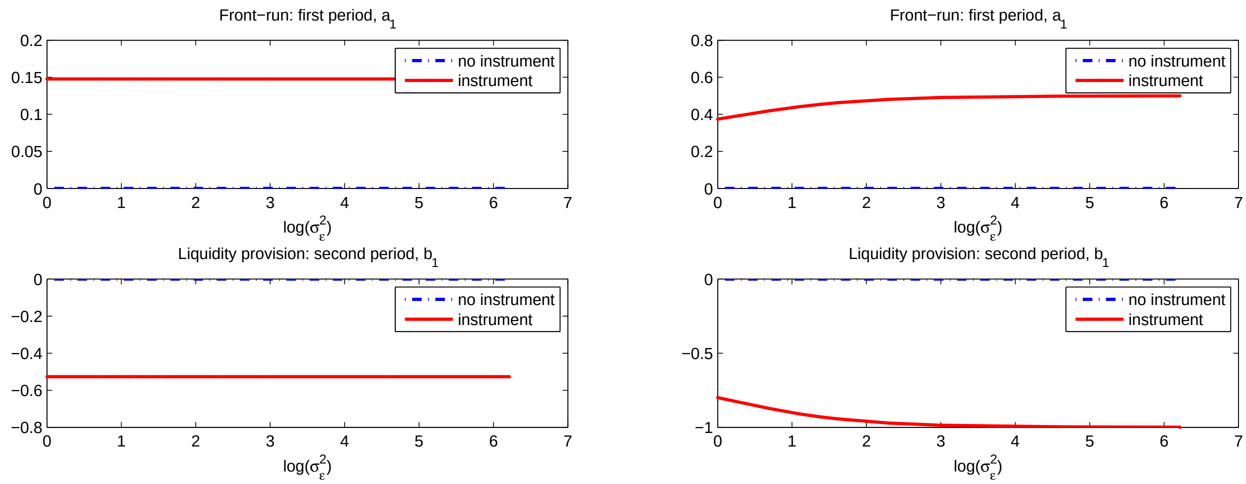

Figure 2: Arbitrageur's strategy: front-run. Left box - no informed traders, right box- with informed traders. Blue line - no instrument, red line - with new instrument. X axis represents logarithm of σ_ε^2- noise in arbitrageur's private signal. Arbitrageur's demand is $x_1 = a_1 u_2 + a_2 s$ and $x_2 = b_1 u_2 + b_2 s$, where $s = \theta + \varepsilon$ and $y_f = \varkappa_{u2} u_2$.

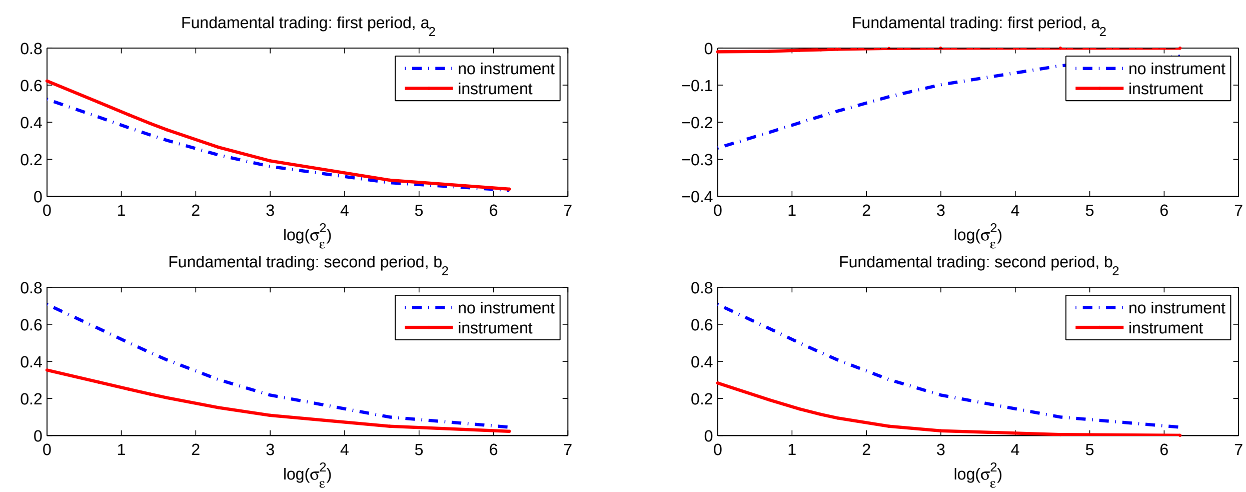

Figure 3: Arbitrageur's strategy: fundamental trading. Left box - no informed traders, right box- with informed traders. Blue line - no instrument, red line - with new instrument. X axis represents logarithm of σ_ε^2- noise in arbitrageur's private signal. Arbitrageur's demand is $x_1 = a_1 u_2 + a_2 s$ and $x_2 = b_1 u_2 + b_2 s$, where $s = \theta + \varepsilon$ and $y_f = \varkappa_{u2} u_2$.

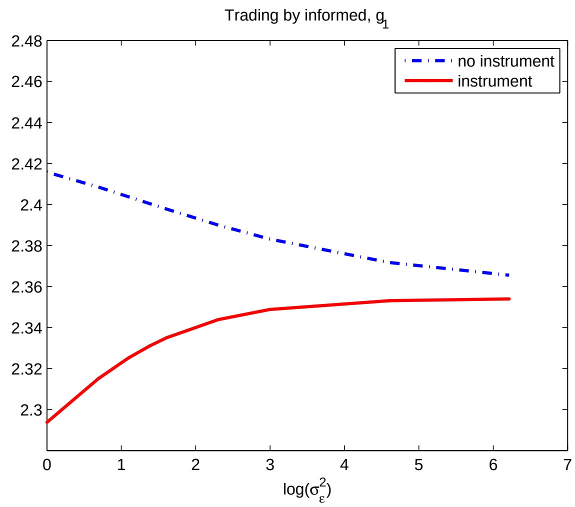

Figure 4: Informed traders combined demand in period 1, g_1, given by $\int_0^1 z_i di = g_1\theta$. Blue line - no instrument, red line - with new instrument. X axis represents logarithm of σ_ε^2- noise in arbitrageur's private signal.

information is getting worse, and eventually offsets the shock u_2 entirely in period 2 and trades half of it in period 1.

Figure 3 shows that when there are no informed traders the arbitrageur trades in the direction of fundamentals in both periods, and gradually ceases his trading as his private signals gets more noisy. The introduction of the new instrument significantly decreases fundamental trading in period 2 and increases in period 1. When informed traders are present but there is no instrument, second period trading is exactly the same for any σ_ε^2, blue lines coincide. However, when instrument is present, the arbitrageur front-runs and cuts noise in period 2 and more so when informed traders are active. Thus he also has to cut fundamental trading more, when instrument is introduced and informed traders are present, which is confirmed by red line on the left being close to zero, than on the right. First period trading is very different. Under the parameters that we have chosen, the arbitrageur initially trades against fundamentals in period 1 to offset extensive fundamental trading by other informed agents. As signal gets noisier the arbitrageur stops trading, and a_2 goes to zero. When instrument is introduced, the arbitrageur actually starts trading in the direction of fundamentals. Figure 4 also shows, that by doing this, the arbitrageur actually "crowds out" informed traders, they trade less when instrument is available.

Figure 5 shows how changes in arbitrageur's strategy affect prices. Figure displays the response of first and second period prices to a 1 st.dev. shock u_2. First consider the case without informed. Without new instrument first price is unaffected. In period 2 when shock is realized, the arbitrageur does not fully offset it and it propagates to the price, but less than $\frac{1}{3}$ of shock. And as σ_ε^2 increases, prices become less informative, market makers decrease sensitivity to order flow and thus price impact of u_2 decreases as well. When new instrument is introduced, the second period price responds almost in the same way. But now first period price also reflects the shock, as the arbitrageur front-runs. Thus we see the propagation of the shock when front-run is possible.

Now let's look at the case with informed traders. When there is no instrument, prices respond

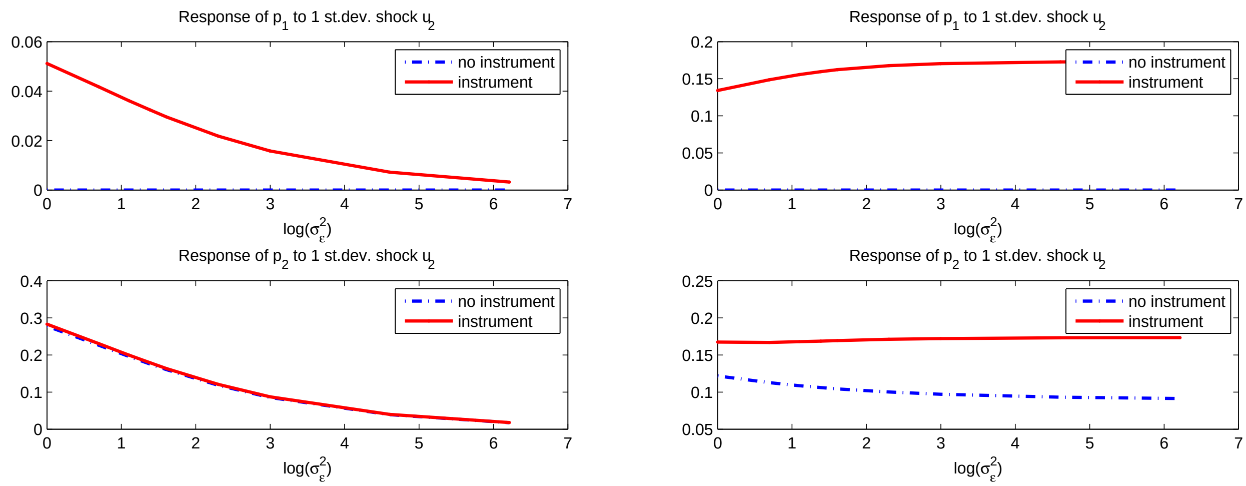

Figure 5: Response of prices to one standard deviation shock of u_2. Left box - no informed traders, right box- with informed traders.

similarly - no reaction in period 1 and decaying reaction in period 2. The only difference, is that in second period price deviations are smaller. With informed traders market makers get much more information from period 1, and thus respond less to second period order flow, and thus less of uninformed demand shock gets reflected in second period price. However, when instrument is introduced the dynamics changes considerably. Two things should be noted: first, period 1 price responds much stronger, second, as arbitrageur's information gets worse both prices respond stronger to u_2 shock, there is no decay. Noticeably, even though the arbitrageur cuts the shock entirely in period 2, second period price reacts much stronger when instrument is available, red line way above the blue line. That is a direct consequence of arbitrageur's stronger front-running activity in the presence of informed traders. Market makers continue being "fooled" by first period large order flow, partially originated from front-running transactions. Thus we observe stronger and more prolonged effect of uninformed demand shocks on prices, similar to what we have observed in the oil market.

Figure 6 further disentangles the effect of information leakage on price responses. Consider second period price. Response of p_2 to u_2 is given by $[\lambda_{12}a_1 + \lambda_{22}(1 + b_1)]\, u_2$, and thus is defined by i) arbitrageur's offsetting strategy in period 2, b_1; ii) arbitrageur's front-running strategy in period 1, a_1; and iii) market makers price setting strategy. We do the following partial equilibrium exercise: we calculate hypothetical price responses taking only one variable from a new equilibrium with instrument, and keeping the rest at the equilibrium level with no instrument. Thus black line on figure 6 is price response when a_1 and b_1 are new, but $\lambda - s$ are kept unchanged - it shows the partial equilibrium effect of new arbitrageur's strategy. One can see that first period price responds to the shock as the arbitrageur front-runs, whereas second period price actually responds less, as the arbitrageur offsets the shock considerably. The fact that black line is substantially below red line when there are no informed traders, shows the quantitative importance of changes in sensitivities triggered by the arbitrageur new strategy when considering the overall effect of the new instrument. That is also confirmed by the green line being well above the red one - green line represents the case with new lambdas but initial arbitrageur's strategy. Period 1 price is unaffected

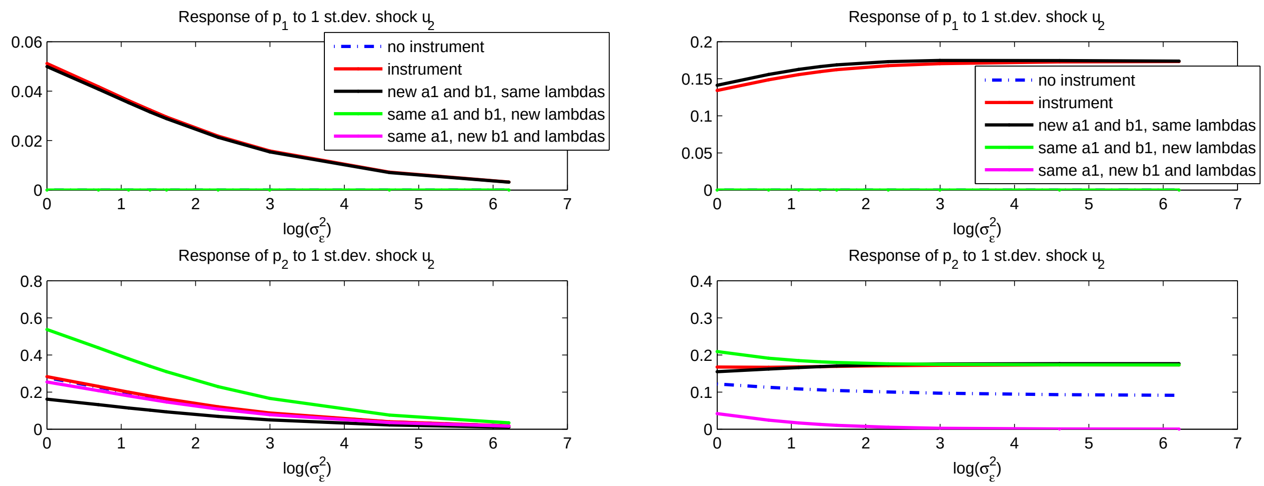

Figure 6: Partial equilibrium exercise to disentangle response of prices to one standard deviation shock of u_2. Left box - no informed traders, right box- with informed traders.

as there is no front-running possible in equilibrium with no instrument, whereas period 2 price responds strongly to the shock, as sensitivity is much larger,λ_{22} is large. Finally, pink line shows the case in which there is no front-running, only offsetting of the shock under new market makers price setting function. Thus in the case with no informed, front-running in period one adds little to the final result: pink and red line are close. Whereas, in the case with informed, pink line is substantially lower, and the arbitrageur front-runs a lot in period 1 and that substantially increases price response. To sum up, in both cases changes in $\lambda - s$ are important, less so when informed are present and in both cases front-running is important - more so when there are no informed.

Taken together the picture suggests that general equilibrium effects of front-run are extremely important. Reallocation of noise and changes in fundamental trading cause market makers to reconsider their price setting strategies, making second period price much more sensitive to unpredictable part of second period order flow, and thus implying larger price response. Front-run partially reveals u_2 in period 1, thus unpredictable part of the order flow, $y_2 - \rho y_1$ depends much less on u_2 and hence its non-fundamental part of variance decreases considerably. Even though fundamental part also decreases, $b_2 - \rho a_2$ becomes smaller, overall second period order flow is more predictable given first period order flow and contains smaller noise. And thus market makers respond more to it and hence u_2 propagates to p_2 more. Hence, absorbing capacity of the market changes in response to front-run.

Figure 7 further shows that second period price, which can be considered as settlement price or the final price, becomes less informative when new instrument is introduced and informed traders are present. That adds to the price response results: notice that at the same time we have less informative period 2 price and still larger response to u_2 shock.

Finally Figure 8 displays trading losses of uninformed and Figure 9 displays profit of arbitrageur. When there are no informed traders, additional information about future u_2 has little effect on trading losses of uninformed or corresponding profit of the arbitrageur (which in addition to trading losses of uninformed active in period 2 also includes losses of uninformed active in period 1). Whereas introduction of new instrument and corresponding leakage of information about future

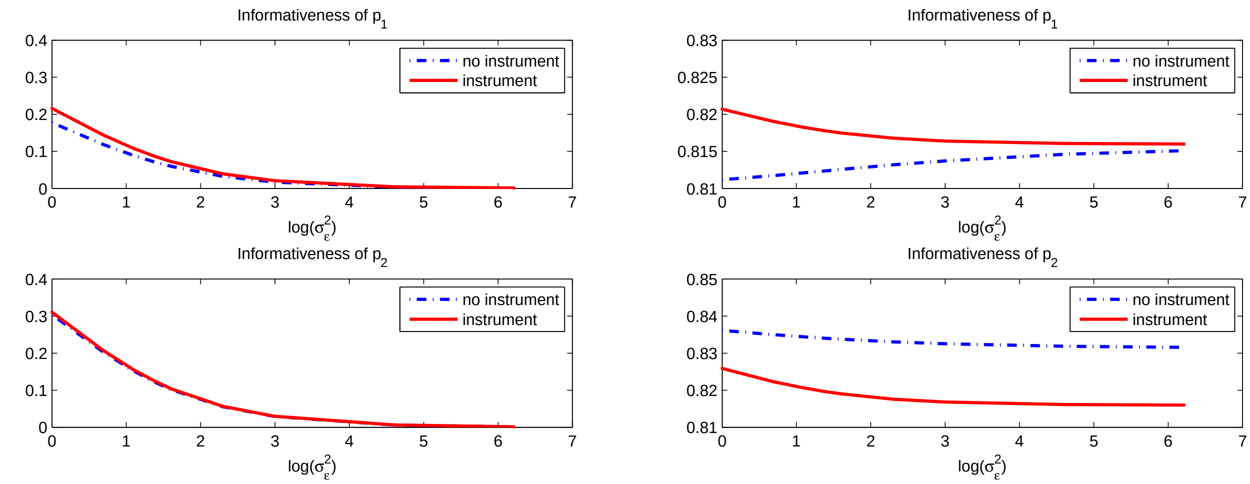

Figure 7: Informativeness of prices. Left box - no informed traders, right box- with informed traders.

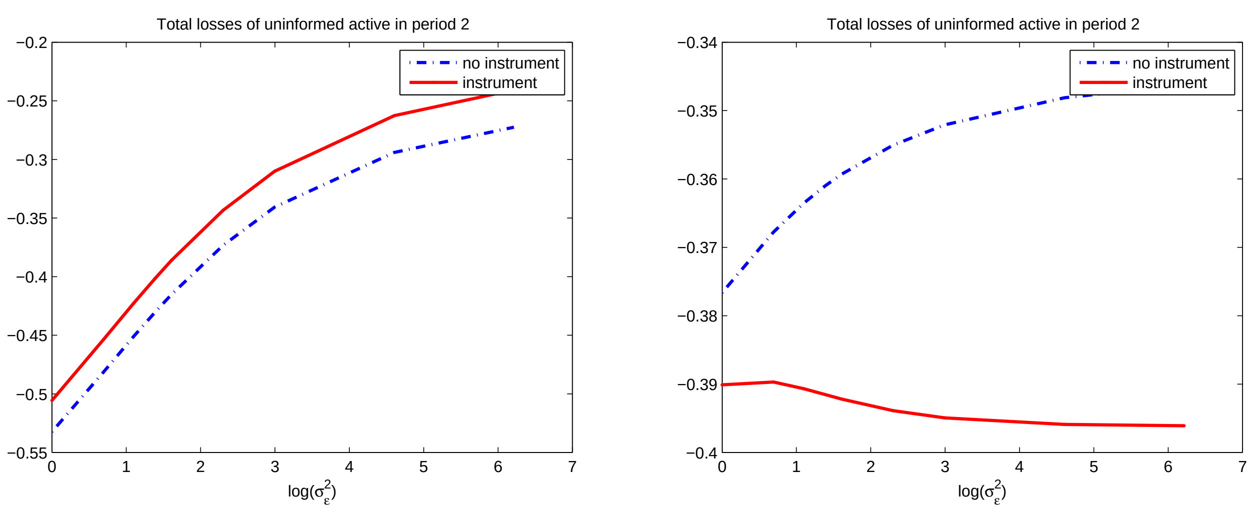

Figure 8: Total losses of uninformed active in period 2. Left box - no informed traders, right box- with informed traders.

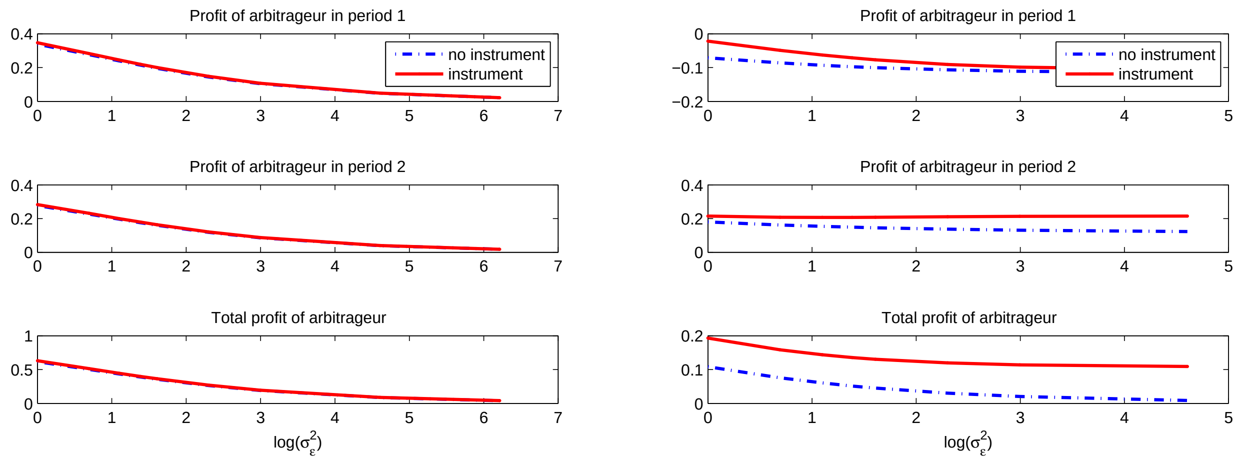

Figure 9: Profit of arbitraguer. Left box - no informed traders, right box- with informed traders.

uninformed demand u_2 has large effect when informed traders are present. The relative increase is losses/profit is larger when front-run is stronger, i.e. when σ_ε^2 is larger. Thus when fundamental trading is not so attractive to the arbitrageur both due to large demand of informed traders, and bad private information, the arbitrageur prefers to switch and front-run, what increases losses of uninformed. Whereas when there is no instrument and thus no other sources of profit, the arbitrageur simply gradually decreases fundamental trading and thus losses of uninformed also decrease.

6.2 Numerical results when there are informed traders in both periods, but only uninformed traders invest in the new instrument (case III)

Next, we consider the case in which informed trader are active in both periods, but we do not allow second period informed traders to invest in the new instrument. Thus as before the arbitrageur is able to learn u_2 without mistake from investment in the new instrument, but now he has to face competitors in both periods. To better see the importance of informed traders we fix $\sigma_\varepsilon^2 = 3$ and present the results for all values of μ - fraction of informed active in period 1. Thus Case II is a particular case when $\mu = 1$ - no informed in period 2.

Figure 10 shows how arbitrageur's strategy is shaped by the presence of informed traders. One can see limited front-run when there are not so many informed traders in period 1, μ is small, and almost complete front-run when μ goes to 1. In there are a lot of informed traders already present in period 1, front-run represents a better trading alternative, thus the arbitrageur largely decreases fundamental trading in period one and switched to front-running.

Figure 11 displays decrease in informed demand when instrument is present. As the arbitrageur gets extra information, he is able to "crowd out" informed traders.

Figure 12displays price informativeness on the left. As there are more informed traders already revealing information in period 1, first period price increases with μ. When the new instrument is introduced, first period price is unaffected, no matter what μ is. However, second period becomes more informative when μ is small and front-run is limited and less informative when μ is larger

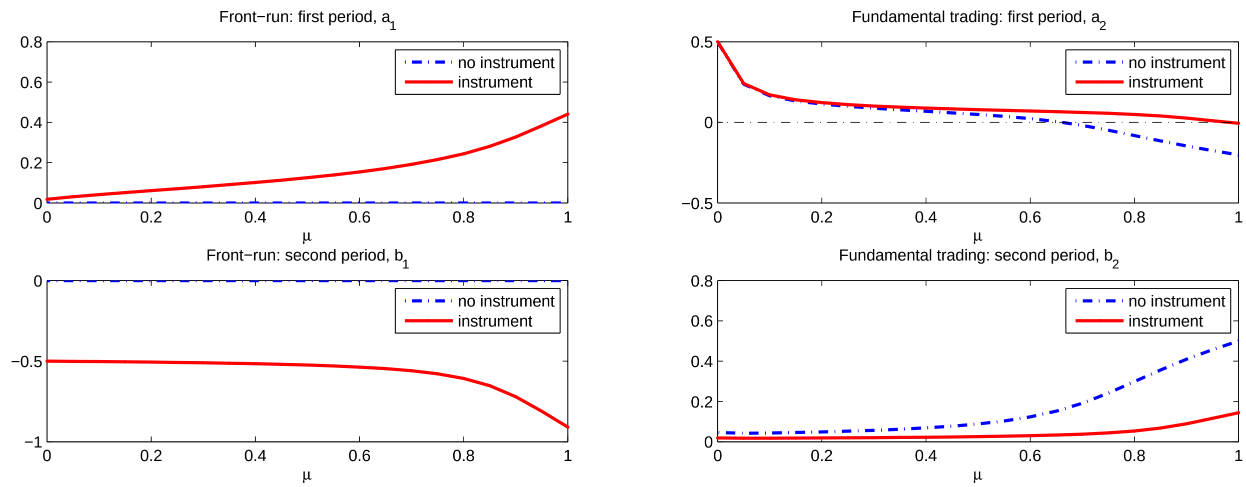

Figure 10: Arbitrageur's strategy in case III as a function of μ- fraction of informed active in period 1.

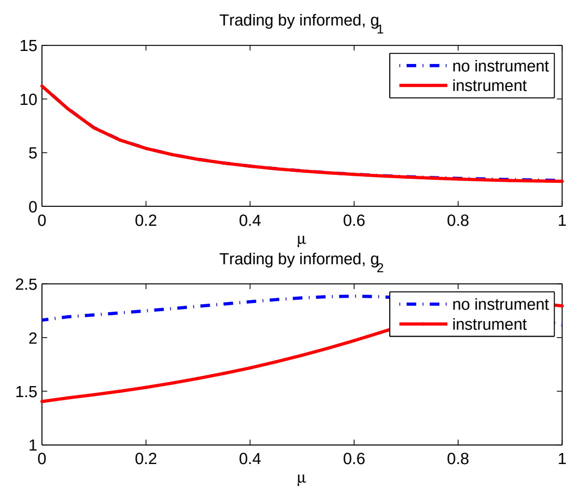

Figure 11: Informed traders demand as a function of μ- fraction of informed active in period 1.

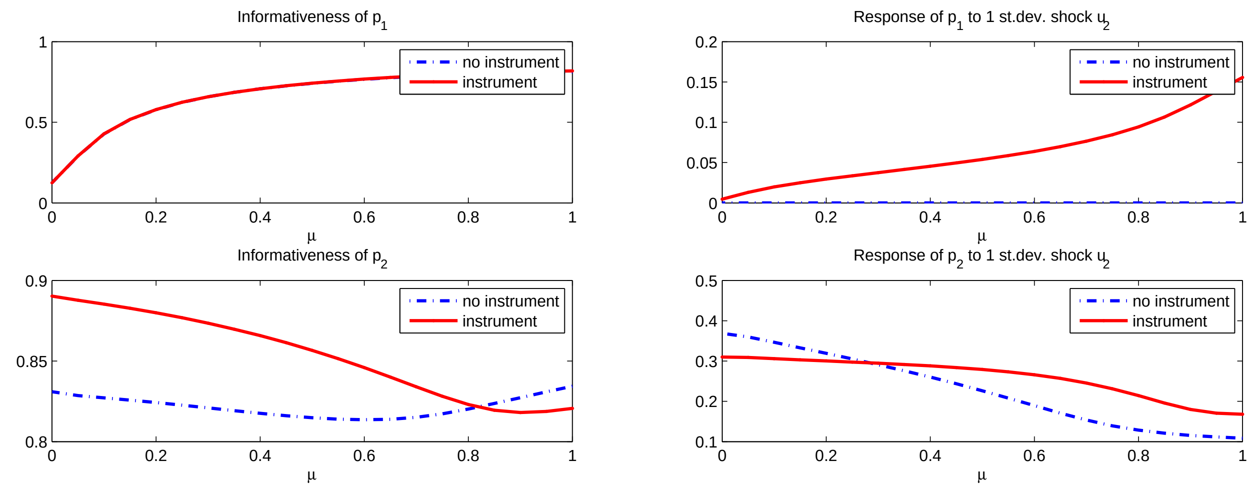

Figure 12: Price informativeness and price response as a function of μ- fraction of informed active in period 1.

and front-run is stronger. Similarly, as front-run increases first period price starts to reflect shock u_2. In period 2 while front-run is relatively mild, second period price actually reflects u_2 less, as the arbitrageur cuts half of the shock. However, as front-run increases and u_2 propagates to p_1, the effect of u_2 on price is larger than when there is no information revelation associated with the new instrument, and does not go to zero, even when the arbitrageur cuts all of the shock in period 2 ($b_1 \to 1$). That happens because first period price now reflects u_2 and through market maker being not able to distinguish it from fundamentals, it also slips into p_2.

Finally, figure 13 shows changes in trading losses and total losses of uninformed and profit of the arbitrageur in response to the new instrument. Notice, as before trading losses are the same ($E[-p_2u_2] = E[-fu_2]$) no matter whether uninformed invest in the instrument or not. However, some of uninformed traders, those with largest costs can economize, while trading the new instrument. When $\mu$is small and thus front-run is mild, introduction of the new instrument is beneficial to uninformed traders active in period 2. Instrument at the same time allows to economize on costs and decrease their trading losses - because the arbitrageur cuts half of the shock and does not really front-run, and thus price less reflect the shock, and hence uninformed gain. But when front-run gets stronger, trading losses increases and at some point, the negative effect of trading losses exceeds positive savings on costs. And instrument is detrimental.

6.3 Numerical results for the general case (case IV)

Finally, we consider the case when informed traders active in period 2 are also allowed to invest in the new instrument. Now investment in the new instrument represents a mixed signal about θ and u_2. Thus front-run and fundamental trading become even more interconnected - front-running, as offsetting shock u_2 when it comes in period 2 and trading in its direction in period 1, will now also imply extra trading against fundamentals in period 2 and in the direction of fundamentals in period 1. The exact degree would depend on the weight of θ in y_f, which in turn is determined by how many informed traders decide to use the instrument and also how much they decide to trade.

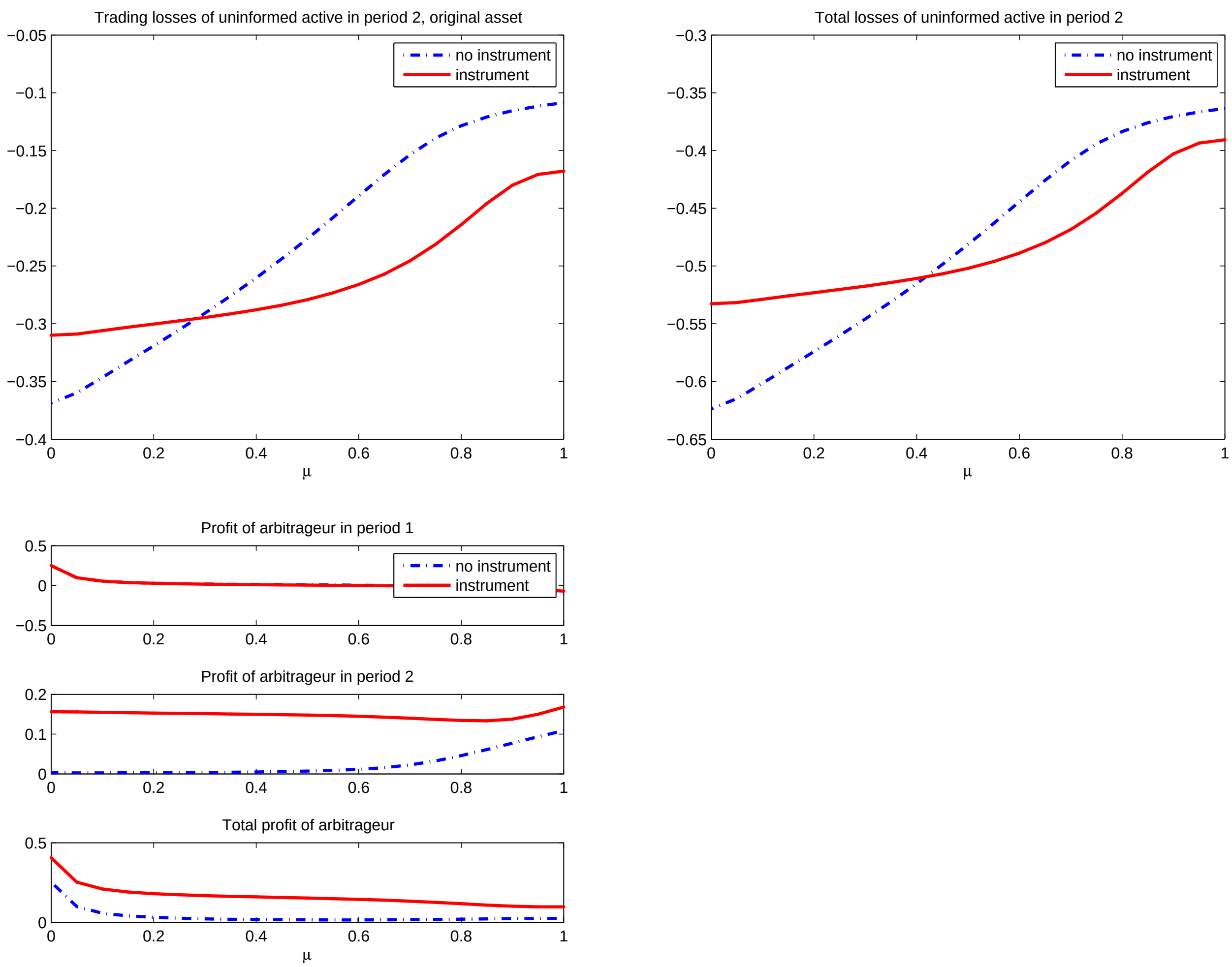

Figure 13: Total losses of uninformed and profit of the arbitrageur as a function of μ- fraction of informed active in period 1.

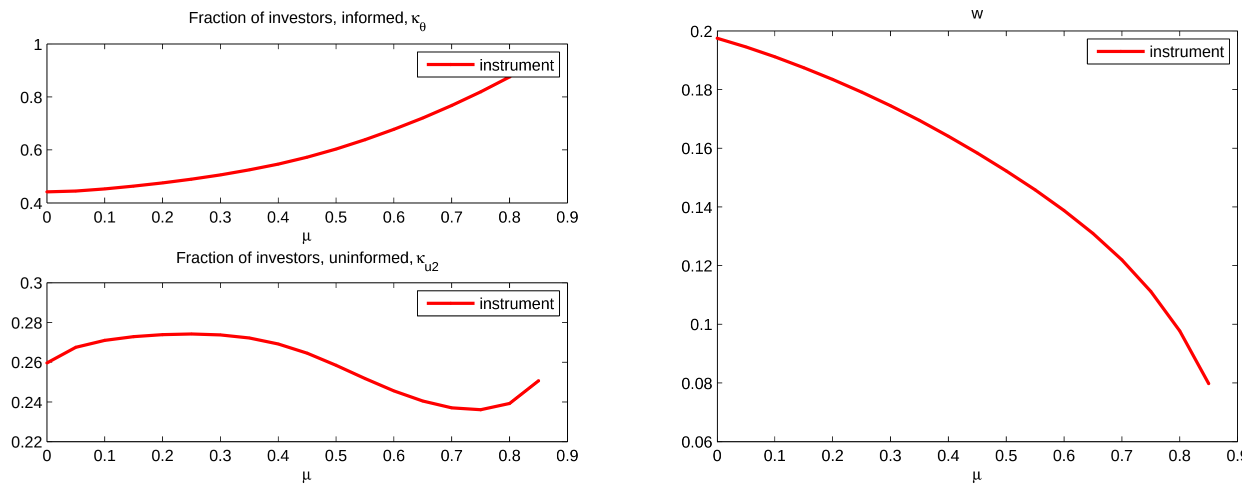

Figure 14: Fraction of investors in the new instrument and price of the new instrument as set by broker per unit order squared as function of μ- fraction of informed active in period 1.

Figure 14 displays fractions of informed and uninformed investors in the fund, thus $\varkappa_\theta = 0.5$ means that half of informed traders invest in the new instrument. Figure shows that larger fraction of informed traders invests in the new instrument when there are more informed traders in period 1, μ is larger. First, as we see price of the instrument decreases. Second, waiting for the second period to trade implies smaller profit, as more information gets be revealed by then. Thus informed investors switch to use the new instrument, even if it may be more costly for them. Figure 15 shows how much informed traders actually trade, and it confirms that as μ increases and the new instrument becomes more attractive, informed investors trade more in total, whereas informed stayers on the original market trade less.

In contrast, the fraction of uninformed stays almost constant, but displays some non-linear dynamics, because for them forces work in the opposite direction, price of the instrument decreases, but trading losses increase.

Arbitrageur's strategy is displayed in figure 16. When μ is zero, there are not so many informed investors and investment in the new instrument, y_f, is a reasonably good signal about future u_2. Thus the arbitrageur front-runs as before. But notice, that he now cuts less of the shock and trades in its direction much stronger. Moreover, in period 2 the arbitrageur actually decreases fundamental trading and in period 1 increase, exactly the contrary to what we had before. That is because the arbitrageur uses his own signal to partially cut out or offset θ part in y_f signal.

Figure 17 displays a decrease in second period price informativeness when the periods are equal. A number of first period informed traders is strong enough to make fundamental trading less attractive, and the new instrument still offers a good signal about future u_2. Again, while front-run is strong, price informativeness suffers. At the same time we see a response of first period price to u_2, and a larger response of p_2 to u_2 when instrument is introduced. Notice that maximum coincides with largest drop in price informativeness.

Finally, figure 18 displays losses of uninformed, both investors in the new instrument and those who stays on the asset market. Few things should be noted. First, all traders experience losses, even those who stay on the asset market and do not invest in the new instrument. Second, investors

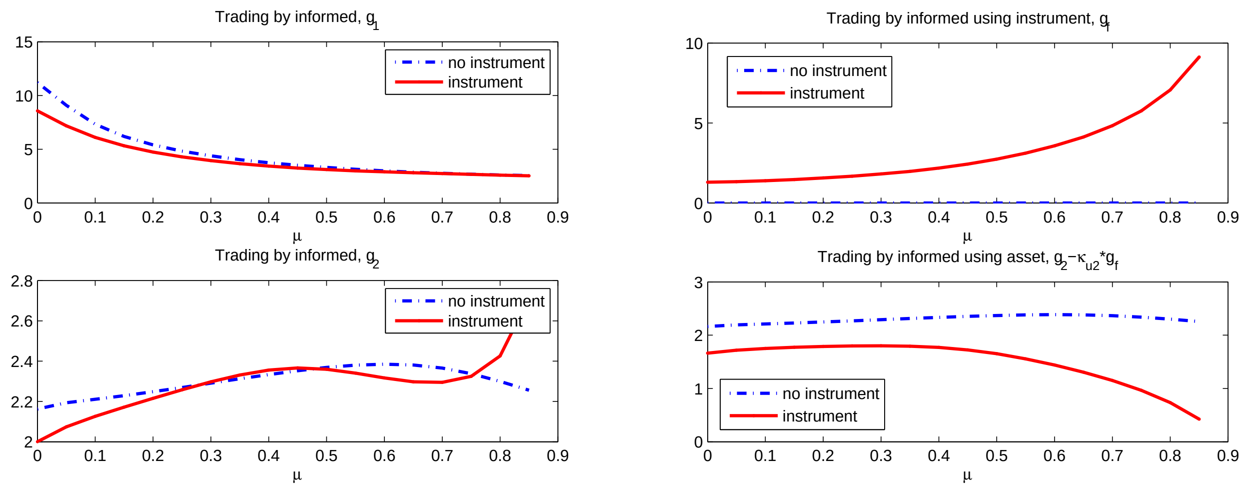

Figure 15: Informed traders demand as a function of μ- fraction of informed active in period 1.

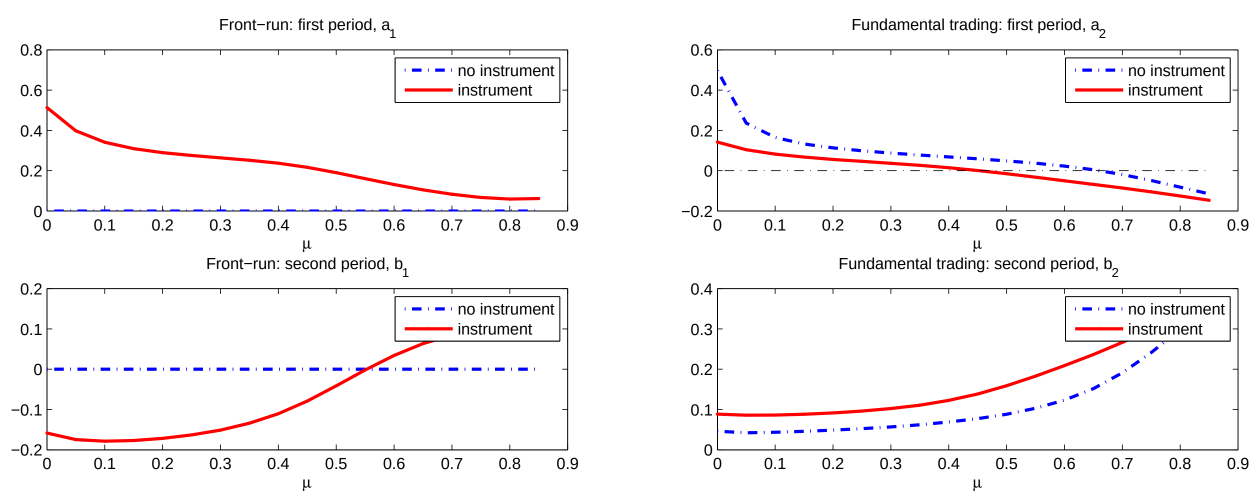

Figure 16: Arbitrageur's strategy in case III as a function of μ- fraction of informed active in period 1.

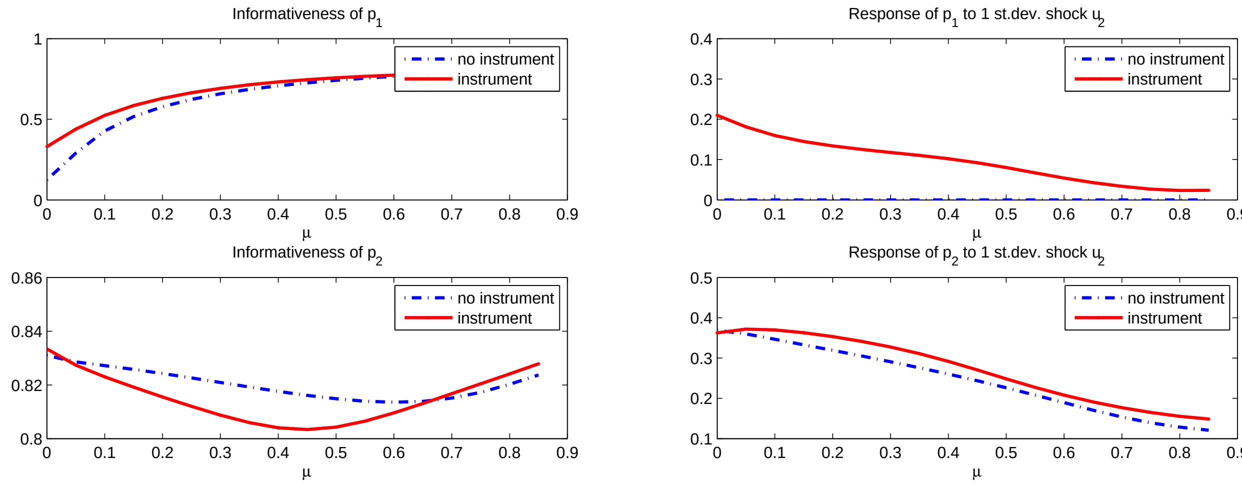

Figure 17: Price informativeness and price response as a function of μ- fraction of informed active in period 1.

lose much more. Moreover, in total as a group uninformed investors lose when new instrument is introduced. Whereas the arbitrageur profits from extra information.

7 Empirical evidence

Our mechanism predicts the detrimental effect of the introduction of the new instrument on the underlying market. The front-running may amplify mispricings triggerred by rebalancing practices of the new instruments. We start with presenting two replicating instruments that require regular rebalancing. Our empirical exercise aims to document a significant price pressure, consistent with rebalancing transactions, that these instruments impose on the underlying market. Observed price impact would suggest limited arbitrage on the underlying market. Given limited arbitrage we may suspect to observe front-running, as front-running arbitrageurs intent to only partially offset the rebalancing shock. Without account level data it seems hard to directly confirm front-running. However, front-running also affects the underlying market in the periods preceding the rebalancing period. Thus, we intent to relate flow of investment in the replicating instrument to observable changes in the underlying market in the period before the rebalancing. Additional investment in the new instrument inevitably increases the rebalancing demand, and thus makes front-running more attractive. We expect to see the effect of extra front-running transactions on the prices of the underlying assets.

Thus the goal of this section is to relate investment in synthetized securities, whose structure allows for predictability in demand, to changes in the underlying market. We consider two special instruments: i) crude oil market and the United States Oil Fund - exchange traded fund (USO); ii) VIX market and exchange traded notes issued by Barclays and linked to linked to the S&P VIX Short-Term Futures index, (VXX). Both entities require regular rebalancing. In the oil market monthly rebalancing is required to replace expiring futures contracts and avoid physical delivery of oil. In the volatility market daily rebalancing is necessary to achieve a constant maturity volatility index. Namely, a particular fraction of the nearby futures contracts must be sold and the next fu-

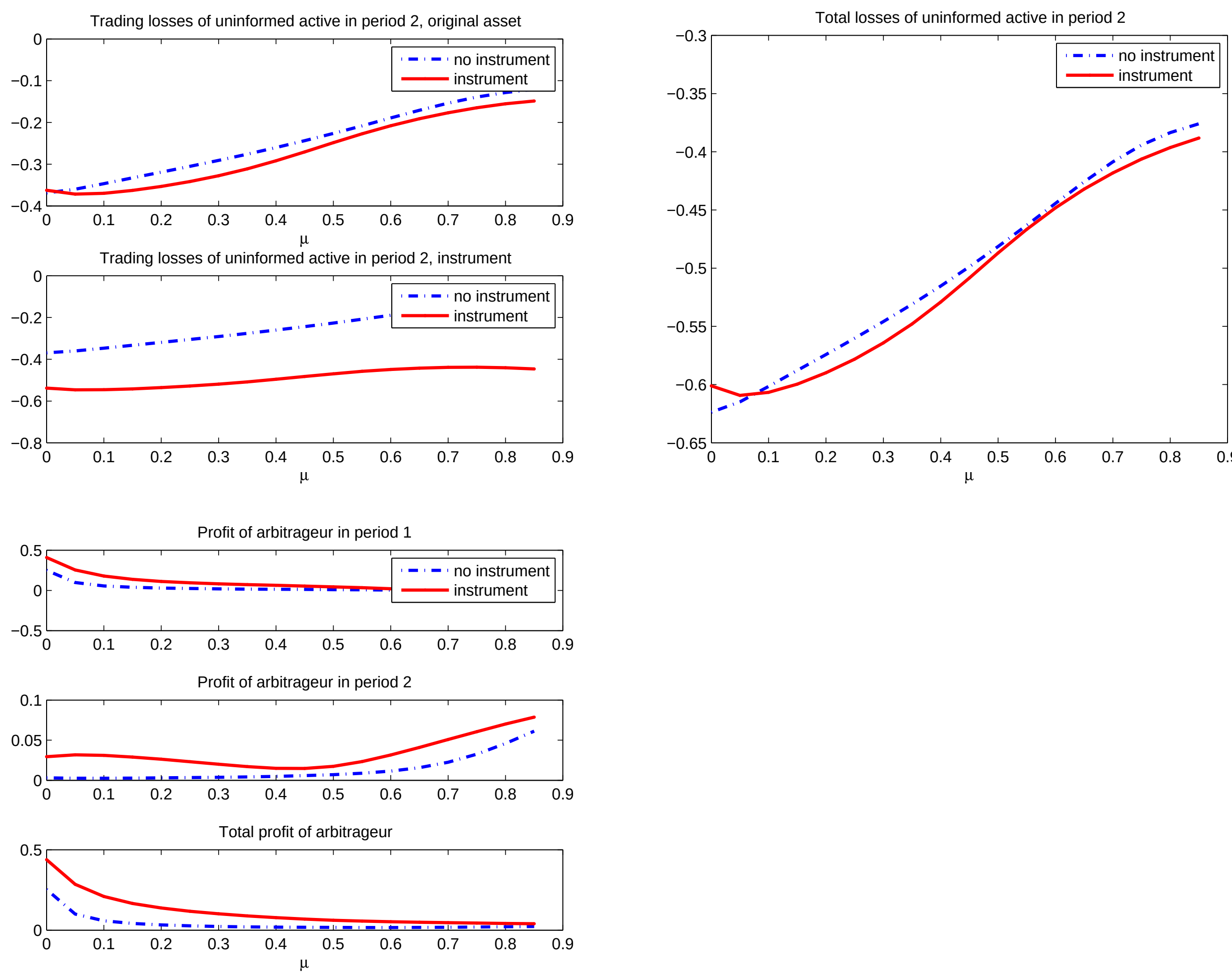

Figure 18: Total losses of uninformed and trading losses (no costs) as a function of μ- fraction of informed active in period 1.

tures contracts must be bought every day. In both cases rebalancing is predictable. USO specifies rolling days in its prospectus a year in advance and publishes the holdings every day on the official website. Barclays, the issuer of VXX, is expected to replicate the underlying volatility index to hedge his own exposure. Observing total investment in VXX at its official website, Barclay's daily rebalancing can also be calculated. Thereby, any sophisticated agent may predict future rebalancing demand and use that information to front-run. In both cases front-running assumes sale of the nearby futures and purchase of the next out futures. That transaction tends to increase the difference between the prices of the second month and the nearby futures, sometimes called contango. We aim to relate changes in the investment in the new instruments with changes in contango. We work with daily data, thus in the case of VXX we can only document the combined price impact of both front-running and rolling, as rebalancing frequency is also daily. In the case of USO we can document front-running, analyzing the days that precede the rolling.

7.1 Data sources

We use several sources to get daily information about flows of investment in and out of exchange traded products. Our goal is to transform money flows into the position occupied either directly by managers of the fund as in the case of USO, or indirectly, as a hedge position likely to be obtained by the issuer of ETN. Generally data on positions are scarce and stale, which can be a serious problem for us, as we would like to work with daily frequencies. Therefore, we focus on one ETF and one ETN and cross-check the data on shares outstanding obtained from multiple sources.

Shares outstanding data

1. Compustat.
2. Monthly brokerage commissions and assets for USO - sec filings, 8-k forms.
3. Creation and redemption flows - etf.com, data provider- FactSet Research Systems, Inc.
4. USO daily holdings positions from its website - 46 days.

Other data

1. Treasury notes.
2. WTI Futures prices, weekly data on storage in Cushing and Working Storage Capacity from EIA.
3. VIX futures prices and weights from vixcentral.com and CBOE[13].

7.2 The new instrument in the oil market

We focus on the United States Oil Fund - the largest exchange traded fund that invests in short term crude oil futures.

[13] http://cfe.cboe.com/data/historicaldata.aspx#VX.

Background United States oil fund holds nearby futures contracts on crude light oil WTI, traded on NYMEX. The Fund is an exchange traded fund (ETF), fund issues shares are traded at NYSE (ticker USO). Shares price is connected to the price of underlying futures contract by the actions of authorized participants(AP). Those are usually large financial institutions that obtained the right to exchange oil futures for Fund's shares. Thus the overall number of shares is not constant, but rather depends on the market conditions. For example, if USO price is large relative to oil futures price, authorized participant may buy oil futures, bring them to the Fund, exchange for new USO shares, and sell them on the secondary market. This arbitrage possibility creates the link that connects USO shares price with oil price.

Sample choice We need to know what contracts the Fund trades at each particular day. Futures have expiration date, thus they have to be replaced in advance in order to avoid the delivery of physical oil, the operation known as rolling. Expiration of contracts and rolling create an investment cycle: the Fund holds the nearby contract, replaces it with second month contract during rolling period, hold second futures until the nearby futures expires at which point second futures becomes the nearby futures and the cycle continues. Expiration usually happens at 18-22 day of each month. The Fund rolls its porfolio usually from 5th to 9th day of each month. Until March, 2009 USO used to roll the entire portfolio in just one day, nowadays the rolling lasts 3-5 business days. Thus to have a consistent estimation we limit our sample to expiration-rolling period only, defined as from 24th day of one month to 4th day of the next month. That would imply, that contango is always defined as the price difference between the next out and the nearby futures contracts, and that the Fund holds and/or trades only the first futures contracts. Moreover, as the rolling period is excluded, if changes in USO position predict future changes in contango, that can be attributed to front-running, rather than direct price impact of rolling. Indeed, any extra contracts that the Fund purchases are to be rolled over in the future, therefore arbitrageurs can front-run a bit more, selling the nearby contract and purchasing the next out contract, and as a result putting upward pressure on contango.

The total sample covers period from April 10, 2006 to March 20, 2015. We drop first 100 days after the Fund was established, as most of its shares belonged to APs, and Compustat does not show any changes in the number of shares outstanding.

USO position We use shares outstanding and shares price data to calculate number of contracts bought or sold by the Fund. Main source of data is Compustat, however, it is well known that shares outstanding data are stale and sometimes incomplete. Thus we also use our own dataset on USO daily holdings for 46 days in 2009 and 2015, flows data from etf.com provided by FactSet Research Systems, Inc. , and end-of-month data from 8-k forms that the Fund submits to the SEC.

Each USO daily holdings file shows the number of shares outstanding as of the beginning of the day, the closing shares price, and pending number of shares redeemed or created during the day and accompanied number of contracts bought or sold by the Fund during the day. Figure 21 displays daily holdings file as of March 24, 2009, as well as Compustat data and EIA futures price. The Fund redeemed 1,400,000 shares and sold 828 contracts at \$53.98 dollars per barrel, which was the closing price from the trading floor of the NYMEX, provided by the EIA(at the bottom), $F_{1,t}$. Figure indicates one day delay in shares outstanding in Compustat - number of actual shares outstanding at the beginning of the day as reported by the Fund is 107,500,000, but is reflected in Compustat on March 25,2009. However closing USO shares price is correct. Thus number of contracts bought or sold by the Fund during the day, can be calculated using Compustat and EIA data as $x_t = (shares_{t+2} - shares_{t+1}) * closing\, shares\, price_t/(1000F_{1,t})$, in particular $x_{3/24/2009} = (106,100,000 - 107,500,000) * 31.61/(1000 * 53.98) = -820$ versus actual -828, thus a very good assessment. Other daily holdings files for other days show the same: shares outstanding figures are delayed exactly by 1 day, hence we can use the same formula to find changes in USO position for all days in the sample.

However, in addition to the delay problem, Compustat also shows lack of flows data during few days in 2015 (shares outstanding constant during 8 days period, although daily holding file documents the opposite). To fix the problem we also use flows data on redemption and creation from etf.com provided by FactSet Research Systems, Inc. to fill in missing values[14]. Unfortunately, the data starts in 2012. Finally, we also use end of month data to fill in any further missing values. Thus we will proceed with the analysis, but keep in mind the limitations of our dataset. For the exposition purpose we divide the changes in Fund's position by 1000.

Other variables Physical arbitrage imposes an upper limit on contango. If second month futures is more expensive than the nearby futures, one could borrow money, buy oil using nearby futures or at the spot market, store it for a month and deliver it according to a more distant futures, thus getting risk free return. Of course, possibility to do that and returns depend on the availability and price of oil storage and borrowing costs. The main delivery point associated with oil futures contracts n WTI traded at NYMEX is Cushing, Oklahoma[15]. Conveniently, the EIA (US Energy Information Administration) reports weekly figures on crude oil inventories every Wednesday(as of end of last week). Same agency also reports working storage capacity twice a year. We create an indicator variable that takes value 1 if announcement was made during the day, and a variable that reflects released change in storage in Cushing relative to previous week and normalized by maximum capacity available at that moment. One can expect an upward pressure on contango from released information about a buildup in storage, as market participants may reconsider the value of contango if arbitrage is perceived to be less possible or more expensive. Notice, that even though there is a delay in reporting, we are interested in market reaction to public information on storage, as it might be the only source of information that the vast majority of arbitrageurs has, maybe with exception to midstream companies that may have better information.

We also use 5-years Treasury Notes to measure borrowing costs. Finally, we use two volatility measures to account for possible risk effect, although we do not do formal risk factor analysis. One measure is VIX, an implied index of the stock market volatility. Second is an index of intraday oil market volatility defined as highest minus lowest price of the nearby futures contract normalized by the closing price.

Results We normalize the daily contango by the first futures price, as the cost of physical arbitrage is proportional to the absolute value of oil to be bought and stored. Thus, the change in contango is defined as

[14] We again check it with our data, and turns out that the delay in flows data is the same as in Compustat.

[15] Even though oil may in principle be delivered to any other point, by mutual agreement of counterparties.

$$\Delta c_t \quad = \quad 100\left[\frac{F_{2,t}-F_{1,t}}{F_{1,t}}-\frac{F_{2,t-1}-F_{1,t-1}}{F_{1,t-1}}\right].$$

We are interested in the link between ΔUSO_{t-1} and Δc_t. Table 2 shows the OLS regression with Newey West standard errors of changes in contango on lagged changes in USO position. Panel A uses the total sample from September 1, 2006 to March 20, 2015. We see a significant and positive effect of USO positions. First specification shows that changes in USO position help predict future changes in contango. Other specifications show that lagged changes in USO position are significant even if we account for market reaction to storage data and volatilities and interest rates. To give a perspective, on January 28, 2009 USO increased by 7500 contracts (which is about 1.5 st.dev. of changes in USO position during winter 2008-2009). Our results imply that such a growth in positions may be associated with 18 cents increase in contango the next day, given first futures price of 42 dollars prevalent at that period of time.

Panel B recalculates the regression on a restricted sample, in which two episodes of extreme contango, including few month around, are eliminated, mainly September, 2008 - June, 2009 and March, 2014-March, 2015. One can see the absence of significance in the effect of changes in Fund's positions on contango. Which also confirms the front-run story, as front-run is relevant if the Fund is big enough. Both panels also imply the importance of market reaction to information about storage in Cushing.

Table 3 further investigates the timing of the front-run. We restrict the sample to two episodes of large growth of the Fund where we expect front-run to be strong and active. Panel A represents the calculations given investment window as 24th day of one month and 4th day of the next month. In contrast Panel B displays the case with shorter window - 26th-2nd. First one can notice that in both panels results are stronger than in the case of the total sample in table 2, suggesting stronger front-run. Second, one can also see larger effect in Panel B. As one moves closer to the rolling period, one can assess the total number of contracts to be rolled better, as there is not much time for the Fund to significantly change its position. However, one expect other player to front-run as well, and see stronger price impact already present. Thus when we restrict the sample to cover the period closer to the front-run, but not close enough, we see stronger effect of USO positions, implying larger front-running activity.

Thus the results possibly indicate the presence of a front-run on USO rolling volume information and its significant price impact.

7.3 The new instrument in the volatility market

The CBOE's VIX is a volatility index that is calculated based on prices of a line of options on S&P 500 index and represents an implied volatility index. Roughly speaking it displays market assessment of equity volatility over next 30 days. Traders would like to get an exposure to volatility to hedge their equity portfolio, because volatility tends to go up when stock market goes down and vice verse. Ideally, traders would prefer to trade an instrument that tracks VIX index, however, VIX is not a tradable asset - it is not directly investable. The next best option is futures contract on VIX traded at NYMEX. But there is a little caveat. The price of a VIX futures contract reflects the market expectation for the level of VIX on the settlement date of the contract. Standardized

VIX futures traded at the exchange offer only a limited set of expiration dates. Thus if one wants a tradable measure of the volatility with constant maturity of 1 month, one has to construct it using two nearest futures with appropriate weights. The S&P 500 VIX Short-Term Futures Index was established to do exactly that - the index utilizes prices of the next two near-term VIX futures contracts to replicate a position that rolls the nearest month VIX futures to the next month on a daily basis in equal fractional amounts. This results in a constant one-month rolling long position in first and second month VIX futures contracts. Thus even though VIX futures contracts have cash settlement and physical delivery is not an issue, if one wants a constant maturity volatility index, daily rebalancing is still necessary, and hence related to the mechanism studied in this paper - price impact of predictable rebalancing.

Background VXX is an exchange traded note (ETN) issued by Barclays Bank and linked to the performance of the S&P VIX Short-Term Futures Index mentioned above. VXX represents the largest exchange traded product among VIX-linked ETFs and ETNs (both short and long term). Thus we decided to study front-running on VIX market using investment tied to VXX. Contrary to USO which is ETF, VXX is ETN - a debt security and hence, no underlying contracts are directly held by its management. Barclays Bank issues obligation and all repayments (at maturity or through early redemption) are based on the behavior of VXX indicative value calculated every day based on the performance of the index itself, no tracking error is possible. As an issuer Barclays holds a huge liability and is likely to hedge it, using the exact methodology that underlies the S&P VIX Short-Term Futures Index - Barclays holds a mix of first and second month contracts and rebalances the portfolio every day, moving away from the nearby to the next out futures contract. In contrast with ETFs, contracts tied to investment in VXX will only be shown on Barclays books. But although one cannot directly observe Barclays positions, one can estimate it based on public information on VXX assets under management. Of course, we may assume a certain degree of internal, in-the-house netting of positions, or hedging through OTC transactions, but in that case eventually (after a few hedging chains) contracts would be brought by someone to the exchange.

VXX position VXX shares are traded at NYSE on a liquid secondary market. The total number of shares outstanding is not fixed, market participants may intervene and create or redeem new shares to keep VXX price in line with the underlying asset. But that process is not as simple as with ETFs, no exchange of underlying security happens. If the number of shares changes due to early redemption, for example, the issuer has to change his own portfolio in order to continue hedge his own exposure. Therefore, issuers of ETN implement restrictions on the minimum number of ETNs and date restrictions for redemptions. According to the VXX filings and prospectus[16] the mechanics of the redemption is the following. First, the owner of ETNs must notify the issuer no later than 4 pm and deliver a signed confirmation on the same day. The next day will be called the valuation date when the price per security will be settled at the closing indicative value minus a fee. The actual trade will happen on the redemption data which shall be the third business day after the valuation date. Thus there is 4 days lag between the redemption decision and actual redemption. Given that the price is settled on the valuation day, the issuer has to sell the contracts on the valuation date as well in order to achieve perfect hedge. Therefore, we get a 4

[16]FORM 8-A http://secfilings.nyse.com/filing.php?doc=1&attach=ON&ipage=6096464&rid=23
Prospectus http://www.ipathetn.com/US/16/en/contentStore.app?id=5149530

days delay between the actual redemption decision and the reflection of that transaction in the books, and thus most likely 3 days between the issuer's hedging operation (on the valuation date) and the reflection of the redemption. In contrast with ETFs, creation of new ETNs is in issuer's sole discretion, and we cannot assess when the hedging operations are conducted, and whether our net flow data is correct. Barclays may issue new shares and sell them to the market participants. However, it might take some time to accumulate the contracts before creating new shares and we may expect to see similar delay between actual purchase of contracts for hedging purposes and reflection of new shares in the book. Thus the problem with delays is worse for VIX ETNs. We use net flows data from etf.com, but we cannot separate redemptions from creations and treat them differently, as only net numbers are presented. Despite the limitations mentioned, we decided to shift the sample 4 days back to account for 3 days delay due to redemption procedure and 1 day delay in reporting the data on etf.com that we have noticed before working with USO data.

The weights were obtained from vixcentral.com and show a particular mix of the first and second month contracts that needs to be held at day t to replicate the S&P VIX Short-Term Futures Index and define the fraction of assets to be invested in each contract. We use the weights to calculate extra number of contracts that hedging of new redemptions or creations brings at the end of day t, ΔVXX_t. Thus if day t experiences net order flow of a_t, then

$$\Delta VXX_t \quad = \quad w_t \frac{a_t}{F_{1,t}} + (1 - w_t) \frac{a_t}{F_{2,t}}.$$

[17]The extra contracts will be added to the total number of contracts that Barclays holds in its hedging portfolio. When it rebalances it tomorrow, those extra contracts will impose additional price impact on the market, as larger number of contracts will have to be replaced. Thus we would like to test if a rise in today's VXX size implies larger change in contango tomorrow. We also calculate total number of contracts associated with investment in VXX(one has to account not only for the flow in and out of fund, but also for deterioration of value of investment in VXX), where we take initial number of shares outstanding and shares prices from Compustat, in addition to flows data.

The daily data on flows start in 2012 and our sample covers January, 2012 - March, 2015. Normalized contango is calculated as $\dfrac{F_{2,t} - F_{1,t}}{F_{1,t}}$ for each day t. Table 4 presents summary statistics. Mean total number of contracts supposedly held by Barclays as a hedge of his exposure to investment in VXX is 60,000 contracts, with a maximum of 111,000 reached in March, 2012. Daily changes in positions can be substantial - as large as 20,000 contracts a day in both ways. Price difference $F_{2,t} - F_{1,t}$ in VIX futures market varies from -3 dollars to 5.5 dollars, with the market being in contango more than 90% of time. For representation purpose we divide the changes in the number of VXX contracts by 1000.

Results First, we observe a large correlation of contango with total number of contracts - 0.66, see table 5. Figure 22 further shows that contango and contracts seem to be very closely connected. Fit is especially good for the down troughs, thus justifying our shares calculation procedure and

[17]ΔVXX_t is divided by 1000 to account for the size of the futures contract

suggesting the link between contracts and contango. Correlation in first differences is largest when first lag of contracts is used, and equals 0.11.

We perform the analysis using daily data on contango in VIX-futures market and calculated VXX position. Results are presented in Table 6. We see the link between concurrent flow of investment in VXX with current and future changes in normalized contango. The results suggest that an increase in VXX contracts by 10000 predicts an increase in normalized contango by 0.0079 the next day, and if we take first futures price to be equal $16.4 - mean value over the sample, an increase in absolute contango would be 13 cents. Mean level of absolute contango over the period is 1.14. Daily inflow to the fund reached maximum of 20,000 contracts. Thus a price impact of daily rebalancing and associated front-running may be substantial. To account for possible mistakes in shifting and preparing the data we also test if more distant lags also have predictive power, and the results show significant impact up until forth lag when the estimator changes sign and becomes insignificant. Thus even if we did not shift our sample for 3 days to account for redemption delay, we would still observe significant contemporaneous effect of changes in positions on contango

Our results document front-running and price impact of rebalancing activity in the oil and volatility markets. Our results are directly related to predictive analysis in table 13 from Neuhierl and Thompson (2014) that tests if managers' positions help predict future returns to momentum in contango strategy. And also related to regression results in table 7 from Mou (2011) that test the link between total value of investment tied to SP-GSCI and allocated to each individual commodity and return to front-running strategy, where identification comes from using a control group of commodities that are not exposed to index investment and thus the Goldman roll. However, as far as we aware VIX market has not been examined for the price impact of the rebalancing activity on the term structure.

Finally, Mou (2011) documents that commodities markets are nowadays more likely to be in contango state, in a dramatic contrast with the 90s. That is consistent with our story, because front-running on the rolling of long position tends to increase contango. Neuhierl and Thompson (2014) in addition to documenting the excess returns on momentum strategy in spreads, also document large autocorrelations. That is also consistent with our mechanism, as our model predicts larger excess correlation of prices (correlation of first and second period prices conditional on fundamental value).

8 Literature review

Our paper is related to a number of literature strands. The first important question is whether information about future demand should be revealed to the market. There is no consensus in the literature. One side considers benevolent arbitrageurs and claims that uninformed traders may profit from 'sunshine trades'. They argue that arbitrage capital is limited and slow moving, thus if a trade is announced in advance, arbitrage capital will have time to adjust and to step in to absorb the shock. At the end, additional liquidity is brought to the market when needed, reducing the transaction costs. Among papers based on that idea is Bessembinder (2014), that illustrates how predictable orders should typically have minimal effects on prices because they are not motivated by fundamentals. From an empirical point of view, USO's rolling that we focus on is also studied in Bessembinder et all. (2014). That paper employs data on individual orders and trades in crude oil futures made available by the Chicago Mercantile Exchange, which owns and operates the NYMEX market. The authors compare the roll and non-roll dates. The evidence indicates narrower bid-ask spreads, greater order book depth, and more trading accounts provide liquidity on roll dates. But, our mechanism is consistent with providing liquidity on roll dates, as the front-running arbitrageur partially offsets the actual rebalancing shock when it comes. Thus there is a lot of trading in each individual futures contract on roll dates, but the realized price difference between the two futures contracts, realized contango, may be already large as a result of previous front-running activity.

An alternative view is based on unwillingness to consider benevolent arbitrageurs. Brunnermeier and Pedersen (2005) introduce a term 'predatory trading' to describe a situation in which an information about future demand leaks to the market and triggers harmful behavior of arbitrageurs. Consider an agent that with some probability will have to liquidate his portfolio. Then the front-running actions of other market participants may not only decrease the profit obtained by the agent from liquidation, but may actually increase probability of the liquidation. Potential harmful role of arbitrageurs and/or financial innovations in the presence of belief disagreement is also studied in Simsek (2013) and Weyl (2007).

Our work is also related to the discussion of non-redundancy of replicating instruments. The idea that a real asset and its synthetic counterpart may have different informational properties, was first mentioned in Grossman (1988). In that paper a real option and dynamic strategy that replicates the option's payoff imply a different degree of initial information revelation. In Grossman (1988) arbitrage capital is beneficial, it absorbs demand shocks and decreases price volatility. However, the analysis in that paper abstracts from the fact that allocation of arbitrage capital can be strategic and not always beneficial. Among other papers that focus on informational effects of derivative markets and/or other sources of non-redundancy Easley, O'Hara, and Srinivas (1998), Goldstein, Li, and Yang (2014), and Oehmke and Zawadowski (2014). Massa (2002) determines the conditions under which the introduction of a new derivative may dampen information acquisition.

The paper that is the closest to ours is Bernhardt and Taub (2008). In a two period Kyle model, a strategic arbitrageur happens to learn second period uninformed demand and front-runs on that knowledge. The authors prove that in a dynamic setup, knowledge of future demand shock changes the equilibrium outcomes, whereas as shown by Rochet and Vila (1994) is static settings that does not happen and both the information content of prices and profits are unaffected. In contrast, we emphasize that the front-running behavior of the arbitrageur depends crucially on the presence and trading activity of other informed traders. In Bernhardt and Taub (2008) information leakage is exogenous. In our model information revelation is caused by the introduction of the new replicating instrument. This allows us to put the losses that uninformed traders experience as a result of trading with front-running arbitrageur in perspective. We can compare the trading losses with the direct gains from trading the new instrument. While introducing the new instrument as a way to lower transaction costs by netting out the orders, we also relate to research on internalization of order flow and its impact on market quality and consumer's trading losses, studied, for example, in Chakravarty and Sarkar (2001) and in Battalio and Holden (2001).

Financial innovation has been particularly important in the commodities markets. The commodities markets have seen a surge of investment in the last decade, a process known as financialization. A number of paper study financial innovations related to financialization of commodities markets, and possible mechanisms of the effect of financialization on the futures market. The main role in that literature is devoted to commodity index investment, buying pressure from speculators, and limited capacity of financial market to absorb such pressure. Our paper builds on general equilibrium effects of front-running and endogenous changes in market absorbing capacity. In a Kyle model limited absorbing capacity originates in the inability of market makers to distinguish informed orders from uninformed. Thus non-fundamental demand shocks propagate to prices. Limited absorbing capacity may also arise in a model with risk averse traders, who would require compensation in order to step in and take the other side of the deal. Thus financialization may also affect futures prices by changing the equilibrium risk premium as modeled in Hamilton and Wu (2015) and Goldstein and Yang (2015). Hamilton and Wu (2015) consider financialization as a large demand shock that needs to be absorbed by risk averse traders, and thus expect to see an increase in risk premium. Empirically, authors do not find evidence that amount of index investment helps predict futures prices. In contrast Goldstein and Yang (2015) associate financialization with an inflow of risk averse financial traders eager to step in and absorb the hedging demand (coming from production side), thus they conclude that financialization is actually beneficial.

Papers that attempt to estimate the impact of financialization, including Hamilton and Wu (2015), Sanders and Irwin (2011), Brunetti et al. (2011), Singleton (2013), Tang and Xiong (2012), Stoll and Whaley (2010), usually do not find a strong effect. Many papers use CFTC aggregate public reports to estimate the investment in individual commodities tied with index investment. Irwin and Sanders (2012) raise concerns about Singleton's method, based on Master's idea, for inferring the crude oil positions of index-fund traders. Even though Hamilton and Wu (2015) generalize the method to mitigate some of the criticisms, the level of aggregation, measurement errors, and timing issues impose severe limitations on usage of such data. In contrast Henderson et al. (2015) study the impact of issues of commodity-linked notes on commodity prices and find a significant impact on futures prices. Although the vast majority of papers focus on the impact of financialization on price levels and volatility, a few papers differ and work with the term structure. In Selezneva (2010) I document unprecedented contango in oil futures market in 2008-2009, show failure of fundamental explanations, and relate contango to the surge of investment in USO and front-running on information about future rolling. Limited arbitrage originates from limits on open positions that arbitrageurs face and thus have to strategically decide on how to better use it. Mou (2011) shows significant excess return of a strategy devised to exploit the price impact of the Goldman roll on futures prices. The Goldman roll is named after the rolling procedure of the Standard and Poor's - Goldman Sachs Commodity Index (SP-GSCI) - the first commercially available and most popular commodity index. The Goldman roll happens from the fifth to ninth business day in each month. Similarly Neuhierl and Thompson (2014) show significant excess return of a strategy that aims to capitalize on the continuance of existing trends in contango across a number of commodity groups. Thus both papers provide evidence of the price impact of rolling. Moreover, Mou (2011) documents that commodities markets are nowadays more likely to be in contango state, in a dramatic contrast with the 90s. That is consistent with our mechanism, because front-running on the rolling of long position tends to increase contango. Neuhierl and Thompson (2014) in addition to documenting the excess returns on momentum strategy in spreads, also document large autocorrelations. That is also consistent with our mechanism, as our model predicts larger excess correlation of prices (correlation of first and second period prices conditional on fundamental value). Both papers also try to connect excess returns with the magnitude of rolling. Mou (2011) uses CFTC's report on index investment and the Masters and White (2008) procedure to find the exact amount of investment tied to the SP-GSCI, and finds significant effect of investment on excess returns (using panel regressions for each commodity). Neuhierl and Thompson (2014) use the money managers net long positions from the CFTC Commitment of Traders (COT) report and test if money managers' position help predict futures returns to momentum strategy in spreads, and find only weak results. However, both procedures suffer from severe data issues.

There are four main features that distinguish our empirical approach from the literature. First, we focus on the effect of investment in the new instrument on the term structure and not on the price level. Mainly, our choice is justified by the nature of rebalancing that is an integral part of the instruments that we study. Second, contrary to the existing literature we do not attempt to assess the total rebalancing demand. Instead, we focus on two special instruments that tend to occupy a substantial market position, thus allowing us to neglect all other positions. The unprecedented growth of popularity of both instruments coincided with significant disruptions in the underlying markets. Moreover, as investment strategies of the two instruments are known, we can observe the dollar amount of investment in the new instruments and relate it with actual positions in the underlying contracts. Thus we avoid measurement errors associated with the usage of aggregate data. Third, we work with high frequency data. We use daily data to relate positions to changes in the underlying market, thus we are able to avoid timing issues also common in the empirical literature. In sum, while limiting the sample to two instruments, we sacrifice extrapolative power, but in return we are able to work with data of better quality. Finally, we consider the effects of rebalancing on the VIX market, which has not been studied before under that perspective[18].

9 Conclusion

Our research aims to raise concerns about potential side effects of replicating instruments. In this paper we show detrimental effects of information leakage associated with the introduction of the new instruments that bundle existing securities. When arbitrage is limited, strategic arbitrageur finds it profitable to front-run on knowledge of future demand shock. In equilibrium, fundamental trading by the arbitrageur and informed traders, as well as price setting functions adjust to the presence of front-running. We find that front-running is associated with less informative prices, larger losses of uninformed traders, and larger propagation of non-fundamental shocks to the prices of the underlying securities. The uninformed investors in the new instrument lose more on trading with front-running the arbitrageur than gain from the lower transaction costs that instrument offers. Therefore, the introduction of the new instrument may be detrimental.

Two real world examples are used to illustrate the mechanism, namely USO in the oil market and VXX in the volatility market. We document a significant effect of the rebalancing practices and associated front-running activity on the markets for the underlying securities. In the oil mar-

[18]VIX market is studied in Bollen, O'Neill and Whaley (2013) that link VIX ETP hedging demand with VIX futures prices, and in Mixon and Onur (2014).

ket front-running on knowledge of future rebalancing has been particularly large and might have caused temporary violation of a no-arbitrage condition.

In the paper the effect of the new instrument is limited to the effect on the financial market. The prices of the underlying securities may be disturbed, but not the fundamental value of the security, as it is assumed to be exogenous. In principle, propagation of non-fundamental shocks to the financial market may change the fundamental value itself. Consider the oil market. First, oil futures are used by real producers to hedge the production. Second, the futures price on WTI is viewed as the world benchmark in oil pricing, thus financial market also performs price discovery role, that can be affected by the introduction of the new instruments and associated front-running, but that question requires further research.

Our paper advocates the necessity of further research and supervision of financial innovation process. The stricter regulation of financial market may be essential, at least to avoid obvious regulatory arbitrage. In particular, if the underlying market is regulated, than the replicating instrument should at least comply with similar restrictions. For example, the US Oil Fund was exempted from position limits due to bona fide hedging, which given our analysis was a mistake which should be avoided. Finally, the incentives of financial institutions to innovate a particular type of instruments should be taken into consideration, but that again stays for further research.

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10 Figures

Figure 19: Number of contracts rolled by the United States Oil Fund and normalized contango, $\frac{F_2 - F_1}{F_1}$. The Fund is an exchange-traded fund that invests in one month futures contracts on WTI traded at NYMEX and rolls its entire portfolio each month. The rolling procedure tends to increase the price difference between the second month and the maturing futures contracts, $F_2 - F_1$, called contango. The figure shows that months in which the Funds rolls significant number of contracts are associated with large contango.

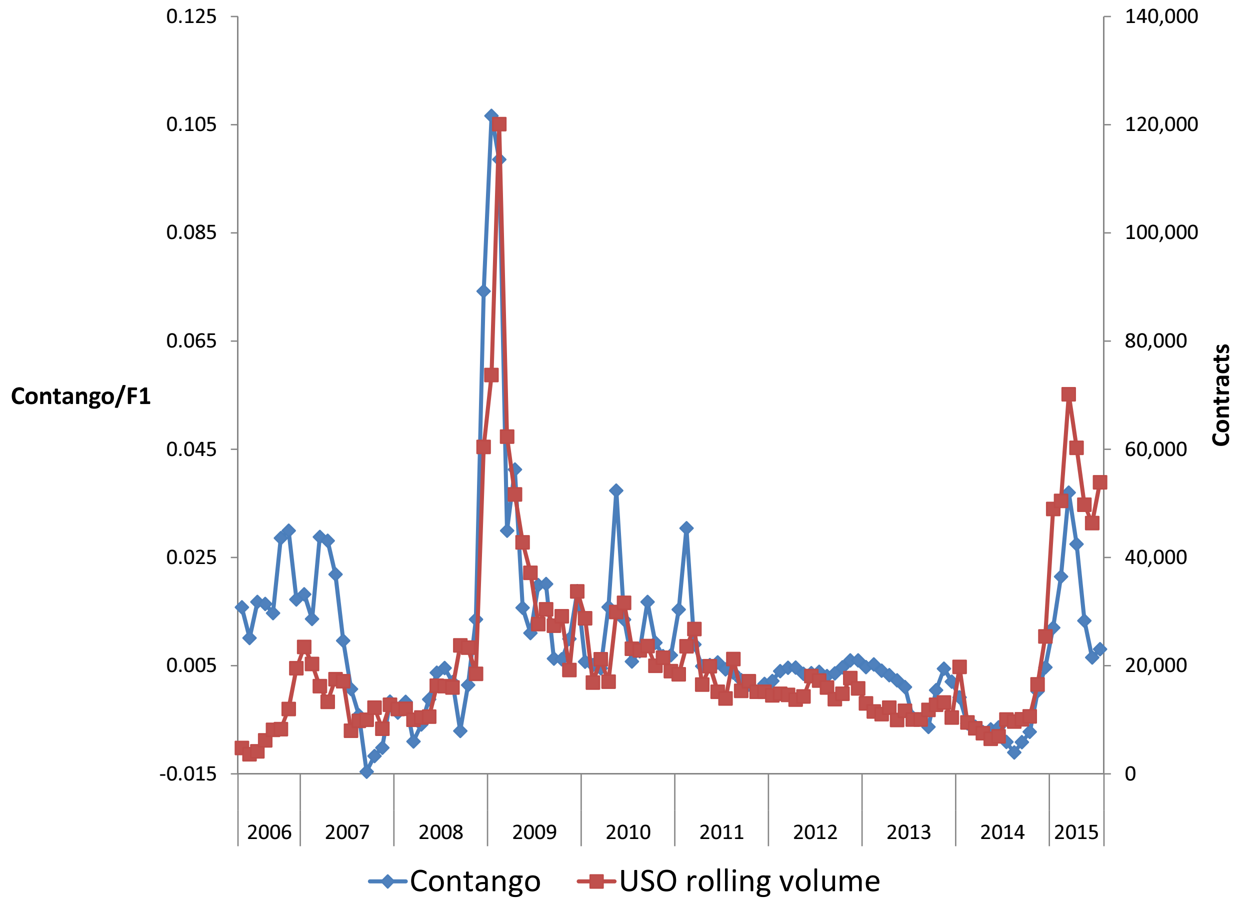


Figure 20: Oil market in contango. The figure displays contango defined as $F_2 - F_1$ in dollars per barrel in two oil futures markets - NYMEX and ICE. Blue line - contango at the NYMEX where WTI is traded, and red line - at the ICE where Brent is traded. Futures on WTI are the ones that the United States Oil Fund invests in. The figure shows dramatic increase in contango in WTI market not observed at Brent market. Vertical grey lines depict rolling days when the fund sells soon to expire contract and buys the second month contract, thus putting upward pressure on contango. First, contango on WTI market is larger than 2 dollars per barrel - no physical arbitrage line. Second, frequency of peaks coincide with expiration/rolling frequency. Third, contango does not only increases during and after rolling days, but starts to increase before that, thus suggesting front-run of future rolling.

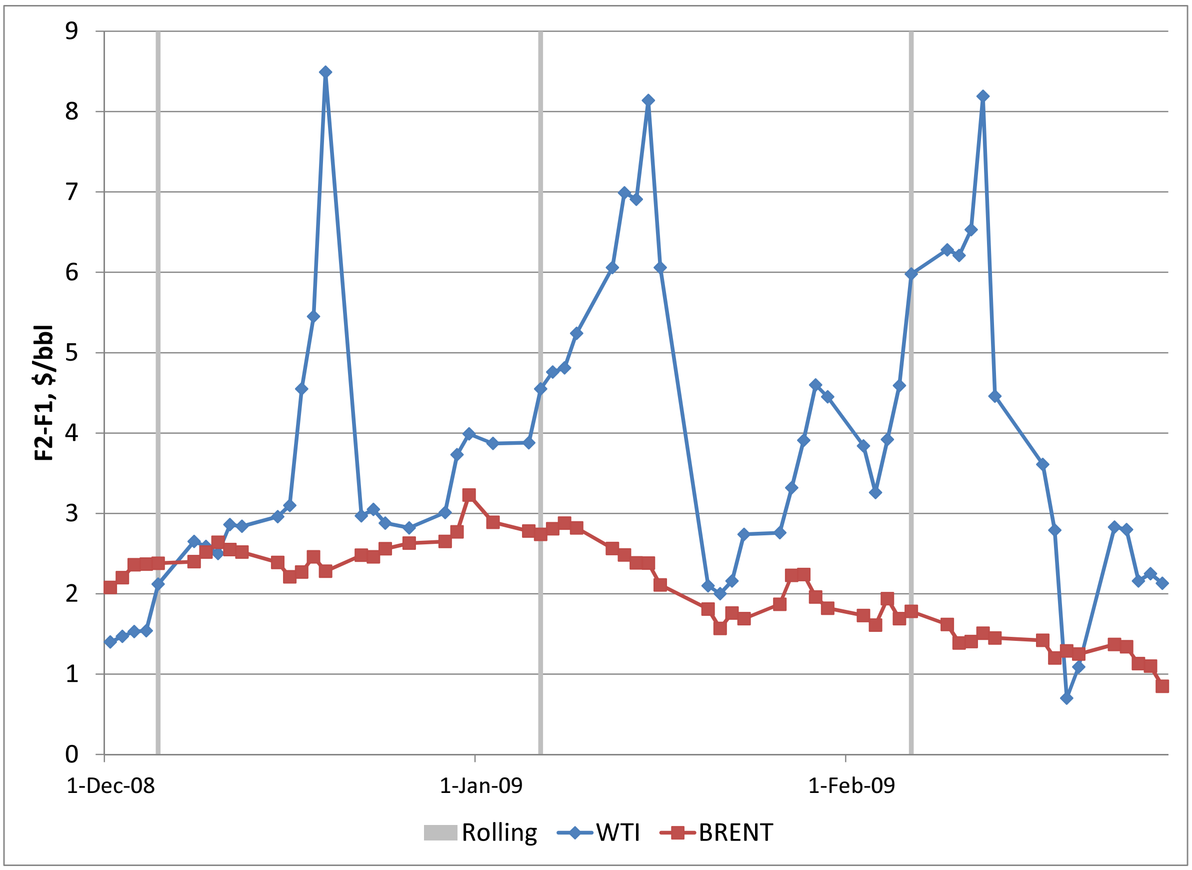


Figure 21: United States Oil Fund daily holdings as of March 24, 2009, comparison with Compustat and EIA data. The Fund redeemed 1,400,000 shares and sold 828 contracts at $53.98 dollars per barrel, which was the closing price from the trading floor of the NYMEX, provided by the EIA(at the bottom), $F_{1,t}$. Analysis of Compustat data indicates one day delay in shares outstanding figures - number of actual shares outstanding at the beginning of the day as reported by the Fund is 107,500,000, but is reflected in Compustat on March 25,2009. However closing USO shares price is correct, $31.61. Thus number of contracts bought or sold by the Fund during the day, can be calculated using Compustat and EIA data as $x_t = (shares_{t+2} - shares_{t+1}) * closing\,shares\,price_t/(1000F_{1,t})$, in particular $x_{3/24/2009} = (106,100,000 - 107,500,000) * 31.61/(1000 * 53.98) = -820$ which is almost -828.

Daily Holdings

As of 3/24/2009

NAV	$31.92		
Units Outstanding	107,500,000	Closing Price	$31.61
Total Net Assets	3,431,360,307.41		

Current Holdings *(subject to change)*

Type	Security	Quantity	Price	Market Value
Oil Futures and Other Oil Interests	F/C WTI CRUDE FUTURE ICE MAY09	28,182	53.98	1,521,264,360
	F/C WS CRUDE FUTURE MAY09	4,000	53.98	215,920,000
	F/C WTI CRUDE FUTURE MAY09	31,390	53.98	1,694,432,200
US Treasuries				
Cash	US DOLLARS	3,125,302,716	1.00	3,125,302,716

Pending Trades — Shares Created (Redeemed) -1,400,000

Type	Security	Quantity	Price	
Sold	CL May'09 Crude	-828	53.98	

Estimated Annualized Yield on Cash Holdings*	0.28 %
Estimated Annualized Management Expenses**	0.45 %

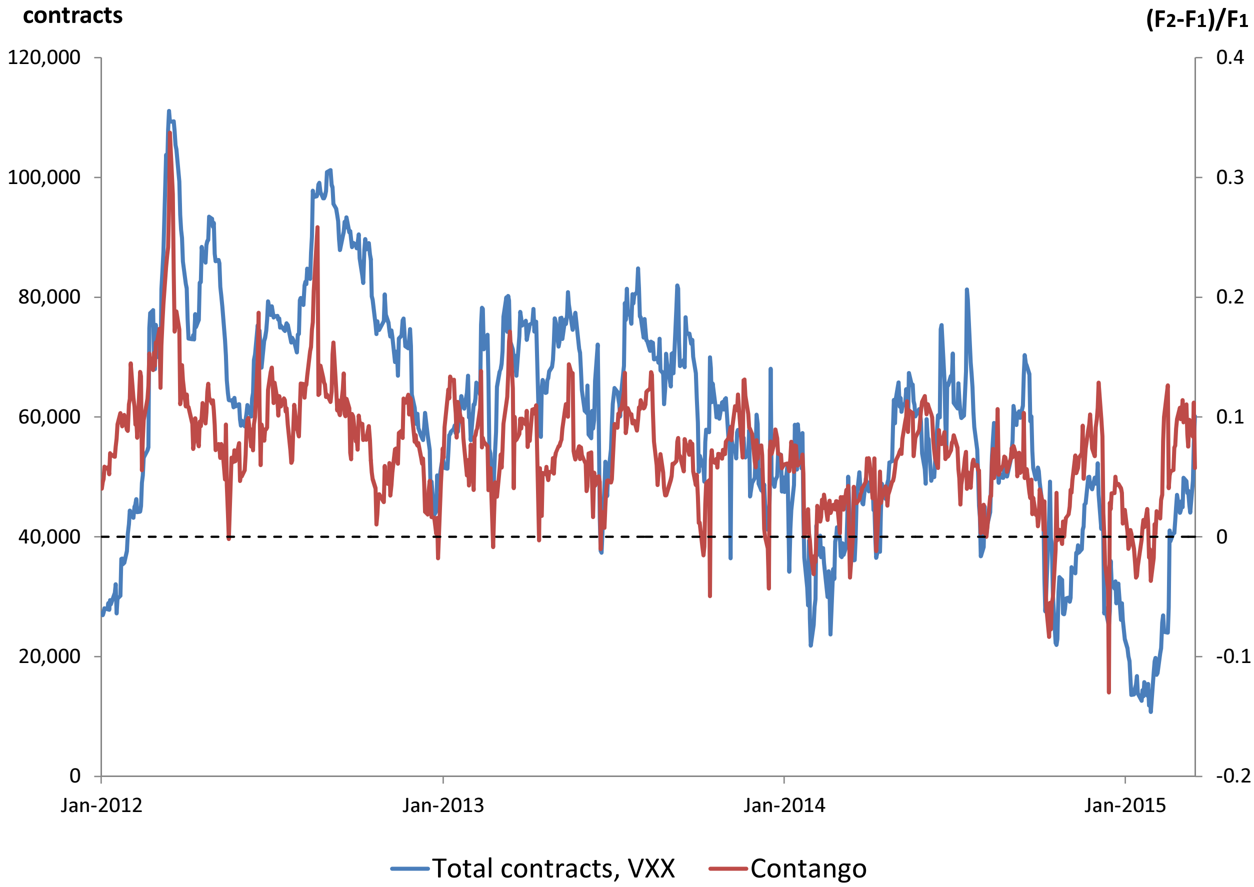
Compustat

Date	Fund	cusip		Shares outstanding	Closing price
20090324	USO	91232N108	UNITED STATES OIL FUND LP	108200000	31.61
20090325	USO	91232N108	UNITED STATES OIL FUND LP	107500000	31.21
20090326	USO	91232N108	UNITED STATES OIL FUND LP	106100000	32.02

EIA

	Crude Oil Future Contract 1 (Dollars per Barrel)
Mar 24, 2009	53.98

Figure 22: VXX and contango. The blue line shows estimated number of contracts tied with investment in VXX - an exchange traded note (ETN) issued by Barclays Bank and linked to the performance of an index S&P VIX Short-Term Futures (left axis). The red line shows contango on VIX market defined as price difference $\frac{F_{2,t} - F_{1,t}}{F_{1,t}}$ of two closest futures contracts. The figure shows that days with large inflows on investment in VXX are associated with larger contango.



11 Tables

Table 1: Summary statistics: daily position of the United States Oil Fund (number of contracts) and contango in WTI oil futures market, $F_{2,t} - F_{1,t}$, where $F_{1,t}$ - the price of a nearby futures contract, $F_{2,t}$ - the price of the next out contract.

	Mean	Std.Dev	Min	Max
USO	17323	15029	2949	105581
contango	0.53	1.04	-2.66	8.49
Cushing occupancy	0.62	0.16	0.27	0.91

Table 2: USO and contango. Change in contango is given by $\Delta c_t = 100\left[\frac{F_{2,t}-F_{1,t}}{F_{1,t}} - \frac{F_{2,t-1}-F_{1,t-1}}{F_{1,t-1}}\right]$, changes in USO position are defined as the number of contracts acquired or sold by the Fund to account for shares created or redeemed, normalized by 1000 for exposition purpose. Specification I test if changes in USO positions help predict futures changes in contango. Specifications II-VI test if lagged changes in USO positions are relevant for changes in contango, where we account for market reaction to news about storage - indicator of announcement I_t and announced change in volumes of oil stored at Cushing relative to maximum capacity, oil market intraday volatility defined as high minus low normalized by the closing price of $F_{1,t}$, stock market volatility index VIX and 5-years Treasury Note rate. Panel A: total sample from September 1, 2006 to March 20, 2015. Panel B: restricted sample, in which two episodes of growth of the Fund and contango, including few month around, are eiliminated, mainly September, 2008 - June, 2009 and March, 2014-March, 2015. Standard errors are computed using Newey-West procedure with 5 lags, other numbers of lags produce similar results. To give a perspective, a maximum one-day change in USO position that happened during the cycle (24th-4th days) was on January 28, 2009, when the Fund bought 7500 of contracts. Our results imply that such a growth in positions may be associated with 18 cents increase in contango the next day, given first futures price of 42 dollars prevalent during that period.

Panel A: Total sample

	I	II	III	IV	V	VI
ΔUSO_{t-1}	.058**(.029)	.055*(.028)	.056**(.028)	.056*(.029)	.056**(.028)	.057**(.028)
Δc_{t-1}	.062(.091)	.066(.089)	.064(.088)	.069(.089)	.065(.089)	.073(.086)
I_t		-.038(.03)	-.038(.03)	-.038(.03)	-.038(.03)	-.038(.03)
$\Delta Cushing_t$		3.934***(1.205)	3.88***(1.243)	3.923***(1.203)	3.98***(1.209)	3.906***(1.253)
$Oil\,volatility_t$				-.251(.784)		-.764(.949)
Trn_t					.008(.005)	.013(.008)
VIX_t			.001(.001)			.002(.001)
const	.023***(.008)	.031***(.01)	.004(.023)	.035**(.016)	.015(.015)	-.021(.025)
N	577	577	577	577	577	577
R^2 adj	0.037	0.058	0.059	0.057	0.057	0.060

Panel B: No peaks

	I	II	III	IV	V	VI
ΔUSO_{t-1}	.011(.011)	.01(.01)	.011(.01)	.01(.01)	.011(.01)	.014(.01)
Δc_{t-1}	.049(.05)	.046(.051)	.045(.051)	.046(.05)	.044(.051)	.049(.052)
I_t		-.034*(.019)	-.034*(.019)	-.034*(.019)	-.034*(.019)	-.034*(.019)
$\Delta Cushing_t$		3.329***(.975)	3.301***(.98)	3.329***(.971)	3.38***(.975)	3.316***(.96)
$Oil\,volatility_t$				.003(.57)		-.972(.788)
Trn_t					.008(.005)	.015**(.007)
VIX_t			-.001(.001)			0(.001)
const	.016***(.006)	.024***(.007)	.038*(.021)	.024***(.009)	.007(.011)	.004(.02)
N	451	451	451	451	451	451
R^2 adj	-0.000085	0.042	0.040	0.040	0.044	0.044

* p<0.1; ** p<0.05; *** p<0.01

Table 3: USO and contango - two episodes of large growth of Fund and contango, mainly September, 2008 - June, 2009 and March, 2014-March, 2015. Change in contango is given by $\Delta c_t = 100\left[\frac{F_{2,t}-F_{1,t}}{F_{1,t}} - \frac{F_{2,t-1}-F_{1,t-1}}{F_{1,t-1}}\right]$, changes in USO position are defined as the number of contracts acquired or sold by the Fund to account for shares created or redeemed, normalized by 1000 for exposition purpose. Specification I test if changes in USO positions help predict futures changes in contango. Specifications II-VI test if lagged changes in USO positions are relevant for changes in contango, where we account for market reaction to news about storage - indicator of announcement I_t and announced change in volumes of oil stored at Cushing relative to maximum capacity, oil market intraday volatility defined as high minus low normalized by the closing price of $F_{1,t}$, stock market volatility index VIX and 5-years Treasury Note rate. Panel A: benchmark cycle - 24th day of one month to 4th day of the next month. Panel B: shorter cycle 26th day of one month to 2th day of the next month.

Panel A: Peaks, 24th-4th

	I	II	III	IV	V	VI
ΔUSO_{t-1}	.094**(.042)	.088**(.043)	.089**(.042)	.088**(.043)	.089**(.044)	.087**(.041)
Δc_{t-1}	.049(.121)	.061(.117)	.055(.116)	.074(.116)	.061(.117)	.071(.112)
I_t		-.063(.12)	-.061(.121)	-.062(.119)	-.062(.121)	-.06(.12)
$\Delta Cushing_t$		4.799(4.13)	4.28(4.432)	4.828(4.162)	4.837(4.161)	4.072(4.627)
$Oil\, volatility_t$				-.829(1.725)		-1.175(1.94)
Trn_t					.018(.043)	-.035(.078)
VIX_t			.002(.002)			.003(.003)
const	.041(.031)	.05(.042)	-.006(.046)	.074(.072)	.016(.109)	.075(.153)
N	125	125	125	125	125	125
R^2 adj	0.064	0.060	0.058	0.055	0.053	0.047

Panel B: Peaks, 26th-2d

	I	II	III	IV	V	VI
ΔUSO_{t-1}	.221***(.062)	.218***(.061)	.233***(.055)	.209***(.057)	.236***(.057)	.223***(.044)
Δc_{t-1}	-.101(.203)	-.095(.191)	-.166(.202)	-.01(.16)	-.11(.191)	-.075(.166)
I_t						
$\Delta Cushing_t$		2.69(4.445)	2.145(4.28)	2.169(3.46)	3.166(4.465)	1.477(3.123)
$Oil\, volatility_t$				-2.952(2.277)		-3.385(2.281)
Trn_t					.123**(.055)	-.005(.106)
VIX_t			.006*(.003)			.007(.004)
const	.057(.048)	.057(.063)	-.11*(.063)	.135*(.08)	-.176(.144)	-.027(.178)
N	65	65	65	65	65	65
R^2 adj	0.220	0.196	0.232	0.211	0.198	0.244

* p<0.1; ** p<0.05; *** p<0.01

Table 4: Summary statistics: daily number of contracts associated with investment in VXX and contango in VIX futures market defined as $F_{2,t} - F_{1,t}$, where $F_{1,t}$ - the price of a nearby futures contract, $F_{2,t}$ - the price of the next out contract.

	Mean	Std. Dev.	Min	Max
$F_{2,t} - F_{1,t}$	1.14	0.83	-3	5.45
VXX	59476	19533	10758	111122
ΔVXX_t	262	4121	-21873	20318

Table 5: Correlation matrix: VXX and normalized contango, $c_t = \frac{F_{2,t} - F_{1,t}}{F_{1,t}}$, both in levels and in first differences. Daily data.

	VXX_t
c_t	0.6619

	ΔVXX_t	ΔVXX_{t-1}	ΔVXX_{t-2}	ΔVXX_{t-3}	ΔVXX_{t-4}
ΔVXX_t	1				
ΔVXX_{t-1}	0.0041	1			
ΔVXX_{t-2}	0.0081	0.0022	1		
ΔVXX_{t-3}	-0.0456	0.0065	-0.0002	1	
ΔVXX_{t-4}	-0.0697	-0.0453	0.0070	0.0003	1

	ΔVXX_t	ΔVXX_{t-1}	ΔVXX_{t-2}	ΔVXX_{t-3}	ΔVXX_{t-4}
Δc_t	0.0968	0.1140	0.0634	0.0643	-0.0317

Table 6: Predictive analysis: VXX and contango in VIX market. Change in normalized contango is given by $\Delta c_t = 100 \left[\frac{F_{2,t} - F_{1,t}}{F_{1,t}} - \frac{F_{2,t-1} - F_{1,t-1}}{F_{1,t-1}} \right]$, daily data. This specification tests if changes in VXX position help predict future movements of contango. Number of contracts associated with investment in VXX is divided by 1000 for exposition purposes. Trn_t is 5years Treasury notes rate. Regressions also include and 4 lags of contango changes(omitted). Standard errors are computed using Newey-West procedure with 5 lags, other numbers of lags produce similar results.

	I	II	III	IV	V	VI
ΔVXX_t	.053**(.023)					
ΔVXX_{t-1}		.079***(.022)				.079***(.022)
ΔVXX_{t-2}			.053***(.02)			
ΔVXX_{t-3}				.051***(.019)		
ΔVXX_{t-4}					-.005(.023)	
Trn_t						.08(.169)
cons	-.039(.076)	-.044(.075)	-.039(.077)	-.039(.077)	-.029(.077)	-.142(.225)
adj R^2	0.046	0.057	0.046	0.045	0.037	0.036
N	756	756	756	756	756	756

* p<0.1; ** p<0.05; *** p<0.01

12 Appendix

12.1 Proposition 1. No instrument: general solution

Informed traders active in the first period trade

$$z_i = \frac{E\left[(\theta - p_1)|s_i\right]}{2c_i} \equiv \frac{1}{c_i} d_1 s_i$$

where d_1 carries all information extraction part.

Fraction μ of informed agents is active in the first period, we lable them by $i \in [0, \mu]$ from lowest to largest costs c_i,which we can parametrize by $c_i = \frac{\bar{c} - \underline{c}}{\mu} i + \underline{c}$. Total informed order flow in the first period is given by

$$\int_0^\mu z_i di = \int_0^\mu \frac{1}{c_i} d_1 s_i di = d_1 \int_0^\mu \frac{1}{c_i}(\theta + \nu_i) di = d_1 \theta \frac{\mu}{\bar{c} - \underline{c}} \int_{\underline{c}}^{\bar{c}} \frac{1}{c_i} dc_i = \frac{1}{\bar{c} - \underline{c}} ln \left(\frac{\bar{c}}{\underline{c}} \right) \mu d_1 \theta \equiv \mu g_1 \theta$$

Thus total informed order flow is proportional to fundamental value θ with coefficient of proportionality g_1.

Similarly, informed traders active in the second period are labeled by $i \in [\mu, 1]$ trade

$$z_i = \frac{E[(\theta - p_2)|s_i]}{2c_i} \equiv \frac{1}{c_i} d_2 s_i$$

and aggregation gives

$$\int_\mu^1 z_i di = \int_\mu^1 \frac{1}{c_i} d_2 s_i di = \frac{1-\mu}{\bar{c} - \underline{c}} d_2 \int_{\underline{c}}^{\bar{c}} \frac{1}{c_i} (\theta + \nu_i) \, dc_i = \frac{1}{\bar{c} - \underline{c}} ln \left(\frac{\bar{c}}{\underline{c}} \right) (1-\mu) d_2 \theta \equiv (1-\mu) g_2 \theta$$

Uninformed hedgers trade trade $u_t + \xi_{j,t}$ each in period t, thus aggregation gives total uninformed demand u_1 in the first period and u_2 in the second. Now we can write the combined order flow:

$$\begin{aligned} y_1 &= \mu g_1 \theta + u_1 + x_1 \\ y_2 &= (1-\mu) g_2 \theta + u_2 + x_2 \end{aligned}$$

Now we can find d_1 and d_2:

$$p_1 = \lambda_{11} y_1 = \lambda_{11} \left(\mu g_1 \theta + u_1 + as \right)$$

$$d_1 = \frac{E\left[(\theta - p_1)|s_i\right]}{2s_i} = \frac{\left[1 - \lambda_{11} \mu g_1 - \lambda_{11} a\right]}{2} \frac{\sigma_\theta^2}{\sigma_\theta^2 + \sigma_\nu^2}$$

because $g_1$is a function of d_1 and because we are lookign for a symmetric equilibrium,

$$d_1 = \frac{1}{2}\left[1-\lambda_{11}\mu\frac{1}{\bar{c}-\underline{c}}ln\left(\frac{\bar{c}}{\underline{c}}\right)d_1-\lambda_{11}a\right]\frac{\sigma_\theta^2}{\sigma_\theta^2+\sigma_\nu^2}$$

$$d_1 = \frac{1}{2}\frac{[1-\lambda_{11}a]}{\left(1+\frac{1}{2}\lambda_{11}\mu\frac{1}{\bar{c}-\underline{c}}ln\left(\frac{\bar{c}}{\underline{c}}\right)\frac{\sigma_\theta^2}{\sigma_\theta^2+\sigma_\nu^2}\right)}\frac{\sigma_\theta^2}{\sigma_\theta^2+\sigma_\nu^2}$$

Similarly

$$\begin{aligned} d_2 &= \frac{E\left[(\theta-p_2)|s_i\right]}{2s_i} \\ &= \frac{E\left[(\left[1-\lambda_{12}\mu g_1-\lambda_{22}(1-\mu)g_2\right]\theta-\lambda_{12}u_1-\lambda_{12}a(\theta+\varepsilon)-\lambda_{22}u_2-\lambda_{22}b(\theta+\varepsilon))|s_i=\theta+\nu_i\right]}{2s_i} \\ &= \frac{\left[1-\lambda_{12}\left(\mu g_1+a\right)-\lambda_{22}\left((1-\mu)g_2+b\right)\right]}{2}\frac{\sigma_\theta^2}{\sigma_\theta^2+\sigma_\nu^2} \end{aligned}$$

and

$$d_2 = \frac{1}{2}\frac{\left[1-\lambda_{12}\left(\mu g_1+a\right)-\lambda_{22}b\right]\frac{\sigma_\theta^2}{\sigma_\theta^2+\sigma_\nu^2}}{1+\frac{1}{2}\lambda_{22}(1-\mu)\frac{1}{\bar{c}-\underline{c}}ln\left(\frac{\bar{c}}{\underline{c}}\right)\frac{\sigma_\theta^2}{\sigma_\theta^2+\sigma_\nu^2}}$$

calculate the integrals.

Now we are ready to solve arbitrageur's problem:

$$\pi_{arbitrageur} = E\left[(\theta-p_1)x_1+(\theta-p_2)x_2|s\right]-cx_1^2-cx_2^2$$

Arbitrageur knows that market makers use linear pricing strategies, and know how order flow is defined, although they cannot observe it. Thus they know that

$$\begin{aligned} p_1 &= \lambda_{11}y_1 = \lambda_{11}\left(\mu g_1\theta+u_1+x_1\right) \\ p_2 &= \lambda_{12}y_1+\lambda_{22}y_2 = \lambda_{12}\left(\mu g_1\theta+u_1+x_1\right)+\lambda_{22}\left((1-\mu)g_2\theta+u_2+x_2\right) \end{aligned}$$

Putting that back and using the facto that signal s is informative only about θ and defining $\hat{\theta}=E[\theta|s]=\frac{\sigma_\theta^2}{\sigma_\theta^2+\sigma_\varepsilon^2}=i_{\theta s}s$ gives

$$\begin{aligned} \pi_{arbitrageur} &= \left(\left[1-\lambda_{11}\mu g_1\right]\hat{\theta}-\lambda_{11}x_1\right)x_1 \\ &\quad+\left(\left[1-\lambda_{12}\mu g_1-\lambda_{22}(1-\mu)g_2\right]\hat{\theta}-\lambda_{12}x_1-\lambda_{22}x_2\right)x_2-cx_1^2-cx_2^2 \end{aligned}$$

first order conditions imply

$$x_1 = \frac{[1-\lambda_{11}\mu g_1]\,\hat{\theta} - \lambda_{12}x_2}{2(\lambda_{11}+c)}$$

$$x_2 = \frac{[1-\lambda_{12}\mu g_1 - \lambda_{22}(1-\mu)g_2]\,\hat{\theta} - \lambda_{12}x_1}{2(\lambda_{22}+c)}$$

and finally

$$x_1 = \frac{2(\lambda_{22}+c)\,[1-\lambda_{11}\mu g_1] - \lambda_{12}\,[1-\lambda_{12}\mu g_1 - \lambda_{22}(1-\mu)g_2]}{4(\lambda_{22}+c)(\lambda_{11}+c)-\lambda_{12}^2}\hat{\theta}$$

$$x_2 = \frac{2(\lambda_{11}+c)\,[1-\lambda_{12}\mu g_1 - \lambda_{22}(1-\mu)g_2] - \lambda_{12}\,[1-\lambda_{11}\mu g_1]}{4(\lambda_{22}+c)(\lambda_{11}+c)-\lambda_{12}^2}\hat{\theta}$$

Therefore,

$$a = \frac{2(\lambda_{22}+c)\,[1-\lambda_{11}\mu g_1] - \lambda_{12}\,[1-\lambda_{12}\mu g_1 - \lambda_{22}(1-\mu)g_2]}{4(\lambda_{22}+c)(\lambda_{11}+c)-\lambda_{12}^2}i_{\theta s}$$

$$b = \frac{2(\lambda_{11}+c)\,[1-\lambda_{12}\mu g_1 - \lambda_{22}(1-\mu)g_2] - \lambda_{12}\,[1-\lambda_{11}\mu g_1]}{4(\lambda_{22}+c)(\lambda_{11}+c)-\lambda_{12}^2}i_{\theta s}$$

Finaly, we can solve problem of market makers

$$p_1 = E\left[\theta|y_1\right] = E\left[\theta|\left(\mu g_1 + a\right)\theta + u_1 + a\varepsilon\right] = \frac{\left(\mu g_1 + a\right)\sigma_\theta^2}{\left(\mu g_1 + a\right)^2\sigma_\theta^2 + \sigma_{u1}^2 + a^2\sigma_\varepsilon^2}y_1 \equiv \lambda_{11}y_1$$

For the second period price we can use the recursive formula

$$\begin{aligned} p_2 &= E[\theta|y_1,y_2] \\ &= E[\theta|y_1] + E\left[\theta - E[\theta|y_1]\middle|y_2 - E[y_2|y_1]\right] \\ &= \lambda_{11}y_1 + E\left[\theta\middle|y_2 - E[y_2|y_1]\right] \end{aligned}$$

Notice that because $E[\theta|y_1]$ is profortional to y_1, and projection error $y_2 - E[y_2|y_1]$ is orthogonal to y_1, the last equality follows. Define ρ as coefficient of projection of second period demand on first period demand

$$E[y_2|y_1] = \rho y_1$$

$$E[y_2|y_1] = E[((1-\mu)g_2 + b)\,\theta + u_2 + b\varepsilon|\,(\mu g_1 + a)\,\theta + u_1 + a\varepsilon]$$

$$\rho = \frac{(\mu g_1 + a)\,((1-\mu)g_2 + b)\,\sigma_\theta^2 + ab\sigma_\varepsilon^2}{(\mu g_1 + a)^2\,\sigma_\theta^2 + \sigma_{u1}^2 + a^2\sigma_\varepsilon^2}$$

Similarly, λ_{22} as projection of fundamental value on projection error $y_2 - E[y_2|y_1]$

$$E\left[\theta\middle|y_2 - E[y_2|y_1]\right] = \lambda_{22}\,(y_2 - \rho y_1)$$

$$\begin{aligned} y_2 - \rho y_1 &= ((1-\mu)g_2 + b)\,\theta + u_2 + b\varepsilon - \rho\,((\mu g_1 + a)\,\theta + u_1 + a\varepsilon) \\ &= ((1-\mu)g_2 + b - \rho\,(\mu g_1 + a))\,\theta + u_2 - \rho u_1 + (b - \rho a)\,\varepsilon \end{aligned}$$

$$\lambda_{22} = \frac{((1-\mu)g_2 + b - \rho\,(\mu g_1 + a))\,\sigma_\theta^2}{((1-\mu)g_2 + b - \rho\,(\mu g_1 + a))^2\,\sigma_\theta^2 + \sigma_{u2}^2 + \rho^2\sigma_{u1}^2 + (b - \rho a)^2\sigma_\varepsilon^2}$$

Then second period price equates

$$p_2 = \lambda_{11} y_1 + \lambda_{22}\,(y_2 - \rho y_1) = \lambda_{12} y_1 + \lambda_{22} y_2$$

where $\lambda_{12} = \lambda_{11} - \rho\lambda_{22}$.

Equilibrium in of the model without the instrument may be summarized by the following system of equations:

$$\lambda_{11} = \frac{(\mu g_1 + a)\,\sigma_\theta^2}{(\mu g_1 + a)^2\,\sigma_\theta^2 + \sigma_{u1}^2 + a^2\sigma_\varepsilon^2}$$

$$\rho = \frac{(\mu g_1 + a)\,((1-\mu)g_2 + b)\,\sigma_\theta^2 + ab\sigma_\varepsilon^2}{(\mu g_1 + a)^2\,\sigma_\theta^2 + \sigma_{u1}^2 + a^2\sigma_\varepsilon^2}$$

$$\lambda_{22} = \frac{((1-\mu)g_2 + b - \rho\,(\mu g_1 + a))\,\sigma_\theta^2}{((1-\mu)g_2 + b - \rho\,(\mu g_1 + a))^2\,\sigma_\theta^2 + \sigma_{u2}^2 + \rho^2\sigma_{u1}^2 + (b - \rho a)^2\sigma_\varepsilon^2}$$

$$\lambda_{12} = \lambda_{11} - \rho\lambda_{22}$$

$$a = \frac{2(\lambda_{22}+c)\left[1-\lambda_{11}\mu g_1\right] - \lambda_{12}\left[1-\lambda_{12}\mu g_1 - \lambda_{22}(1-\mu)g_2\right]}{4(\lambda_{22}+c)(\lambda_{11}+c)-\lambda_{12}^2} i_{\theta s}$$

$$b = \frac{2(\lambda_{11}+c)\left[1-\lambda_{12}\mu g_1 - \lambda_{22}(1-\mu)g_2\right] - \lambda_{12}\left[1-\lambda_{11}\mu g_1\right]}{4(\lambda_{22}+c)(\lambda_{11}+c)-\lambda_{12}^2} i_{\theta s}$$

$$g_1 = \frac{1}{2}\frac{1}{\bar{c}-\underline{c}} ln\left(\frac{\bar{c}}{\underline{c}}\right) \frac{\left[1-\lambda_{11}a\right]}{\left(1+\frac{1}{2}\lambda_{11}\mu\frac{1}{\bar{c}-\underline{c}} ln\left(\frac{\bar{c}}{\underline{c}}\right)\frac{\sigma_\theta^2}{\sigma_\theta^2+\sigma_\nu^2}\right)} \frac{\sigma_\theta^2}{\sigma_\theta^2+\sigma_\nu^2}$$

$$g_2 = \frac{1}{2}\frac{1}{\bar{c}-\underline{c}} ln\left(\frac{\bar{c}}{\underline{c}}\right) \frac{\left[1-\lambda_{12}\left(\mu g_1 + a\right) - \lambda_{22}b\right]\frac{\sigma_\theta^2}{\sigma_\theta^2+\sigma_\nu^2}}{1+\frac{1}{2}\lambda_{22}(1-\mu)\frac{1}{\bar{c}-\underline{c}} ln\left(\frac{\bar{c}}{\underline{c}}\right)\frac{\sigma_\theta^2}{\sigma_\theta^2+\sigma_\nu^2}}$$

12.1.1 Proposition 2

In any linear equilibrium

1. Prices become more informative over time and underreact to fundamental value θ, in particular $I(p_t) = \lambda_{\theta,t}$ and $0 < \lambda_{\theta,1} \leq \lambda_{\theta,2} \leq 1$.

2. The arbitrageur trades in the direction of fundamentals in period 2, or $b > 0$.

3. The arbitrageur and informed traders profit at the expense of uninformed traders.

Arbitrageur may trade against fundamentals in period 1, if g_1 is sufficiently large.

Proof:

$p_1 = \lambda_{11}\left(\mu g_1 + a\right)\theta + shocks$

$p_2 = \left[\lambda_{12}\left(\mu g_1 + a\right)\theta + \lambda_{22}\left((1-\mu)g_2 + b\right)\right]\theta + shocks = \lambda_{\theta,2}\theta + shocks$

Consider first period. As long as there is some uninformed trading, market makers would be afraid to mix up uninformed demand swings with informed trading, and thus would never fully react to order flow. Let's find $I(p_1)$, given that $p_1 = \lambda_{11}\left(\mu g_1 + a\right)\theta + \lambda_{11}\left(u_1 + a\varepsilon\right)$. Indeed, λ_{11}^2 cancels out and

$$I(p_1) = \frac{\left(\mu g_1 + a\right)^2 \sigma_\theta^2}{\left(\mu g_1 + a\right)^2 \sigma_\theta^2 + \sigma_{u1}^2 + a^2\sigma_\varepsilon^2} = \lambda_{11}\left(\mu g_1 + a\right),$$

where we use the formula for λ_{11} from proposition 1. Therefore, $I(p_1) = \lambda_{\theta,1} = \lambda_{11}\left(\mu g_1 + a\right) < 1$.

Similarly,

$$\begin{aligned}
\lambda_{\theta,2} &= \lambda_{12}\left(\mu g_1 + a\right) + \lambda_{22}\left((1-\mu)g_2 + b\right) \\
&= \lambda_{11}\left(\mu g_1 + a\right) + \lambda_{22}\left((1-\mu)g_2 + b - \rho\left(\mu g_1 + a\right)\right) \\
&= \frac{\left(\mu g_1 + a\right)^2 \sigma_\theta^2}{\left(\mu g_1 + a\right)^2 \sigma_\theta^2 + \sigma_{u1}^2 + a^2\sigma_\varepsilon^2} + \frac{\left((1-\mu)g_2 + b - \rho\left(\mu g_1 + a\right)\right)^2 \sigma_\theta^2}{\left((1-\mu)g_2 + b - \rho\left(\mu g_1 + a\right)\right)^2 \sigma_\theta^2 + \sigma_{u2}^2 + \rho^2\sigma_{u1}^2 + (b-\rho a)^2\sigma_\varepsilon^2}
\end{aligned}$$

and

$$\rho = \frac{\left(\mu g_1 + a\right)\left((1-\mu)g_2 + b\right)\sigma_\theta^2 + ab\sigma_\varepsilon^2}{\left(\mu g_1 + a\right)^2 \sigma_\theta^2 + \sigma_{u1}^2 + a^2\sigma_\varepsilon^2}$$

one can substitute for ρ and rearrange to find expression for $\lambda_{\theta,2}$ and compare it with $I(p_2)$. Notice also that $\lambda_{\theta,2}$ equals $\lambda_{\theta,1}$ plus some positive term, thus informativeness is larger in period 2.

If $\sigma_\varepsilon^2 = 0$, we can use first order condition that defines x_2, and substitute equilibrium strategy for $x_1 = as = a\hat{\theta}$, then

$$\begin{aligned}
x_2 &= b\hat{\theta} = \frac{\left[1 - \lambda_{12}\mu g_1 - \lambda_{22}(1-\mu)g_2\right]\hat{\theta} - \lambda_{12}a\hat{\theta}}{2\lambda_{22}} \\
&> \frac{\left[1 - \lambda_{12}\left(\mu g_1 + a\right) - \lambda_{22}\left((1-\mu)g_2 + b\right) + \lambda_{22}b\right]\hat{\theta}}{2\lambda_{22}} > \frac{b\hat{\theta}}{2}
\end{aligned}$$

last inequality follows, because second price underreacts to fundamentals, $\lambda_{\theta,2} = \lambda_{12}\left(\mu g_1 + a\right) + \lambda_{22}\left((1-\mu)g_2 + b\right) < 1$. Hence, $b > 0$. If $\sigma_\varepsilon^2 > 0$, in the appendix.

Finally, if g_1 is sufficiently large, than the arbitrageur may trade against fundamentals in the first period

12.2 Proposition 3: instrument - general solution

The combined order flow for the asset in the second period and for the instrument.

$$y_f = \varkappa_\theta \frac{1}{w} d_f (1-\mu)\theta + \varkappa_{u2} u_2$$

$$\begin{aligned}
y_2 &= y_f + \frac{1}{\bar{c} - \underline{c}} d_2 ln\left(\frac{wA_1}{\underline{c}}\right)(1-\mu)\theta + (1-\varkappa_{u2})u_2 + x_2 \\
&= \left(\varkappa_\theta \frac{1}{w} d_f + \frac{1}{\bar{c} - \underline{c}} d_2 ln\left(\frac{wA_1}{\underline{c}}\right)\right)(1-\mu)\theta + u_2 + x_2
\end{aligned}$$

Notice that order flow for the instrument will be brought back to the market of the risky asset in the second period. To simplify the calculations, rewrite

$$y_f = \varkappa\left((1-\mu)g_f\theta + u_2\right)$$

where $g_f = \frac{\varkappa_\theta}{\varkappa_{u2}}\frac{1}{w}d_f$ and $\varkappa_{u2} = \varkappa$. And similarly

$$y_2 = (1-\mu)g_2\theta + u_2 + x_2$$

where $g_2 = \frac{1}{\bar{c}-\underline{c}}d_2 ln\left(\frac{wA_1}{\underline{c}}\right) + \varkappa_\theta\frac{1}{w}d_f$. Notice, that y_2 has exactly the same form as before, only the definition of g_2 is a bit different.

First period order flow can be found as before, where $g_1 = d_1\frac{1}{\bar{c}-\underline{c}}ln\left(\frac{\bar{c}}{\underline{c}}\right)$

$$y_1 = \mu g_1\theta + u_1 + x_1$$

As before we will be looking for a linear equilibrium:

$$\begin{aligned} p_1 &= \lambda_{11}y_1 \\ p_2 &= \lambda_{12}y_1 + \lambda_{22}y_2 \end{aligned}$$

and arbitrageur's trading function is linear in his signals

$$\begin{aligned} x_1 &= a_1\frac{1}{\varkappa}y_f + a_2 s \\ x_2 &= b_1\frac{1}{\varkappa}y_f + b_2 s \end{aligned}$$

arbitrageur does not trade the new instrument but may observe, y_f - we may think of it as of total asset under management - the statistics that is required to be published every day. Thus the arbitrageur extracts information from both y_f and s and trades based on that information. As before, we assume that the arbitrageur decides how much to trade before the asset market opens - thus x_2 does not depend on information revealed in the first period, the arbitrageur does not update his information.

Now we can find d_1 and d_2, and d_f. Algebra gives

$$d_1 = \frac{1}{2}\frac{\left[1-\lambda_{11}\left(a_1(1-\mu)g_f + a_2\right)\right]\frac{\sigma_\theta^2}{\sigma_\theta^2+\sigma_\nu^2}}{1+\frac{1}{2}\lambda_{11}\mu\frac{1}{\bar{c}-\underline{c}}ln\left(\frac{\bar{c}}{\underline{c}}\right)\frac{\sigma_\theta^2}{\sigma_\theta^2+\sigma_\nu^2}}$$

$$d_f = \frac{1}{2}\frac{\frac{\sigma_\theta^2}{\sigma_\theta^2+\sigma_\nu^2}}{1+\frac{1}{2}\lambda_f(1-\mu)\varkappa_\theta\frac{1}{w}\frac{\sigma_\theta^2}{\sigma_\theta^2+\sigma_\nu^2}}$$

$$d_2 = \frac{1}{2}\frac{\left[1-\lambda_{12}\left(\mu g_1+a_1(1-\mu)g_f+a_2\right)-\lambda_{22}\left((1-\mu)\left(\varkappa_\theta\frac{1}{w}d_f+b_1 g_f\right)+b_2\right)\right]\frac{\sigma_\theta^2}{\sigma_\theta^2+\sigma_\nu^2}}{1+\frac{1}{2}\lambda_{22}(1-\mu)\frac{1}{\bar{c}-\underline{c}}ln\left(\frac{wA_1}{\underline{c}}\right)\frac{\sigma_\theta^2}{\sigma_\theta^2+\sigma_\nu^2}}$$

Similarly, the constant the define threshold value A_1 and B_0 can be found.[19]
Finally, we need to solve arbitrageur's problem:

$$\pi = E\left[(\theta-p_1)x_1+(\theta-p_2)x_2|y_f,s\right]-cx_1^2-cx_2^2$$

Solution takes the form

$$x_1 = \frac{2(\lambda_{22}+c)\hat{\theta}_1-\lambda_{12}\hat{\theta}_2}{4(\lambda_{22}+c)(\lambda_{11}+c)-\lambda_{12}^2}+\frac{\lambda_{12}\lambda_{22}}{4(\lambda_{22}+c)(\lambda_{11}+c)-\lambda_{12}^2}\hat{u}_2$$

$$x_2 = \frac{2(\lambda_{11}+c)\hat{\theta}_2-\lambda_{12}\hat{\theta}_1}{4(\lambda_{22}+c)(\lambda_{11}+c)-\lambda_{12}^2}-\frac{2(\lambda_{11}+c)\lambda_{22}}{4(\lambda_{22}+c)(\lambda_{11}+c)-\lambda_{12}^2}\hat{u}_2$$

where

$$\hat{\theta}_1 = (1-\lambda_{11}\mu g_1)\left(i_{\theta y}y_f+i_{\theta s}s\right)$$

$$\hat{\theta}_2 = (1-\lambda_{12}\mu g_1-\lambda_{22}(1-\mu)g_2)\left(i_{\theta y}y_f+i_{\theta s}s\right)$$

[19]

$$A_1 = \frac{\left[1-\lambda_{12}\left(\mu g_1+a_1(1-\mu)g_f+a_2\right)-\lambda_{22}\left((1-\mu)(g_2+b_1 g_f)+b_2\right)\right]^2}{\left(1-\lambda_f\varkappa(1-\mu)g_f\right)^2}$$

$$B_0 = \frac{\left[\lambda_f\varkappa-\lambda_{12}a_1-\lambda_{22}\left(1+b_1\right)\right]\sigma_{u2}^2}{\sigma_{u2}^2+\sigma_\xi^2}$$

$$\hat{u}_2 = i_{uy} y_f + i_{us} s$$

Variables $i_{\theta y}$, $i_{\theta s}$, i_{uy} and i_{us} are defined by information extraction formulas given two signal y_f and s.[20] Given expressions for x_1 and x_2, a_1, a_2, b_1, b_2 can be found.

Market maker's problem slightly changes given that now the arbitrageur reacts to investments in the new instrument and thus shock u_2, for example, leaks into first period price. Repeating the same steps as before, we get

$$\lambda_{11} = \frac{(\mu g_1 + a_1(1-\mu)g_f + a_2)\,\sigma_\theta^2}{(\mu g_1 + a_1(1-\mu)g_f + a_2)^2\,\sigma_\theta^2 + \sigma_{u1}^2 + a_1^2\sigma_{u2}^2 + a_2^2\sigma_\varepsilon^2}$$

$$\lambda_{22} = \frac{((1-\mu)(g_2 + b_1 g_f) + b_2 - \rho\,(\mu g_1 + a_1(1-\mu)g_f + a_2))\,\sigma_\theta^2}{((1-\mu)(g_2 + b_1 g_f) + b_2 - \rho\,(\mu g_1 + a_1(1-\mu)g_f + a_2))^2\,\sigma_\theta^2 + ((1+b_1) - \rho a_1)^2\,\sigma_{u2}^2 + \rho^2\sigma_{u1}^2 + (b_2 - \rho a_2)^2\sigma_\varepsilon^2}$$

$$\rho = \frac{(\mu g_1 + a_1(1-\mu)g_f + a_2)\,((1-\mu)(g_2 + b_1 g_f) + b_2)\,\sigma_\theta^2 + a_1\,(1+b_1)\,\sigma_{u2}^2 + a_2 b_2 \sigma_\varepsilon^2}{(\mu g_1 + a_1(1-\mu)g_f + a_2)^2\,\sigma_\theta^2 + \sigma_{u1}^2 + a_1^2\sigma_{u2}^2 + a_2^2\sigma_\varepsilon^2}$$

But we also need to solve the problem of market makers trading the new instrument. That gives

$$\lambda_f = [\lambda_{12}\,(\mu g_1 + a_1(1-\mu)g_f + a_2) + \lambda_{22}\,((1-\mu)(g_2 + b_1 g_f) + b_2)]\, i_\theta + [\lambda_{12} a_1 + \lambda_{22}(1+b_1)]\, i_{u2}$$

Thus we have found a linear solution as a function of w, and it is defined by the system of equations (numbers). What is left is to solve broker's problem and find optimal w. Given found solution we can rewrite broker's expected profit as

$$E\pi_{broker}(w) = \varkappa_\theta \frac{1}{w} d_f^2 (1-\mu)\left(\sigma_\theta^2 + \sigma_\nu^2\right) + w\varkappa_{u2}\left(\sigma_{u2}^2 + \sigma_\xi^2\right) - c^b\left(\varkappa_\theta^2 \frac{1}{w^2} d_f^2 (1-\mu)^2 \sigma_\theta^2 + \varkappa_{u2}^2 \sigma_{u2}^2\right)$$

That needs to be maximized with respect to w. As w decreases each trader pays less to the broker, however, the number of customers increases as well - more agents choose new instrument

[20]

$$E[\theta|y_f, s] = i_{\theta y} y_f + i_{\theta s} s$$

where $i_\theta = \frac{1}{\varkappa}\frac{(1-\mu)g_f\sigma_\theta^2}{(1-\mu)^2 g_f^2 \sigma_\theta^2 + \sigma_{u2}^2}$; $i_{\theta s} = \frac{(1 - i_\theta\varkappa(1-\mu)g_f)\,\sigma_\theta^2}{(1 - i_\theta\varkappa(1-\mu)g_f)^2\,\sigma_\theta^2 + i_\theta^2\varkappa^2\sigma_{u2}^2 + \sigma_\varepsilon^2}$; $i_{\theta y} = i_\theta\,(1 - i_{\theta s})$

Doing similar steps for u_2 one can find

$$E[u_2|y_f, s] = i_{uy} y_f + i_{us} s$$

where $i_{u2} = \frac{1}{\varkappa}\frac{\sigma_{u2}^2}{(1-\mu)^2 g_f^2 \sigma_\theta^2 + \sigma_{u2}^2}$, $i_{us} = \frac{-\,i_\theta\varkappa\sigma_{u2}^2}{(1 - i_\theta\varkappa(1-\mu)g_f)^2\,\sigma_\theta^2 + i_\theta^2\varkappa^2\sigma_{u2}^2 + \sigma_\varepsilon^2}$; $i_{uy} = i_{u2} - i_{us} i_\theta$

for its lower costs. Moreover, the composition of traders changes, and is a concern for a broker, as agents are not equally attractive to him. Above all, the broker prefers underwater traders - those agents only bring profit to the broker, their orders are completely and fully matched, thus no costly order imbalance. However, when the broker changes w, informed and uninformed traders also start using the new instrument, but their orders carry costs- broker has to bring the imbalance back to the market. Finally, lower w means also that all informed brokers trade more.

Because both composition of brokers changes, as well as amount of informed trading, changes in w have general equilibrium effects - they define strategy of the arbitrageur and market maker, in other words all conditional expectations change when w changes. The question how far can the broker estimate the general equilibrium effects of changes in w.

Consider fraction of informed traders that choose the new instrument, $\varkappa_\theta = \dfrac{\bar{c} - wA_1}{\bar{c} - \underline{c}}$. Of course, the broker should be able to realize the first order effect of w on $\varkappa$, he knows that given A_1, $\dfrac{\partial \varkappa_\theta}{\partial w} = -\dfrac{1}{\bar{c} - \underline{c}} A_1$. Similarly, he should be aware of first order effect of w on informed order flow: individual demand $z_i = \dfrac{E[(\theta - f)|s_i]}{2w}$ depends on w, and given fixed$E[(\theta - f)|s_i]$ or in other words given fixed d_f, $\dfrac{\partial E\left[w z_i^2\right]}{\partial w} \propto -\dfrac{1}{w^2}$. But probably, analytical abilities and processing power of the broker should prevent him from being able to access the effect of w on expectation, thus he considers A_1 and d_f as given.

Under the assumption of bounded rationality of the broker, and also assuming interior equilibrium we can find an optimal w taking first order derivative of broker's problem. [21]

Thus

Proposition 3. For a given w a unique linear equilibrium exists and is parametrized by $\{\lambda_{11}, \lambda_{12}, \lambda_{22}, \rho, a_1, b_1, a_2, b_2, d_1, d_2, d_f\}$ that solve the following system of equations

$$A_1 = \frac{\left[1 - \lambda_{12}\left(\mu g_1 + a_1(1-\mu)g_f + a_2\right) - \lambda_{22}\left((1-\mu)(g_2 + b_1 g_f) + b_2\right)\right]^2}{\left(1 - \lambda_f \varkappa (1-\mu) g_f\right)^2}$$

$$\varkappa_\theta = \frac{\bar{c} - wA_1}{\bar{c} - \underline{c}}$$

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[21]

$$\begin{aligned} 0 = \frac{d\pi_{broker}(w)}{dw} &= \frac{-A_1}{\bar{c} - \underline{c}} \frac{1}{w} d_f^2 (1-\mu)\left(\sigma_\theta^2 + \sigma_\nu^2\right) - \frac{\bar{c} - wA_1}{\bar{c} - \underline{c}} \frac{1}{w^2} d_f^2 (1-\mu)\left(\sigma_\theta^2 + \sigma_\nu^2\right) \\ &+ \frac{\bar{c} - 2w - B_0}{\bar{c} - \underline{c}}\left(\sigma_{u2}^2 + \sigma_\xi^2\right) + \frac{\bar{c} - 2w}{\bar{c}} \sigma_\eta^2 \\ &+ c^b 2 \frac{A_1}{\bar{c} - \underline{c}}\left(\frac{\bar{c} - wA_1}{\bar{c} - \underline{c}}\right) \frac{1}{w^2} d_f^2 (1-\mu)^2 \sigma_\theta^2 + 2c^b \left(\frac{\bar{c} - wA_1}{\bar{c} - \underline{c}}\right)^2 \frac{1}{w^3} d_f^2 (1-\mu)^2 \sigma_\theta^2 \\ &+ 2c^b \frac{1}{\bar{c} - \underline{c}}\left(\frac{\bar{c} - w - B_0}{\bar{c} - \underline{c}}\right) \sigma_{u2}^2 \end{aligned}$$

$$B_0 = \frac{\left[\lambda_f \varkappa - \lambda_{12} a_1 - \lambda_{22}\left(1+b_1\right)\right]\sigma_{u2}^2}{\sigma_{u2}^2 + \sigma_\xi^2}$$

$$\varkappa = \varkappa_{u2} = \frac{\bar{c} - w - B_0}{\bar{c} - \underline{c}}$$

$$\lambda_{11} = \frac{\left(\mu g_1 + a_1(1-\mu)g_f + a_2\right)\sigma_\theta^2}{\left(\mu g_1 + a_1(1-\mu)g_f + a_2\right)^2\sigma_\theta^2 + \sigma_{u1}^2 + a_1^2\sigma_{u2}^2 + a_2^2\sigma_\varepsilon^2}$$

$$\lambda_{22} = \frac{\left((1-\mu)(g_2+b_1g_f) + b_2 - \rho\left(\mu g_1 + a_1(1-\mu)g_f + a_2\right)\right)\sigma_\theta^2}{\left((1-\mu)(g_2+b_1g_f) + b_2 - \rho\left(\mu g_1 + a_1(1-\mu)g_f + a_2\right)\right)^2\sigma_\theta^2 + \left((1+b_1) - \rho a_1\right)^2\sigma_{u2}^2 + \rho^2\sigma_{u1}^2 + (b_2 - \rho a_2)^2\sigma_\varepsilon^2}$$

$$\rho = \frac{\left(\mu g_1 + a_1(1-\mu)g_f + a_2\right)\left((1-\mu)(g_2+b_1g_f) + b_2\right)\sigma_\theta^2 + a_1\left(1+b_1\right)\sigma_{u2}^2 + a_2b_2\sigma_\varepsilon^2}{\left(\mu g_1 + a_1(1-\mu)g_f + a_2\right)^2\sigma_\theta^2 + \sigma_{u1}^2 + a_1^2\sigma_{u2}^2 + a_2^2\sigma_\varepsilon^2}$$

$$\lambda_{12} = \lambda_{11} - \rho\lambda_{22}$$

$$\lambda_f = \left[\lambda_{12}\left(\mu g_1 + a_1(1-\mu)g_f + a_2\right) + \lambda_{22}\left((1-\mu)(g_2+b_1g_f) + b_2\right)\right] i_\theta + \left[\lambda_{12}a_1 + \lambda_{22}(1+b_1)\right] i_{u2}$$

$$a_1\frac{1}{\varkappa} = \frac{\left[2\lambda_{22}\left(1-\lambda_{11}\mu g_1\right) - \left(1 - \lambda_{12}\mu g_1 - \lambda_{22}(1-\mu)g_2\right)\lambda_{12}\right] i_{\theta y} + \lambda_{12}\lambda_{22} i_{uy}}{4\lambda_{22}\lambda_{11} - \lambda_{12}^2}$$

$$a_2 = \frac{\left[2\lambda_{22}\left(1-\lambda_{11}\mu g_1\right) - \left(1 - \lambda_{12}\mu g_1 - \lambda_{22}(1-\mu)g_2\right)\lambda_{12}\right] i_{\theta s} + \lambda_{12}\lambda_{22} i_{us}}{4\lambda_{22}\lambda_{11} - \lambda_{12}^2}$$

$$b_1\frac{1}{\varkappa} = \frac{i_{\theta y}\left(1-\lambda_{11}\mu g_1\right) - 2(\lambda_{11}+c)a_1\frac{1}{\varkappa}}{\lambda_{12}}$$

$$b_2 = \frac{i_{\theta s}\left(1-\lambda_{11}\mu g_1\right) - 2(\lambda_{11}+c)a_2}{\lambda_{12}}$$

$$d_1 = \frac{1}{2}\frac{\left[1 - \lambda_{11}\left(a_1(1-\mu)g_f + a_2\right)\right]\frac{\sigma_\theta^2}{\sigma_\theta^2+\sigma_\nu^2}}{1 + \frac{1}{2}\lambda_{11}\mu\frac{1}{\bar{c}-\underline{c}} ln\left(\frac{\bar{c}}{\underline{c}}\right)\frac{\sigma_\theta^2}{\sigma_\theta^2+\sigma_\nu^2}}$$

$$d_2 = \frac{1}{2}\frac{\left[1-\lambda_{12}\left(\mu g_1 + a_1(1-\mu)g_f + a_2\right) - \lambda_{22}\left((1-\mu)\left(\varkappa_\theta \frac{1}{w}d_f + b_1 g_f\right) + b_2\right)\right]\frac{\sigma_\theta^2}{\sigma_\theta^2+\sigma_\nu^2}}{1+\frac{1}{2}\lambda_{22}(1-\mu)\frac{1}{\bar{c}-\underline{c}}ln\left(\frac{wA_1}{\underline{c}}\right)\frac{\sigma_\theta^2}{\sigma_\theta^2+\sigma_\nu^2}}$$

$$d_f = \frac{1}{2}\frac{\frac{\sigma_\theta^2}{\sigma_\theta^2+\sigma_\nu^2}}{1+\frac{1}{2}\lambda_f(1-\mu)\varkappa_\theta\frac{1}{w}\frac{\sigma_\theta^2}{\sigma_\theta^2+\sigma_\nu^2}}$$

12.3 Case I. No informed traders.

The solution method involves writing the solution as a function of ρ, where ρ is defined as $E[y_2|y_1] = \rho y_1$. We will show that ρ will depend only on the ratio of variances of correlated part of uninformed demands in two periods $\frac{\sigma_{u1}^2}{\sigma_{u2}^2}$.

12.3.1 No instrument

First, we normalize $\sigma_\theta^2 = 1$. When there are no informed traders, the system in proposition 1 simplifies to

$$\lambda_{11} = \frac{a}{a^2+\sigma_{u1}^2+a^2\sigma_\varepsilon^2}$$

$$\rho = \frac{ab(1+\sigma_\varepsilon^2)}{a^2+\sigma_{u1}^2+a^2\sigma_\varepsilon^2} = \lambda_{11}b(1+\sigma_\varepsilon^2)$$

$$\lambda_{22} = \frac{(b-\rho a)}{(b-\rho a)^2+\sigma_{u2}^2+\rho^2\sigma_{u1}^2+(b-\rho a)^2\sigma_\varepsilon^2}$$

$$\lambda_{12} = \lambda_{11} - \rho\lambda_{22}$$

$$a = \frac{2\lambda_{22}-\lambda_{12}}{4\lambda_{22}\lambda_{11}-\lambda_{12}^2}\frac{1}{1+\sigma_\varepsilon^2}$$

$$b = \frac{2\lambda_{11} - \lambda_{12}}{4\lambda_{22}\lambda_{11} - \lambda_{12}^2} \frac{1}{1+\sigma_\varepsilon^2}$$

We conjecture that in any equilibrium $b = \frac{1}{\sqrt{1+\sigma_\varepsilon^2}}\sigma_{u2}$. Then given an expression for ρ

$$\rho = \frac{ab(1+\sigma_\varepsilon^2)}{a^2 + \sigma_{u1}^2 + a^2\sigma_\varepsilon^2} = \lambda_{11} b(1+\sigma_\varepsilon^2)$$

we can write λ_{11} as a function of ρ

$$\lambda_{11} = \frac{1}{\sqrt{1+\sigma_\varepsilon^2}\sigma_{u2}}\rho \equiv C_a \rho$$

where $C_a = \frac{1}{\sqrt{1+\sigma_\varepsilon^2}\sigma_{u2}}$. Next, using expression for λ_{11} we can solve for a

$$\lambda_{11} = \frac{a}{a^2(1+\sigma_\varepsilon^2) + \sigma_{u1}^2} = C_a \rho$$

$$a^2 C_a \rho (1+\sigma_\varepsilon^2) - a + C_a \rho \sigma_{u1}^2 = 0$$

Quadratic equation can have two solutions, but one can show that only the following one can be true in equilibrium

$$a = \frac{1 - \sqrt{1 - 4\frac{1}{\sigma_{u2}^2}\rho^2\sigma_{u1}^2}}{2C_a\rho(1+\sigma_\varepsilon^2)}$$

Let's use formula for $\lambda_{22}$and substitute for a and b

$$\lambda_{22} = \frac{(b-\rho a)}{(b-\rho a)^2 (1+\sigma_\varepsilon^2) + \sigma_{u2}^2 + \rho^2\sigma_{u1}^2}$$

One can get

$$\lambda_{22} = C_a \frac{\frac{1}{2}\left(1 + \sqrt{1 - 4\frac{1}{\sigma_{u2}^2}\rho^2\sigma_{u1}^2}\right)}{\frac{1}{2}\left(1 + \sqrt{1 - 4\frac{1}{\sigma_{u2}^2}\rho^2\sigma_{u1}^2}\right) + 1}$$

Then $\lambda_{12} = \lambda_{11} - \rho\lambda_{22} = C_a \rho \dfrac{1}{\frac{1}{2}\left(1 + \sqrt{1 - 4\frac{1}{\sigma_{u2}^2}\rho^2\sigma_{u1}^2}\right) + 1}$

Thus we were able to write $\{\lambda_{11}, \lambda_{22}, \lambda_{12}, a, b\}$ as functions of exogenous parameters and ρ. We have conjectured that $b = \frac{1}{\sqrt{1+\sigma_\varepsilon^2}}\sigma_{u2}$, thus we need that

$$b = \frac{2\lambda_{11} - \lambda_{12}}{4\lambda_{22}\lambda_{11} - \lambda_{12}^2}\frac{1}{1+\sigma_\varepsilon^2} = \frac{1}{\sqrt{1+\sigma_\varepsilon^2}}\sigma_{u2}$$

hence we can substitute $\{\lambda_{11}, \lambda_{22}, \lambda_{12}\}$ and find an expression for ρ that needs to be valid for our conjecture to be true

$$\rho = \frac{1}{2}\sqrt{1 - 4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}}\left(3 + \sqrt{1 - 4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}}\right)$$

thus

$$\sqrt{1 - 4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}} = \frac{-3 + \sqrt{9+8\rho}}{2}$$

and finally

$$\frac{\sigma_{u1}^2}{\sigma_{u2}^2} = \frac{3\sqrt{9+8\rho} - 7 - 4\rho}{8\rho^2}$$

One can check that expression for $a = \frac{2\lambda_{22} - \lambda_{12}}{4\lambda_{22}\lambda_{11} - \lambda_{12}^2}\frac{1}{1+\sigma_\varepsilon^2}$ gives exactly the same equation that connects ρ with $\frac{\sigma_{u1}^2}{\sigma_{u2}^2}$.

Therefore, we found an equilibrium. All variables are functions of ρ only that in turn is solely defined by $\frac{\sigma_{u1}^2}{\sigma_{u2}^2}$.

Finally, we need to solve for optimal w

$$\begin{aligned} E\pi_{broker}(w) &= w\varkappa_{u2}\left(\sigma_{u2}^2 + \sigma_\xi^2\right) - c^b\varkappa_{u2}^2\sigma_{u2}^2 \\ &= w\frac{\bar{c} - w}{\bar{c} - \underline{c}}\left(\sigma_{u2}^2 + \sigma_\xi^2\right) - c^b\left(\frac{\bar{c} - w}{\bar{c} - \underline{c}}\right)^2\sigma_{u2}^2 \end{aligned}$$

and therefore

$$w = \frac{1}{2}\bar{c}\left(1 + \frac{c^b \frac{1}{(\bar{c} - \underline{c})}\sigma_{u2}^2}{\left(\sigma_{u2}^2 + \sigma_\xi^2\right) + c^b \frac{1}{(\bar{c} - \underline{c})}\sigma_{u2}^2}\right)$$

Arbitrageur's strategy is defined as

$$\frac{\sigma_{u1}^2}{\sigma_{u2}^2} = \frac{3\sqrt{9 + 8\rho} - 7 - 4\rho}{8\rho^2}$$

$$b = \frac{1}{\sqrt{1 + \sigma_\varepsilon^2}}\sigma_{u2}$$

$$a = \frac{1 - \sqrt{1 - 4\frac{1}{\sigma_{u2}^2}\rho^2\sigma_{u1}^2}}{2\rho}\frac{1}{\sqrt{1 + \sigma_\varepsilon^2}}\sigma_{u2}$$

Trading losses of uninformed active in period 2 are the same whether they trade the new instrument or the risky asset and equal

$$-E[p_2 u_2] = -\lambda_{22}\sigma_{u2}^2 = -\frac{1}{\sqrt{1 + \sigma_\varepsilon^2}}\frac{2(1 + \rho)}{3 + 2\rho + \sqrt{9 + 8\rho}}\sigma_{u2}$$

Similarly, trading losses of uninformed active in period 1

$$-E[p_1 u_1] = -\lambda_{11}\sigma_{u1}^2 = -\frac{1}{\sqrt{1 + \sigma_\varepsilon^2}\sigma_{u2}}\rho\sigma_{u1}^2 = -\frac{1}{\sqrt{1 + \sigma_\varepsilon^2}}\frac{3\sqrt{9 + 8\rho} - 7 - 4\rho}{8\rho}\sigma_{u2}$$

Informativeness of prices

$$I(p_1) = \lambda_{11}a = \frac{1 - \sqrt{1 - 4\frac{1}{\sigma_{u2}^2}\rho^2\sigma_{u1}^2}}{2}\frac{1}{1 + \sigma_\varepsilon^2}$$

$$I(p_2) = \lambda_{12}a + \lambda_{22}b = \frac{1}{\frac{1}{2}\left(1 + \sqrt{1 - 4\frac{1}{\sigma_{u2}^2}\rho^2\sigma_{u1}^2}\right) + 1}\frac{1}{1 + \sigma_\varepsilon^2}$$

12.3.2 Instrument

Let's conjecture that $\dfrac{b_2}{\rho a_2} = 2$ in any arbitrary equilibrium, so that

$$\frac{b_2}{\rho a_2} = \frac{-\lambda_{12} + 2\lambda_{11}}{\rho\left(2\lambda_{22} - \lambda_{12}\right)} = \frac{\lambda_{11} + \rho\lambda_{22}}{\rho\left(2\lambda_{22} - \lambda_{11} + \rho\lambda_{22}\right)} = 2$$

Thus we can find $\dfrac{\lambda_{22}}{\lambda_{11}}$ as a function of ρ

$$\frac{\lambda_{22}}{\lambda_{11}} = \frac{1 + 2\rho}{\rho(3 + 2\rho)}$$

We can use that to find other ratios

$$\lambda_{12} = \lambda_{11} - \rho\lambda_{22} = \lambda_{11}\frac{2}{3 + 2\rho}$$

$$\frac{\lambda_{12}}{2\lambda_{11}} = \frac{1}{3 + 2\rho}$$

$$\frac{\lambda_{12}}{2\lambda_{22}} = \frac{\rho}{1 + 2\rho}$$

Notice that $\dfrac{\lambda_{12}}{2\lambda_{11}} < \dfrac{\lambda_{12}}{2\lambda_{22}}$ as long as $\rho > \dfrac{1}{2}$.

Using that we can solve for arbitrageur's optimal strategy (where we utilized the formulas derived before)

$$a_1 = \frac{\lambda_{12}\lambda_{22}}{4\lambda_{11}\lambda_{22} - \lambda_{12}^2} = \frac{(1 + 2\rho)}{2(4\rho + 3)(\rho + 1)}$$

$$a_2 = \frac{2\lambda_{22} - \lambda_{12}}{4\lambda_{11}\lambda_{22} - \lambda_{12}^2}\frac{1}{1 + \sigma_\varepsilon^2} = \frac{(3 + 2\rho)}{\lambda_{11}}\frac{1}{2(4\rho + 3)}\frac{1}{1 + \sigma_\varepsilon^2}$$

$$b_1 = -\frac{2\lambda_{11}\lambda_{22}}{4\lambda_{11}\lambda_{22} - \lambda_{12}^2} = -\frac{(1 + 2\rho)(3 + 2\rho)}{2(4\rho + 3)(\rho + 1)}$$

$$b_2 = \frac{-\lambda_{12} + 2\lambda_{11}}{4\lambda_{11}\lambda_{22} - \lambda_{12}^2}\frac{1}{1 + \sigma_\varepsilon^2} = \frac{(3 + 2\rho)}{\lambda_{11}}\frac{\rho}{(4\rho + 3)}\frac{1}{1 + \sigma_\varepsilon^2}$$

Let's assume that $C_a(\sigma_\varepsilon^2, \sigma_{u2}^2, \rho)$ is a function such that

$$\lambda_{11} = C_a \frac{(3+2\rho)}{(4\rho+3)}\rho$$

Then

$$\lambda_{12} = C_a \frac{2\rho}{(4\rho+3)}$$

$$\lambda_{22} = C_a \frac{1+2\rho}{(4\rho+3)}$$

$$\lambda_f \equiv C_a \frac{(2\rho+3)(2\rho+1)^2}{(4\rho+3)^2 2(\rho+1)} \frac{1}{\varkappa}$$

We have three more equations to find C_a and ρ, mainly equations for λ_{11}, λ_{22} and λ_{12} as determined by market makers

- $\sigma_{u1}^2(\rho, \sigma_{u2}^2, \sigma_\varepsilon^2)$ using equation that defines λ_{11}

  First, we can express $a_2 = \frac{(3+2\rho)}{\lambda_{11}} \frac{1}{2(4\rho+3)} \frac{1}{1+\sigma_\varepsilon^2} = \frac{1}{C_a\rho} \frac{1}{2} \frac{1}{1+\sigma_\varepsilon^2}$

  Using information extraction formula and arbitrageur's strategy just found we get

$$\begin{aligned} \lambda_{11} &= \frac{a_2}{a_2^2(1+\sigma_\varepsilon^2) + \sigma_{u1}^2 + a_1^2\sigma_{u2}^2} \\ &= \frac{\frac{1}{C_a\rho}\frac{1}{2}\frac{1}{1+\sigma_\varepsilon^2}}{\frac{1}{C_a^2\rho^2}\frac{1}{4}\frac{1}{(1+\sigma_\varepsilon^2)} + \sigma_{u1}^2 + \left(\frac{(1+2\rho)}{2(4\rho+3)(\rho+1)}\right)^2 \sigma_{u2}^2} = C_a \frac{(3+2\rho)}{(4\rho+3)}\rho \end{aligned}$$

$$\frac{1}{C_a^2\rho^2} \frac{(4\rho+3)}{2(3+2\rho)} \frac{1}{1+\sigma_\varepsilon^2} = \frac{1}{C_a^2\rho^2}\frac{1}{4}\frac{1}{(1+\sigma_\varepsilon^2)} + \sigma_{u1}^2 + \left(\frac{(1+2\rho)}{2(4\rho+3)(\rho+1)}\right)^2 \sigma_{u2}^2$$

  We can solve for σ_{u1}^2:

$$\sigma_{u1}^2 = \frac{1}{C_a^2\rho^2}\left(\frac{(4\rho+3)}{2(3+2\rho)} - \frac{1}{4}\right)\frac{1}{1+\sigma_\varepsilon^2} \quad - \frac{(1+2\rho)^2}{4(4\rho+3)^2(\rho+1)^2}\sigma_{u2}^2$$

$$\sigma_{u1}^2 = \frac{1}{C_a^2 \rho^2} \frac{1}{2} \left(\frac{(8\rho + 6 - 3 - 2\rho)}{2(3 + 2\rho)} \right) \frac{1}{1 + \sigma_\varepsilon^2} - \frac{(1 + 2\rho)^2}{4(4\rho + 3)^2 (\rho + 1)^2} \sigma_{u2}^2$$

$$\sigma_{u1}^2 = \frac{1}{C_a^2} \frac{1}{1 + \sigma_\varepsilon^2} \frac{1}{4\rho^2} \frac{3(2\rho + 1)}{(3 + 2\rho)} - \frac{(1 + 2\rho)^2}{4(4\rho + 3)^2 (\rho + 1)^2} \sigma_{u2}^2$$

- $\sigma_{u1}^2(\rho, \sigma_{u2}^2, \sigma_\varepsilon^2)$ from λ_{22}
  - We can write $b_2 = \frac{1}{C_a} \frac{1}{1 + \sigma_\varepsilon^2}$

  Similarly, one could use expression for λ_{22} to get a formula for σ_{u1}^2

$$\lambda_{22} = \frac{(b_2 - \rho a_2)}{(b_2 - \rho a_2)^2 (1 + \sigma_\varepsilon^2) + ((1 + b_1) - \rho a_1)^2 \sigma_{u2}^2 + \rho^2 \sigma_{u1}^2}$$

$$\sigma_{u1}^2 = \frac{1}{C_a^2} \frac{1}{1 + \sigma_\varepsilon^2} \frac{1}{4\rho^2} \frac{6\rho + 5}{(1 + 2\rho)} - \frac{(2\rho + 3)^2}{4\rho^2 (4\rho + 3)^2} \sigma_{u2}^2$$

- We can equalize two expressions to characterize the solution even further. That gives C_a as a function of ρ, σ_{u2}^2, σ_ε^2

$$\frac{1}{C_a^2} \frac{1}{1 + \sigma_\varepsilon^2} = \frac{(4\rho^2 + 6\rho + 3)(2\rho + 3)(2\rho + 1)}{4(\rho + 1)^2 (4\rho + 3)^2} \sigma_{u2}^2$$

- $\sigma_{u1}^2(\rho, \sigma_{u2}^2, \sigma_\varepsilon^2)$ from ρ

$$\rho = \frac{a_2 b_2 (1 + \sigma_\varepsilon^2) + a_1 (1 + b_1) \sigma_{u2}^2}{a_2^2 (1 + \sigma_\varepsilon^2) + \sigma_{u1}^2 + a_1^2 \sigma_{u2}^2}$$

$$\sigma_{u1}^2 = \frac{1}{C_a^2} \frac{1}{4\rho^2} \frac{1}{1 + \sigma_\varepsilon^2} + \frac{(1 + 2\rho)(2\rho + 3)}{4\rho (4\rho + 3)^2 (\rho + 1)} \sigma_{u2}^2$$

- we can use the expression from above to substitute for the first term and confirm that indeed, ρ depends only on the ratio $\frac{\sigma_{u1}^2}{\sigma_{u2}^2}$ and solves

$$\frac{\sigma_{u1}^2}{\sigma_{u2}^2} = \frac{(2\rho+1)^2(2\rho+3)}{16\rho^2(\rho+1)^2(4\rho+3)}$$

It can be shown that RHS decreases with ρ. Thus we have a unique solution.

One can find informativeness of prices

$$I(p_1) = \lambda_{11}a_2 = \frac{1}{2}\frac{(2\rho+3)}{(4\rho+3)}\frac{1}{1+\sigma_\varepsilon^2}$$

$$I(p_2) = \lambda_{12}a_2 + \lambda_{22}b_2 = 2\frac{\rho+1}{(4\rho+3)}\frac{1}{1+\sigma_\varepsilon^2}$$

Notice that $I(p_1) < I(p_2)$, prices become more informative over time. Moreover, both prices are more informative if the arbitrageur gets better private signal, as he increases trading based on fundamental information. If $\frac{\sigma_{u1}^2}{\sigma_{u2}^2}$ is larger price informativeness increase as well. Because ρ decreases, and C_a is either constant or increases less than ρ decreases thus a_2 increases. If there is more noise in the first period relative to second period, the arbitrageur trades more on fundamentals in the first period. And that increases informativeness of both prices.

If first period uninformed demand relative to second period increases, ρ is smaller and both informativeness increase.

Now we can find uninformed traders losses both on the market of the instrument and on the spot market. Let's write as as a function of σ_{u2}^2.

$$E[-p_2u_2] = E[-fu_2] = -C_a\frac{(2\rho+1)^2(2\rho+3)}{2(4\rho+3)^2(\rho+1)}\sigma_{u2}^2$$

$$E[-p_1u_1] = -C_a\frac{(2\rho+1)^2(2\rho+3)^2}{16\rho(\rho+1)^2(4\rho+3)^2}\sigma_{u2}^2$$

Finally we can solve for optimal w. Broker's profit equals

$$\begin{aligned} E\pi_{broker}(w) &= w\varkappa_{u2}\left(\sigma_{u2}^2+\sigma_\xi^2\right) - c^b\varkappa_{u2}^2\sigma_{u2}^2 \\ &= w\frac{\bar{c}-w}{\bar{c}-\underline{c}}\left(\sigma_{u2}^2+\sigma_\xi^2\right) - c^b\left(\frac{\bar{c}-w}{\bar{c}-\underline{c}}\right)^2\sigma_{u2}^2 \end{aligned}$$

and therefore

$$w = \frac{1}{2}\bar{c}\left(1+\frac{c^b\frac{1}{(\bar{c}-\underline{c})}\sigma_{u2}^2}{\left(\sigma_{u2}^2+\sigma_\xi^2\right)+c^b\frac{1}{(\bar{c}-\underline{c})}\sigma_{u2}^2}\right)$$

12.3.3 Propositions 4 and 5: analysis and comparison

Values of ρ Let's compare ρ-s in two equilibria

no instrument

$$\frac{\sigma_{u1}^2}{\sigma_{u2}^2} = \frac{3\sqrt{9+8\rho}-7-4\rho}{8\rho^2}$$

instrument

$$\frac{\sigma_{u1}^2}{\sigma_{u2}^2} = \frac{(2\rho+1)^2(2\rho+3)}{16\rho^2(\rho+1)^2(4\rho+3)}$$

First of all, it can be shown that both rhs functions are decreasing functions of ρ, in the relevant range. Thus as ratio $\frac{\sigma_{u1}^2}{\sigma_{u2}^2}$ increases, equilibrium values of ρ decrease in both equilibria.

When $\frac{\sigma_{u1}^2}{\sigma_{u2}^2}$ goes to zero

- In the case of no instrument, $3\sqrt{9+8\rho}-7-4\rho \geq 0$, hence, $\rho \leq 2$ and goes to 2 when $\frac{\sigma_{u1}^2}{\sigma_{u2}^2}$ goes to zero
- In the case with instrument, as ratio $\frac{\sigma_{u1}^2}{\sigma_{u2}^2}$ goes to zero, ρ increases, and rhs can be approximated with $\frac{8}{64\rho^2}$, thus ρ goes to infinity and is not limited above.

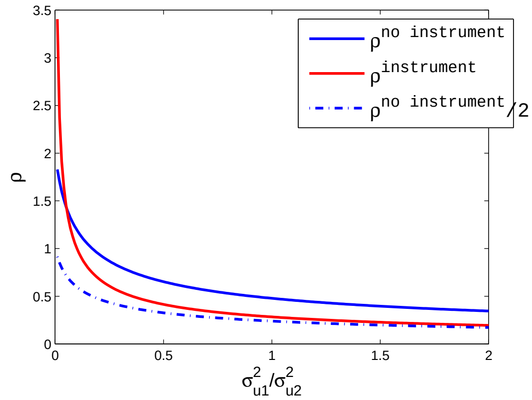
Thus for small ratio $\frac{\sigma_{u1}^2}{\sigma_{u2}^2}$, $\rho^{instrument} > 2 > \rho^{no\,instrument}$

When $\frac{\sigma_{u1}^2}{\sigma_{u2}^2}$ goes to infinity, in both cases ρgoes to zero

- in the case of no instrument, right hand side can be approximated with $\frac{1}{4\rho^2}$
- in the case with instrument, right hand side can be approximated with $\frac{1}{16\rho^2}$

thus for large ratio $\frac{\sigma_{u1}^2}{\sigma_{u2}^2}$, $\rho^{instrument} \approx \frac{\rho^{no\,instrument}}{2} < \rho^{no\,instrument}$

Hence, for $\frac{\sigma_{u1}^2}{\sigma_{u2}^2}$ larger than some threshold value, $\rho^{instrument}$ is smaller than $\rho^{no\,instrument}$. Figure shows the behavior of ρ-s.



Value of C_a

$$\frac{1}{C_a^2}\frac{1}{1+\sigma_\varepsilon^2} = \frac{1}{\bar{C}_a^2}\sigma_{u2}^2$$

where

$$\bar{C}_a^2 = \frac{4(\rho+1)^2(4\rho+3)^2}{(4\rho^2+6\rho+3)(2\rho+3)(2\rho+1)}$$

It can be shown that $\bar{C}_a^2$ is almost constant and approximately equals 4. Hence,

$$C_a \approx 2\frac{1}{\sigma_{u2}}\frac{1}{\sqrt{1+\sigma_\varepsilon^2}}$$

Values of a_1 and b_1

$$a_1 = \frac{(1+2\rho)}{2(4\rho+3)(\rho+1)}$$

$$b_1 = -\frac{(1+2\rho)(3+2\rho)}{2(4\rho+3)(\rho+1)}$$

Obviously, $a_1 > 0$ and $b_1 < 0$. Moreover $b_1 > -1$ as

$$\begin{aligned}\frac{(1+2\rho)(3+2\rho)}{2(4\rho+3)(\rho+1)} &< 1\\ 3+8\rho+4\rho^2 &< 8\rho^2+14\rho+6\\ 0 &< 4\rho^2+7\rho+3\end{aligned}$$

Thus the arbitrageur front-runs. He trades against u_2 in period 2, but never fully eliminates the shock, $-1 < b_1 < 0$. And trades in the direction of the shock in period 1.

Notice also that $a_1 < \frac{1}{2}$ and $b_1 < -\frac{1}{2}$. Thus the arbitrageur offsets more than a half of the shock, but front runs less than that amount.

Values of a and a_2

$$a = \frac{1-\sqrt{1-4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}}}{2\rho}\frac{1}{\sqrt{1+\sigma_\varepsilon^2}}\sigma_{u2}$$

$$a_2 = \frac{1}{2\rho\bar{C}_a}\frac{1}{\sqrt{1+\sigma_\varepsilon^2}}\sigma_{u2}$$

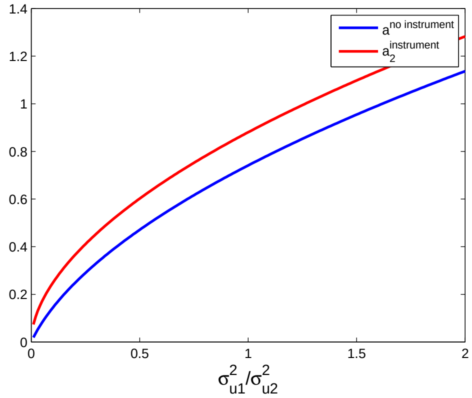
First, both a and a_2 are positive, the arbitrageur always trades in the direction of fundamentals. Obviously, $a_2 > 0$ and $a > 0$ because $\sqrt{1-4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}} = \frac{-3+\sqrt{9+8\rho}}{2}$

$$a = \frac{1-\sqrt{1-4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}}}{2\rho}\frac{1}{\sqrt{1+\sigma_\varepsilon^2}}\sigma_{u2} = \frac{5-\sqrt{9+8\rho}}{4\rho}\frac{1}{\sqrt{1+\sigma_\varepsilon^2}}\sigma_{u2} > 0$$

because $\rho < 2$ (which follows from $\frac{\sigma_{u1}^2}{\sigma_{u2}^2} = \frac{3\sqrt{9+8\rho}-7-4\rho}{8\rho^2} > 0$).

Second, both functions decrease with ρ, and therefore increase with $\frac{\sigma_{u1}^2}{\sigma_{u2}^2}$. If first period is getting noisier than the second one, the arbitrageur trades more based on fundamentals in the first period.

Third, $a < a_2$ - the arbitrageur increases his fundamental trading when he is able to observe future uninformed demand.



Informativeness of prices without instrument

$$I(p_1) = \frac{1}{2}\left(1-\sqrt{1-4\frac{1}{\sigma_{u2}^2}\rho^2\sigma_{u1}^2}\right)\frac{1}{1+\sigma_\varepsilon^2} = \frac{5-\sqrt{9+8\rho}}{4}\frac{1}{1+\sigma_\varepsilon^2}$$

$$I(p_2) = \frac{1}{\frac{1}{2}\left(1+\sqrt{1-4\frac{1}{\sigma_{u2}^2}\rho^2\sigma_{u1}^2}\right)+1}\frac{1}{1+\sigma_\varepsilon^2} = \frac{4}{3+\sqrt{9+8\rho}}\frac{1}{1+\sigma_\varepsilon^2}$$

with instrument

$$I(p_1) = \frac{1}{2}\frac{(2\rho+3)}{(4\rho+3)}\frac{1}{1+\sigma_\varepsilon^2}$$

$$I(p_2) = 2\frac{\rho+1}{(4\rho+3)}\frac{1}{1+\sigma_\varepsilon^2}$$

As we have seen before ρ does not depend on σ_ε^2. Thus informativeness of prices increases when the arbitrageur gets a better private signal.

Second, prices become more informative over time. $I(p_2) > I(p_1)$ in both cases, because market maker do not forget and learn more information over time.

Third, if $\frac{\sigma_{u1}^2}{\sigma_{u2}^2}$ increase, or equivalently if ρ decreases, $I(p_1)$ and $I(p_2)$ increase. If there is more uninformed demand in the first period, the arbitrageur trades more aggresively already in the first period and prices become more informative.

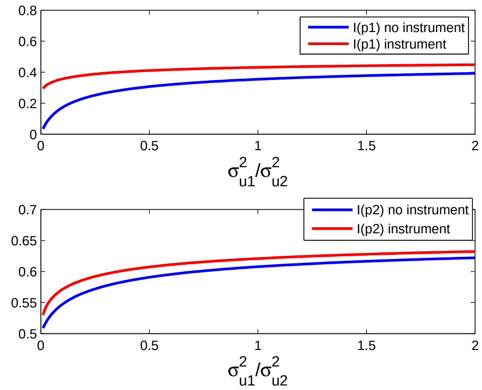
If $\frac{\sigma_{u1}^2}{\sigma_{u2}^2}$ goes to zero,

- without instrument, $\rho \to 2$ and $I(p_1) \to 0$ and $I(p_2) \to \frac{1}{2}$
- with instrument, $\rho \to \infty$, and $I(p_1) \to \frac{1}{4}$ and $I(p_2) \to \frac{1}{2}$

If $\frac{\sigma_{u1}^2}{\sigma_{u2}^2}$ goes to infinity

- without instrument, $\rho \to 0$ and $I(p_1) \to \frac{1}{2}$ and $I(p_2) \to \frac{2}{3}$
- with instrument, $\rho \to 0$, and $I(p_1) \to \frac{1}{2}$ and $I(p_2) \to \frac{2}{3}$

It can also be shown numerically that prices are always more informative when instrument is presents, however, informativeness is the same in the limit.



Losses of uninformed active in period 1 without instrument

$$-E[p_1u_1] = -\lambda_{11}\sigma^2_{u1} = -\frac{1}{\sqrt{1+\sigma^2_\varepsilon\sigma_{u2}}}\rho\sigma^2_{u1} = -\frac{3\sqrt{9+8\rho}-7-4\rho}{8\rho}\frac{1}{\sqrt{1+\sigma^2_\varepsilon}}\sigma_{u2}$$

with instrument

$$E[-p_1u_1] = -\bar{C}_a\frac{(3+2\rho)}{(4\rho+3)}\rho\sigma^2_{u1}\frac{1}{\sigma_{u2}}\frac{1}{\sqrt{1+\sigma^2_\varepsilon}} = -\bar{C}_a\frac{(2\rho+1)^2(2\rho+3)^2}{16\rho(\rho+1)^2(4\rho+3)^2}\frac{1}{\sqrt{1+\sigma^2_\varepsilon}}\sigma_{u2}$$

Losses of uninformed active in period 2 without instrument

$$E[-p_2u_2] = -\lambda_{22}\sigma^2_{u2} = -\frac{2(1+\rho)}{3+2\rho+\sqrt{9+8\rho}}\frac{1}{\sqrt{1+\sigma^2_\varepsilon}}\sigma_{u2}$$

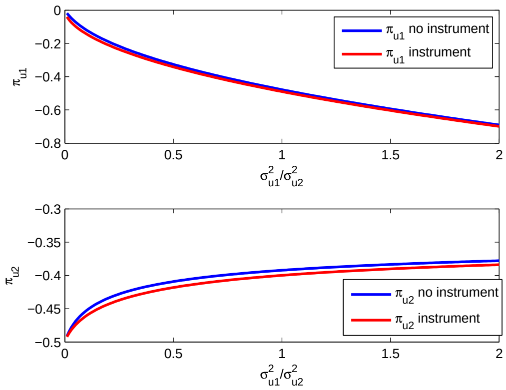
with instrument

$$E[-p_2u_2] = E[-fu_2] = -\bar{C}_a\frac{(2\rho+1)^2(2\rho+3)}{2(4\rho+3)^2(\rho+1)}\frac{1}{\sqrt{1+\sigma^2_\varepsilon}}\sigma_{u2}$$

Above all, lossess of uninformed increase when arbitageur gets better private signal. Second, for a given ρ, lossess increase if $\sigma_{u2}$is larger. Finally, if σ^2_{u1} increases for a given σ^2_{u2}, then ρdecreases, and losses of uninformed become smaller. As the arbitrageur prefers to shift his fundamental demand tothe first period, and price informativeness increases.

Notice that if $\frac{\sigma^2_{u1}}{\sigma^2_{u2}}$ goes to infinity, then in both cases $\rho \to 0$ and losses of uninformed $E[-p_2u_2] \to -\frac{1}{3}\frac{1}{\sqrt{1+\sigma^2_\varepsilon}}\sigma_{u2}$ in both cases. For large enough ratio extra knowledge of $u_2$is of no use to arbitrageur.

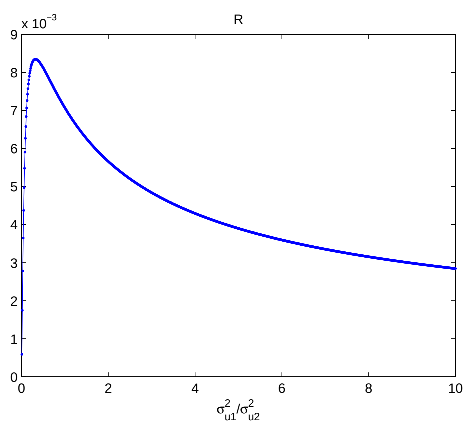
Figure shows losses of uninformed with and without the instrument given $\sigma^2_\varepsilon = 0$ and $\sigma_{u2} = 1$.



Let's denote $\rho_0$as equilibrium value of ρ without the new instrument, and ρ_1 as equilibrium value of ρ with the new instrument, then

$$\begin{aligned} E[-p_2u_2]_{instrument} - E[-p_2u_2]_{no\,instrument} &= -\bar{C}_a\frac{(2\rho_1+1)^2(2\rho_1+3)}{2(4\rho_1+3)^2(\rho_1+1)}\frac{1}{\sqrt{1+\sigma_\varepsilon^2}}\sigma_{u2} + \frac{2(1+\rho_0)}{3+2\rho_0+\sqrt{9+8\rho_0}}\frac{1}{\sqrt{1+\sigma_\varepsilon^2}}\sigma_{u2} \\ &= -\left(\frac{(2\rho_1+1)^2(2\rho_1+3)}{(4\rho_1+3)^2(\rho_1+1)} - \frac{2(1+\rho_0)}{3+2\rho_0+\sqrt{9+8\rho_0}}\right)\frac{1}{\sqrt{1+\sigma_\varepsilon^2}}\sigma_{u2} \\ &\equiv -R\frac{1}{\sqrt{1+\sigma_\varepsilon^2}}\sigma_{u2} < 0 \end{aligned}$$

The R term depends only on the ratio $\dfrac{\sigma_{u1}^2}{\sigma_{u2}^2}$,and thus one can numerically show that the rho term in brackets is postive and always less then $\bar{R} = 0.01$. Hence, although introduction of the new instrument increases trading losses of uninformed active in period 2 (both investors and those who still trades the risky asset), the losses, $\Delta_{trade} = R\frac{1}{\sqrt{1+\sigma_\varepsilon^2}}\sigma_{u2}$, cannot be larger than $\bar{R}\sigma_{u2}$.



Economization on costs Let's calculate how much agents save using the new instrument. To simplify the notation, let's assume $\sigma_\xi^2 = \alpha\sigma_{u2}^2$, $c^b = \beta\bar{c}$, $\underline{c} = 0$. Then

$$w = \frac{1}{2}\left(1 + \frac{\beta}{1+\alpha+\beta}\right)\bar{c} \equiv \omega\bar{c} < \bar{c}$$

Without instrument all uninformed active in period 2 have to pay costs, thus expected costs equal

$$Costs_0 = E[\int_{j=0}^{1} c_j (u_2 + \xi_j)^2 dj] \quad = \quad \frac{1}{\bar{c} - \underline{c}} \int_{0}^{\bar{c}} c_j dc_j \left(\sigma_{u2}^2 + \sigma_{\xi}^2\right) = \frac{\bar{c}}{2} (1 + \alpha)\, \sigma_{u2}^2$$

With instrument, uninformed that decided to trade the risky asset, are those with $j < j^*$ such that $c_{j*} = w$.

$$Costs_{1,asset} = E[\int_{j=0}^{j^*} c_j (u_2 + \xi_j)^2 dj] \quad = \quad \frac{1}{\bar{c}} \int_{0}^{w} c_j dc_j \left(\sigma_{u2}^2 + \sigma_{\xi}^2\right) = \frac{w^2}{2\bar{c}} (1 + \alpha)\, \sigma_{u2}^2 = \frac{\omega^2}{2} \bar{c} (1 + \alpha)\, \sigma_{u2}^2$$

$$Costs_{1,instrument} = E[\int_{j^*}^{1} w (u_2 + \xi_j)^2 dj] \quad = \quad \frac{w}{\bar{c}} \int_{w}^{\bar{c}} dc_j \left(\sigma_{u2}^2 + \sigma_{\xi}^2\right) = w \frac{\bar{c} - w}{\bar{c}} (1 + \alpha)\, \sigma_{u2}^2 = \omega (1 - \omega) \bar{c} (1 + \alpha)\, \sigma_{u2}^2$$

Combined costs paid by uninformed traders when instrument is introduced

$$Costs_1 \quad = \quad \left(\frac{\omega^2 \bar{c}}{2} + \omega \bar{c} (1 - \omega) \right) (1 + \alpha)\, \sigma_{u2}^2 = \left(1 - \frac{\omega}{2} \right) \omega \bar{c}\, (1 + \alpha)\, \sigma_{u2}^2$$

Instrument allows to save on costs, $\omega < 1$, and the exact amount saved equals

$$\Delta_{costs} = Costs_0 - Costs_1 \quad = \quad (1 - \omega)^2 \, \frac{\bar{c}}{2} (1 + \alpha)\, \sigma_{u2}^2$$

Savings are smaller there is less idiosyncrasy, α is smaller, if overall costs are smaller $\bar{c}$ is smaller, and if β is larger.

Comparison of savings on costs on trading losses Let's compare how much agents can save, Δ^+_{costs}, with how much they lose due to arbitrageur's front running behavior Δ^-_{trade}.

$$\begin{aligned} \Delta^+_{costs} &= (1 - \omega)^2 \, \frac{\bar{c}}{2} (1 + \alpha)\, \sigma_{u2}^2 \\ \Delta^-_{trade} &= \bar{R} \sigma_{u2} \end{aligned}$$

When instrument is beneficial for its investors? When

$$\begin{aligned} \Delta^+_{costs} - \Delta^-_{trade} &\geq 0 \\ (1 - \omega)^2 \, \frac{\bar{c}}{2} (1 + \alpha)\, \sigma_{u2}^2 &\geq \bar{R} \sigma_{u2} \end{aligned}$$

$$\frac{(1+\alpha)^3}{(1+\alpha+\beta)^2}\frac{\bar{c}}{8}\sigma_{u2} \geq \bar{R}$$

$$\bar{c}\sigma_{u2} \geq 8\bar{R}\frac{(1+\alpha+\beta)^2}{(1+\alpha)^3}$$

We normalized $\sigma_\theta^2 = 1$. If $\alpha = 1$, so that idiosyncratic component at least as large as correlated, and if $\beta = 0$, broker is absolutely efficient in trading, than rhs equals $4\bar{R} = 0.04$

12.4 Case II. Informed traders in the first period only

With informed traders it is hard to solve the problem analytically. Thus we will look first at the case when $\sigma_\varepsilon^2 = 0$, when the arbitrageur is able to get a perfect signal about fundamentals.

12.4.1 No instrument, $\sigma_\varepsilon^2 = 0$, proposition 6

Assume also that $\sigma_\nu^2 = 0$. We need to solve the following system of equations

$$C_1 = \frac{1}{\bar{c}-\underline{c}} ln\left(\frac{\bar{c}}{\underline{c}}\right)$$

$$g_1 = \frac{1}{2}C_1\frac{[1-\lambda_{11}a]}{\left(1+\frac{1}{2}\lambda_{11}C_1\right)}$$

$$a = \frac{2\lambda_{22}[1-\lambda_{11}g_1]-\lambda_{12}[1-\lambda_{12}g_1]}{4\lambda_{22}\lambda_{11}-\lambda_{12}^2}$$

$$b = \frac{2\lambda_{11}[1-\lambda_{12}g_1]-\lambda_{12}[1-\lambda_{11}g_1]}{4\lambda_{22}\lambda_{11}-\lambda_{12}^2}$$

$$\lambda_{11} = \frac{(g_1+a)}{(g_1+a)^2+\sigma_{u1}^2}$$

$$\lambda_{22} = \frac{(b-\rho(g_1+a))}{(b-\rho(g_1+a))^2+\sigma_{u2}^2+\rho^2\sigma_{u1}^2}$$

$$\lambda_{12} = \lambda_{11} - \rho\lambda_{22}$$

$$\rho = \frac{(g_1 + a)\, b}{(g_1 + a)^2 + \sigma_{u1}^2}$$

We conjecture that $b = \sigma_{u2}$. Then, one can use expression for ρ to write λ_{11} as function of ρ

$$\rho = \frac{(g_1 + a)\, b}{(g_1 + a)^2 + \sigma_{u1}^2} = b\lambda_{11}$$

$$\lambda_{11} = \frac{\rho}{b} = \frac{\rho}{\sigma_{u2}} \equiv C_a\rho$$

Using formula for λ_{11} we can solve for $g_1 + a$

$$\lambda_{11} = \frac{(g_1 + a)}{(g_1 + a)^2 + \sigma_{u1}^2} = C_a\rho$$

$$C_a\rho\,(g_1 + a)^2 - (g_1 + a) + C_a\rho\sigma_{u1}^2 = 0$$

Two cases are possible

$$g_1 + a = \frac{1 \pm \sqrt{1 - 4\rho^2 \frac{\sigma_{u1}^2}{\sigma_{u2}^2}}}{2\rho}\sigma_{u2}$$

Contrary to the case I without informed traders, now both solutions will be observed, depending on costs that informed has to pay. Let's solve the case with a "plus" sign.

We know sum $g_1 + a$ and using formula for g_1 can solve for both g_1 and a and get

$$g_1 = \frac{1}{2}C_1 \left[1 - \frac{1 + \sqrt{1 - 4\rho^2 \frac{\sigma_{u1}^2}{\sigma_{u2}^2}}}{2}\right]$$

$$a = \frac{1 + \sqrt{1 - 4\rho^2 \frac{\sigma_{u1}^2}{\sigma_{u2}^2}}}{2\rho}\sigma_{u2} - \frac{1}{2}C_1 \left[1 - \frac{1 + \sqrt{1 - 4\rho^2 \frac{\sigma_{u1}^2}{\sigma_{u2}^2}}}{2}\right]$$

Using formulas for b and $g_1 + a$ we can find λ_{22} and λ_{12}

$$\lambda_{22} = \frac{\frac{1}{2}\left(1-\sqrt{1-4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}}\right)}{\frac{1}{2}\left(1-\sqrt{1-4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}}\right)+1}\frac{1}{\sigma_{u2}}$$

$$\lambda_{12} = \lambda_{11}-\rho\lambda_{22} = \rho\frac{1}{\sigma_{u2}}\frac{1}{\frac{1}{2}\left(1-\sqrt{1-4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}}\right)+1}$$

Finally we need to check our conjecture that $b=\sigma_{u2}$

$$b = \frac{2\lambda_{11}\left[1-\lambda_{12}g_1\right]-\lambda_{12}\left[1-\lambda_{11}g_1\right]}{4\lambda_{22}\lambda_{11}-\lambda_{12}^2} = \sigma_{u2}$$

That gives a condition

$$\frac{\rho}{\left[\frac{1}{2}\left(1-\sqrt{1-4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}}\right)+1\right]} = \frac{1}{2}\frac{1}{2}C_1\left[1-\sqrt{1-4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}}\right]\frac{\rho}{\sigma_{u2}}-\sqrt{1-4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}}$$

If instead a minus sign is chosen, then the condition becomes

$$\frac{\rho}{\left[\frac{1}{2}\left(1+\sqrt{1-4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}}\right)+1\right]} = \frac{1}{2}\frac{1}{2}C_1\left[1+\sqrt{1-4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}}\right]\frac{\rho}{\sigma_{u2}}+\sqrt{1-4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}}$$

Notice, that if $1-4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}=0$ or $\rho=\frac{1}{2}\frac{\sigma_{u2}}{\sigma_{u1}}$, then solutions coincide. The condition requires that

$$C_1 = \frac{8}{3}\sigma_{u2}$$

One can check the condition for a to see that indeed we have found the solution. Thus

- if $C_1=\frac{8}{3}\sigma_{u2}$,

$$g_1+a = \frac{1}{2\rho}\sigma_{u2}=\sigma_{u1}$$

$$g_1 = \frac{2}{3}\sigma_{u2}$$

$$a = \sigma_{u1} - \frac{2}{3}\sigma_{u2}$$

- If $C_1 > \frac{8}{3}\sigma_{u2}$, which mean that informed traders would demand a lot, and $g_1 + a_2 > \sigma_{u1}$

$$g_1 + a = \frac{1+\sqrt{1-4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}}}{2\rho}\sigma_{u2}$$

$$g_1 = \frac{1}{2}C_1\left[1 - \frac{1+\sqrt{1-4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}}}{2}\right]$$

$$a = \frac{1+\sqrt{1-4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}}}{2\rho}\sigma_{u2} - \frac{1}{2}C_1\left[1 - \frac{1+\sqrt{1-4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}}}{2}\right]$$

where ρ is defined by

$$\frac{\rho}{\left[\frac{1}{2}\left(1-\sqrt{1-4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}}\right)+1\right]} = \frac{1}{2}\frac{1}{2}C_1\left[1-\sqrt{1-4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}}\right]\frac{\rho}{\sigma_{u2}} - \sqrt{1-4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}}$$

- If $C_1 < \frac{8}{3}\sigma_{u2}$, which mean that informed traders do not trade much, and $g_1 + a_2 < \sigma_{u1}$ and

$$g_1 + a = \frac{1-\sqrt{1-4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}}}{2\rho}\sigma_{u2}$$

$$g_1 = \frac{1}{2}C_1\left[1 - \frac{1-\sqrt{1-4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}}}{2}\right]$$

$$a = \frac{1-\sqrt{1-4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}}}{2\rho}\sigma_{u2} - \frac{1}{2}C_1\left[1-\frac{1-\sqrt{1-4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}}}{2}\right]$$

where ρ is defined by

$$\frac{\rho}{\left[\frac{1}{2}\left(1+\sqrt{1-4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}}\right)+1\right]} = \frac{1}{2}\frac{1}{2}C_1\left[1+\sqrt{1-4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}}\right]\frac{\rho}{\sigma_{u2}}+\sqrt{1-4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}}$$

Notice that now ρ depends on absolute value of σ_{u2} as well, not only on the ratio $\frac{\sigma_{u1}^2}{\sigma_{u2}^2}$as before.

Let's first consider the case, when there are not too much of informed traders demand, $C_1 < \frac{8}{3}\sigma_{u2}$. When the arbitrageur is the only informed agent, and if he has perfect info, we trades $a = \frac{1-\sqrt{1-4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}}}{2\rho}\sigma_{u2}$. Whereas when informed traders also trade in the first period, $g_1 + a = \frac{1-\sqrt{1-4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}}}{2\rho}\sigma_{u2}$. the arbitrageur is the only strategic trader, he realizes how his trade affects the price and thus the returns. Whereas informed traders do not account for that effect. When the arbitrageur has perfect information about θ, and when informed traders also get perfect signal, $\sigma_\nu^2 = 0$, the arbitrageur is able to perfectly predict the demand of informed traders. Thus he is able to offset their demand and make the combined fundamental trading eqaul optimal level for a given ρ, therefore g_1+a equals what would the arbitrageur trade if he was the only informed trader who faces equilibrium ρ. And in the second period he trades exactly the same amount, $b = \sigma_{u2}$.

Moreover, if informed traders have relatively small costs, if they trade a lot, C_1 is large(but not too large $< \frac{8}{3}\sigma_{u2}$), then g_1 may be larger than optimality requires. Then the arbitrageur may find it optimal even to trade against fundamentals in the first period $a < 0$, to still bring back amount of fundamental trading to the optimum.

Given that the arbitrageur trades as to bring back the amount of fundamental tradign to the optimum, price informativeness and trading losses of uninformed have the same form as functions of ρ as in the case with no informed traders.

Mainly

$$I(p_1) = \lambda_{11}(g_1+a) = \frac{1-\sqrt{1-4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}}}{2}$$

$$I(p_2) = \lambda_{12}(g_1+a_2)+\lambda_{22}b_2 = \frac{1}{\frac{1}{2}\left(1+\sqrt{1-4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}}\right)+1}$$

and

$$E\left[-p_2 u_{22}\right] \;=\; -\lambda_{22}\sigma_{u2}^2 = -\frac{\frac{1}{2}\left(1-\sqrt{1-4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}}\right)}{\frac{1}{2}\left(1-\sqrt{1-4\rho^2\frac{\sigma_{u1}^2}{\sigma_{u2}^2}}\right)+1}\sigma_{u2}$$

But ρ is not the same, and equilibrium values are different.

If $C_1 = \frac{8}{3}\sigma_{u2}$, $g_1 + a = \sigma_{u1}$ and $a = \left(\frac{\sigma_{u1}}{\sigma_{u2}} - \frac{2}{3}\right)\sigma_{u2}$. Here, it can clearly be seen that for $\frac{\sigma_{u1}}{\sigma_{u2}}$ small enough, the arbitrageur trades agains fundamentals.

Notice, that as C_1 increases, prices become more informative.

12.4.2 Instrument, $\sigma_\varepsilon^2 = \infty$ - unlimited front-run, proposition 7.

When the arbitrageur does not have good private information, he cannot add much to price informativeness,which depends solely on how much informed traders trade. Thus market makers in period 2 do not get much more information relative to what they've learned from informed demaned g_1 in period 1, and $p_2 = E[\theta|y_1, y_2] \approx E[\theta|y_1] \approx \lambda_{11}y_1$. Hence, $\lambda_{12} \approx \lambda_{11}$ and also means that first period order flow is almost uninformative aboout second period order flow, ρ is small.

In that case, similar to proposition 4, expected profit of the arbitrageur from front running, can be written as $E\left(\theta - p_1|s, u_2\right) + E\left(\theta - p_2|s, u_2\right) = -\lambda_{11}a_1^2u_2^2 - \lambda_{12}a_1b_1u_2^2 - \lambda_{22}(1+b_1)b_1u_2^2 = \left[-\lambda_{11}a_1\left(a_1+b_1\right) - \lambda_{22}(1+b_1)b_1\right]u_2^2$, which means that $a_1 = -\frac{b_1}{2}$ and $b_1 = -\frac{2\lambda_{22}}{4\lambda_{22}-\lambda_{11}}$. The magnitude of front run depends on the ratio of λ_{22} to λ_{11}.

Let's conjecture that in equilibrium $\lambda_{22} = \frac{1}{2}\lambda_{11}$ and thus $b_1 = -1$ and $a_1 = -\frac{1}{2}$. ALso assume that $a_2 = 0$, so that it goes to zero faster than $1/\sigma_\varepsilon^2$. Then presence of informed guys means that λ_{11} can still be greater than zero, in particular informed traders submit demand $g_1 = \frac{\frac{1}{2}C_1}{1+\lambda_{11}\frac{1}{2}C_1}$ and λ_{11} then solves

$$\lambda_{11}\left(1+\lambda_{11}\frac{1}{2}C_1\right)^2 \;=\; \frac{\frac{1}{2}C_1}{\sigma_{u1}^2 + \frac{1}{4}\sigma_{u2}^2}$$

Thus $\lambda_{11} > 0$ and $g_1 > 0$ even if $\sigma_\varepsilon^2 \to \infty$. Combined order flow equals

$$\begin{aligned} y_1 &= \left(g_1 + a_2\right)\theta + u_1 + a_1u_2 + a_2\varepsilon = g_1\theta + u_1 + \frac{1}{2}u_2 \\ y_2 &= b_2\theta + (1+b_1)u_2 + b_2\varepsilon = b_2\theta + b_2\varepsilon \end{aligned}$$

First order flow in informative about second order flow only if $b_2 > 0$, in that case $E[y_2|y_1] = \rho y_1$ where $\rho = \dfrac{b_2 g_1}{g_1^2 + \sigma_{u1}^2 + \frac{1}{4}\sigma_{u2}^2}$. Thus ρ and b_2 should decline at the same rate when $\sigma_\varepsilon^2 \to \infty$. Then using $E[\theta|y_2 - \rho y_1] = \lambda_{22}(y_2 - \rho y_1)$ we can find λ_{22}

$$\lambda_{22} = \frac{(b_2 - \rho g_1)}{(b_2 - \rho g_1)^2 + (-\rho a_1)^2 \sigma_{u2}^2 + \rho^2 \sigma_{u1}^2 + b_2^2 \sigma_\varepsilon^2} \approx \frac{(b_2 - \rho g_1)}{b_2^2 \sigma_\varepsilon^2}$$

where all terms that have ρ^2 or b_2^2, or ρb_2 can be neglected relative to $b_2^2 \sigma_\varepsilon^2$. Similarly $\lambda_{11} = \dfrac{g_1}{g_1^2 + \sigma_{u1}^2 + \frac{1}{4}\sigma_{u2}^2} = \dfrac{\rho}{b_2}$ which should equal $2\lambda_{22}$ according to our conjecture. One can then solve that for ρ and b_2 to get

$\rho \approx \frac{1}{2}\frac{1 - \lambda_{11}^2 g_1^2}{\lambda_{11} g_1}\frac{1}{\sigma_\varepsilon^2}$ and $b_2 \approx \frac{1 - \lambda_{11} g_1}{\lambda_{11}}\frac{1}{\sigma_\varepsilon^2}$. Finally, one can use formula for $a_2$from proposition 3 and check that $a_2 = \dfrac{[2\lambda_{22}(1 - \lambda_{11} g_1) - (1 - \lambda_{12} g_1)\lambda_{12}]}{4\lambda_{22}\lambda_{11} - \lambda_{12}^2}\dfrac{1}{1 + \sigma_\varepsilon^2} = 0$ confirming our conjecture.

Thus we have found the solution when σ_ε^2 is large.

Solution: λ_{11} solves

$$\lambda_{11}\left(1 + \lambda_{11}\frac{1}{2}C_1\right)^2 = \frac{\frac{1}{2}C_1}{\sigma_{u1}^2 + \frac{1}{4}\sigma_{u2}^2}$$

which then gives $g_1 \approx \dfrac{\frac{1}{2}C_1}{1 + \lambda_{11}\frac{1}{2}C_1}$. Given g_1 and λ_{11} the solution is given by

Arbitrageur's trategy: $a_2 \approx 0$, $a_1 \approx \frac{1}{2}$, $b_1 \approx -1$, $b_2 \approx \frac{1 - \lambda_{11} g_1}{\lambda_{11}}\frac{1}{1 + \sigma_\varepsilon^2}$

Market makers: $\lambda_{22} \approx \frac{1}{2}\lambda_{11}$, $\lambda_{12} \approx \lambda_{11}$, $\rho \approx \frac{1}{2}\frac{1 - \lambda_{11}^2 g_1^2}{\lambda_{11} g_1}\frac{1}{1 + \sigma_\varepsilon^2}$

Notice that in the limit informativenes of prices coincide

$$I(p_2) = \lambda_{12}(g_1 + a_2) + \lambda_{22} b_2 \approx \lambda_{12} g_1 = \lambda_{11} g_1$$

$$I(p_1) = \lambda_{11}(g_1 + a_2) \approx \lambda_{11} g_1$$

Losses of uninformed

$$E[-p_2u_2] = -\left[\lambda_{12}a_1 + \lambda_{22}(1+b_1)\right]\sigma_{u2}^2 \approx -\lambda_{11}\frac{1}{2}\sigma_{u2}^2$$

12.5 Arbitrageur observes first period price

We will consider the case without informed in period 2, $\mu = 1$.

12.5.1 General solution.

Thus $y_f = \varkappa u_2$ and the arbitrageur perfectly observes u_2. Define $s_p = g_1\theta + u_1$ - new information that the arbitrageur may learn when he observes first period price. As before let's look for a linear equilibrium.

Order flow and prices

$$\begin{aligned} p_1 &= \lambda_{11}y_1 = \lambda_{11}\left(g_1\theta + u_1 + x_1\right) \\ p_2 &= \lambda_{12}y_1 + \lambda_{22}y_2 = \lambda_{12}\left(g_1\theta + u_1 + x_1\right) + \lambda_{22}\left(u_2 + x_2\right) \end{aligned}$$

Signal extraction after period one price is realized will give

$$E[\theta|s,s_p] = i_s\left(1 - g_1 i_{\theta p}\right)s + i_{\theta p}s_p = t_\theta\theta + t_u u_1 + t_\varepsilon\varepsilon$$

$$E[u_1|s,s_p] = -g_1 i_s i_{u1p}s + i_{u1p}s_p = r_\theta\theta + r_u u_1 + r_\varepsilon\varepsilon$$

In period 2 the arbitrageur maximizes

$$\begin{aligned} E\left[\pi_2|s,s_p,u_2,x_1\right] &= E\left[(\theta - \lambda_{12}\left(g_1\theta + u_1 + x_1\right) + \lambda_{22}\left(u_2 + x_2\right))x_2|s,s_p,u_2,x_1\right] \\ &= \left[1 - \lambda_{12}g_1\right]E\left[\theta|s,s_p\right]x_2 - \lambda_{12}E\left[u_1|s,s_p\right]x_2 - \lambda_{12}x_1x_2 - \lambda_{22}u_2x_2 - \lambda_{22}x_2^2 \end{aligned}$$

thus

$$x_2(x_1) = \frac{\left[1 - \lambda_{12}g_1\right]E\left[\theta|s,s_p\right]}{2\lambda_{22}} - \frac{\lambda_{12}}{2\lambda_{22}}E\left[u_1|s,s_p\right] - \frac{\lambda_{12}}{2\lambda_{22}}x_1 - \frac{1}{2}u_2$$

and

$$E\left[\pi_2|s,s_p,u_2,x_1\right] = \frac{\left(\left[1 - \lambda_{12}g_1\right]E\left[\theta|s,s_p\right] - \lambda_{12}E\left[u_1|s,s_p\right] - \lambda_{12}x_1 - \lambda_{22}u_2\right)^2}{4\lambda_{22}}$$

using signal extraction formulas that can be written as

$$
\begin{aligned}
E\left[\pi_2|s,s_p,u_2,x_1\right] &= E\left[\frac{\left(\left[1-\lambda_{12}g_1\right]\left(t_\theta\theta+t_u u_1+t_\varepsilon\varepsilon\right)-\lambda_{12}\left(r_\theta\theta+r_u u_1+r_\varepsilon\varepsilon\right)-\lambda_{12}x_1-\lambda_{22}u_2\right)^2}{4\lambda_{22}}\right] \\
&= \frac{1}{4\lambda_{22}}\left(E\left[B^2\right]-2\lambda_{12}E\left[B\right]x_1+\lambda_{12}^2x_1^2\right)
\end{aligned}
$$

where

$$
B = \left[1-\lambda_{12}g_1\right]\left(t_\theta\theta+t_u u_1+t_\varepsilon\varepsilon\right)-\lambda_{12}\left(r_\theta\theta+r_u u_1+r_\varepsilon\varepsilon\right)-\lambda_{22}u_2
$$

$$
E\left[B|s,u_2\right] = \left[1-\lambda_{12}g_1\right]\left(t_\theta E[\theta|s]+t_\varepsilon E[\varepsilon|s]\right)-\lambda_{12}\left(r_\theta E[\theta|s]+r_\varepsilon E[\varepsilon|s]\right)-\lambda_{22}u_2
$$

Given that in period 1 the arbitrageur maximizes

$$
\begin{aligned}
\pi_{arbitrageur} &= E\left[(\theta-p_1)|s,u_2\right]x_1+E\left[\pi_2(x_1)|s,u_2,x_1\right] \\
&= E\left[\left(\theta-\lambda_{11}\left(g_1\theta+u_1+x_1\right)\right)|s,u_2\right]x_1+\frac{1}{4\lambda_{22}}E\left[B^2|s,u_2\right]-\frac{\lambda_{12}}{2\lambda_{22}}E\left[B|s,u_2\right]x_1+\frac{\lambda_{12}^2}{4\lambda_{22}}x_1^2
\end{aligned}
$$

Hence

$$
x_1 = \frac{\left[1-\lambda_{11}g_1\right]2\lambda_{22}}{4\lambda_{11}\lambda_{22}-\lambda_{12}^2}E\left[\theta|s\right]-\frac{\lambda_{12}}{4\lambda_{11}\lambda_{22}-\lambda_{12}^2}E\left[B|s,u_2\right]
$$

and finally

$$
\begin{aligned}
x_1 = {} & \frac{\left[1-\lambda_{11}\mu g_1\right]2\lambda_{22}}{4\lambda_{11}\lambda_{22}-\lambda_{12}^2}E\left[\theta|s\right]-\frac{\lambda_{12}}{4\lambda_{11}\lambda_{22}-\lambda_{12}^2}\left[1-\lambda_{12}g_1\right]\left(t_\theta E[\theta|s]+t_\varepsilon E[\varepsilon|s]\right) \\
& +\frac{\lambda_{12}^2}{4\lambda_{11}\lambda_{22}-\lambda_{12}^2}\left(r_\theta E[\theta|s]+r_\varepsilon E[\varepsilon|s]\right)+\frac{\lambda_{12}\lambda_{22}}{4\lambda_{11}\lambda_{22}-\lambda_{12}^2}u_2
\end{aligned}
$$

Thus using $E[\theta|s]=i_s s$ and $E[\varepsilon|s]=(1-i_s)s$, arbitrageur's strategy is given by

$$
a_1 = \frac{\lambda_{12}\lambda_{22}}{4\lambda_{11}\lambda_{22}-\lambda_{12}^2}
$$

$$
b_1 = -\frac{2\lambda_{11}\lambda_{22}}{4\lambda_{11}\lambda_{22}-\lambda_{12}^2}
$$

$$a_2 = \frac{[1-\lambda_{11}g_1]\,2\lambda_{22}}{4\lambda_{11}\lambda_{22}-\lambda_{12}^2} i_s - \frac{\lambda_{12}}{4\lambda_{11}\lambda_{22}-\lambda_{12}^2}[1-\lambda_{12}g_1]\,(t_\theta i_s + t_\varepsilon(1-i_s)) + \frac{\lambda_{12}^2}{4\lambda_{11}\lambda_{22}-\lambda_{12}^2}(r_\theta i_s + r_\varepsilon(1-i_s))$$

$$\begin{aligned} b_2 &= \frac{[1-\lambda_{12}g_1]}{2\lambda_{22}} i_s\,(1-g_1 i_{\theta p}) + \frac{\lambda_{12}}{2\lambda_{22}} g_1 i_s i_{u1p} \\ &\quad - \frac{\lambda_{12}}{2\lambda_{22}}\left(\frac{[1-\lambda_{11}g_1]\,2\lambda_{22}}{4\lambda_{11}\lambda_{22}-\lambda_{12}^2} i_s - \frac{\lambda_{12}}{4\lambda_{11}\lambda_{22}-\lambda_{12}^2}[1-\lambda_{12}g_1]\,(t_\theta i_s + t_\varepsilon(1-i_s)) + \frac{\lambda_{12}^2}{4\lambda_{11}\lambda_{22}-\lambda_{12}^2}(r_\theta i_s + r_\varepsilon(1-i_s))\right)\end{aligned}$$

$$b_3 = \left(\frac{[1-\lambda_{12}g_1]}{2\lambda_{22}} i_{\theta p} - \frac{\lambda_{12}}{2\lambda_{22}} i_{u1p}\right)$$

- If perfect info, $\sigma_\varepsilon^2 = 0$, then $t_\theta = 1$ and $r_u = 1$, $E[\theta|s] = \theta$, and $E[\theta|s, s_p] = \theta$ and $E[u_1|s, s_p] = u_1$, and

$$x_1 = \frac{[1-\lambda_{11}g_1]\,2\lambda_{22} - [1-\lambda_{12}g_1]\,\lambda_{12}}{4\lambda_{11}\lambda_{22}-\lambda_{12}^2}\theta + \frac{\lambda_{12}\lambda_{22}}{4\lambda_{11}\lambda_{22}-\lambda_{12}^2}u_2$$

$$x_2(x_1) = \frac{[1-\lambda_{12}g_1]\,\theta}{2\lambda_{22}} - \frac{\lambda_{12}}{2\lambda_{22}}u_1 - \frac{\lambda_{12}}{2\lambda_{22}}x_1 - \frac{1}{2}u_2$$

  only new $-\frac{\lambda_{12}}{2\lambda_{22}}u_1$ term, relative to the case when the arbitrageur does not observe p_1

- If no private info at all, $\sigma_\varepsilon^2 = \infty$, then $i_s = \frac{\sigma_\theta^2}{\sigma_\theta^2 + \sigma_\varepsilon^2} = 0$, $i_{\theta p} = \frac{g_1\sigma_\theta^2}{g_1^2\sigma_\theta^2 + \sigma_{u1}^2}$ and

$$x_1 = \frac{\lambda_{12}\lambda_{22}}{4\lambda_{11}\lambda_{22}-\lambda_{12}^2}u_2$$

$$x_2(x_1) = \frac{[1-\lambda_{12}g_1]\,g_1 - \lambda_{12}}{2\lambda_{22}}\frac{\sigma_{u1}^2}{g_1^2\sigma_\theta^2 + \sigma_{u1}^2}(g_1\theta + u_1) - \frac{2\lambda_{11}\lambda_{22}}{4\lambda_{11}\lambda_{22}-\lambda_{12}^2}u_2$$

  first term is new, and represents the ability of the arbitrageur to learn some information about θ and $u_1$and trade based on that.

- General case. Notice, that for any parameters, a_1 and b_1 have exactly the same form as before. Thus any affect on front-running, may only come through general equilibrium effects.

Market makers

$$\lambda_{11} = \frac{(g_1 + a_2)\,\sigma_\theta^2}{(g_1 + a_2)^2\,\sigma_\theta^2 + \sigma_{u1}^2 + a_1^2\sigma_{u2}^2 + a_2^2\sigma_\varepsilon^2}$$

$$\rho = \frac{(g_1 + a_2)\,(b_2 + b_3 g_1)\,\sigma_\theta^2 + b_3\sigma_{u1}^2 + a_1\,(1 + b_1)\,\sigma_{u2}^2 + a_2 b_2 \sigma_\varepsilon^2}{(g_1 + a_2)^2\,\sigma_\theta^2 + \sigma_{u1}^2 + a_1^2\sigma_{u2}^2 + a_2^2\sigma_\varepsilon^2}$$

$$E\left[\theta \middle| y_2 - E[y_2|y_1]\right] = \lambda_{22}\,(y_2 - \rho y_1)$$

$$\begin{aligned} y_2 - \rho y_1 &= (b_2 + b_3 g_1)\,\theta + b_3 u_1 + (1 + b_1)\,u_2 + b_2\varepsilon - \rho\,((g_1 + a_2)\,\theta + u_1 + a_1 u_2 + a_2\varepsilon) \\ &= (b_2 + b_3 g_1 - \rho\,(g_1 + a_2))\,\theta + (b_3 - \rho)\,u_1 + (1 + b_1 - \rho a_1)\,u_2 + (b_2 - \rho a_2)\,\varepsilon \end{aligned}$$

$$\lambda_{22} = \frac{(b_2 + b_3 g_1 - \rho\,(g_1 + a_2))\,\sigma_\theta^2}{(b_2 + b_3 g_1 - \rho\,(g_1 + a_2))^2\,\sigma_\theta^2 + (b_3 - \rho)^2\,\sigma_{u1}^2 + ((1 + b_1) - \rho a_1)^2\,\sigma_{u2}^2 + (b_2 - \rho a_2)^2\sigma_\varepsilon^2}$$

everything else is as before

$$g_1 = C_1 \frac{1}{2} \frac{[1 - \lambda_{11} a_2]\,\dfrac{\sigma_\theta^2}{\sigma_\theta^2 + \sigma_\nu^2}}{1 + \dfrac{1}{2}\lambda_{11} C_1 \dfrac{\sigma_\theta^2}{\sigma_\theta^2 + \sigma_\nu^2}}$$

$$\begin{aligned} f &= E[p_2|u_2] \\ &= E[\lambda_{12} y_1 + \lambda_{22} y_2 | u_2] \\ &= E\left[\lambda_{12}\,((g_1 + a_2)\,\theta + u_1 + a_1 u_2 + a_2\varepsilon) + \lambda_{22}\,((b_2 + b_3 g_1)\,\theta + b_3 u_1 + (1 + b_1)\,u_2 + b_2\varepsilon) \middle| u_2\right] \\ &= [\lambda_{12} a_1 + \lambda_{22}(1 + b_1)]\,u_2 \end{aligned}$$

$$\lambda_f = [\lambda_{12} a_1 + \lambda_{22}(1 + b_1)]\,\frac{1}{\varkappa}$$

Finally we can solve for optimal w. Broker's profit equals

$$
\begin{aligned}
E\pi_{broker}(w) &= w\varkappa_{u2}\left(\sigma_{u2}^2+\sigma_\xi^2\right)-c^b\varkappa_{u2}^2\sigma_{u2}^2 \\
&= w\frac{\bar{c}-w}{\bar{c}-\underline{c}}\left(\sigma_{u2}^2+\sigma_\xi^2\right)-c^b\left(\frac{\bar{c}-w}{\bar{c}-\underline{c}}\right)^2\sigma_{u2}^2
\end{aligned}
$$

and therefore

$$
w = \frac{1}{2}\bar{c}\left(1+\frac{c^b\frac{1}{(\bar{c}-\underline{c})}\sigma_{u2}^2}{\left(\sigma_{u2}^2+\sigma_\xi^2\right)+c^b\frac{1}{(\bar{c}-\underline{c})}\sigma_{u2}^2}\right)
$$

The case without the fund is a particular case of the one considered here with $a_1 = b_1 = 0$.

Proposition. *Without instrument in any equilibrium $\rho = 0$.*

Even though informed traders are present they are not strategic and the arbitrageur still is able to make order flows completely unpredictable. That is standard Kyle-type result.

12.5.2 Case 1: no instrument and no informed in period 1, $C_1 = 0$.

Thus $g_1 = 0$ and $a_1 = b_1 = 0$.

It can be shown that solution has the following form. Define $z \equiv \frac{\lambda_{22}}{\lambda_{11}}$, then it is related to $\frac{\sigma_{u2}^2}{\sigma_{u1}^2}$ as

$$
\frac{\sigma_{u2}^2}{\sigma_{u1}^2} = \frac{4z-1}{4z^2(2z-1)}
$$

and all other variables are functions of z.

$$
a_2 = \sqrt{\frac{2z-1}{2z}}\frac{\sigma_{u1}}{\sqrt{1+\sigma_\varepsilon^2}}
$$

$$
b_2 = \frac{1}{\sqrt{2z(2z-1)}}\frac{\sigma_{u1}}{\sqrt{1+\sigma_\varepsilon^2}}
$$

$$
b_3 = -\frac{1}{2z}
$$

$$
\lambda_{11} = \frac{\sqrt{2z(2z-1)}}{(4z-1)}\frac{1}{\sqrt{1+\sigma_\varepsilon^2}}\frac{1}{\sigma_{u1}}
$$

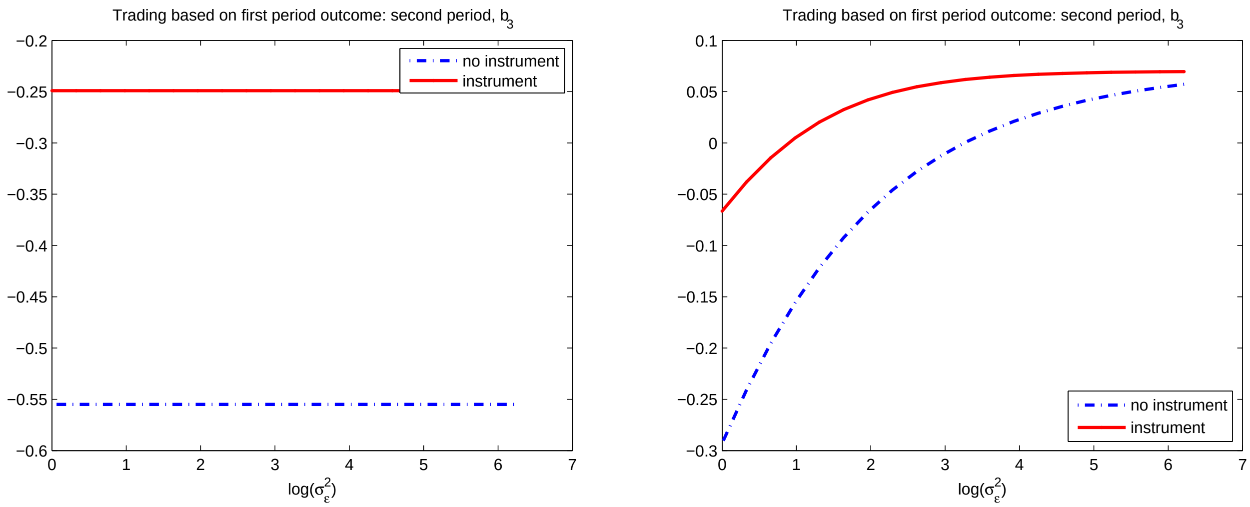


Figure 23: Arbitrageur's strategy: trading based on first period outcome. Left box - no informed traders, right box- with informed traders. Blue line - no instrument, red line - with new instrument. X axis represents logarithm of σ_ε^2- noise in arbitrageur's private signal. Arbitrageur's demand is $x_1 = a_1 u_2 + a_2 s$ and $x_2 = b_1 u_2 + b_2 s + b_3 s_p$, where $s = \theta + \varepsilon$,and $y_f = \varkappa_{u2} u_2$, and $s_p = g_1 \theta + u_1$.

$$\lambda_{22} = \frac{z\sqrt{2z(2z-1)}}{(4z-1)} \frac{1}{\sqrt{1+\sigma_\varepsilon^2}} \frac{1}{\sigma_{u1}}$$

$$\rho = 0$$

$z > \frac{1}{2}$, $b_3 > -1$

12.5.3 Numerical analysis of back run case

Consider same parameters as in the benchmark case.

First of all, figure 23 shows how the arbitrageur uses extra information obtained from observation of first period outcomes. When there are no informed traders, the arbitrageur learns perfectly the realization of u_1 from first period price or order flow. First period u_1 propagates to the price, as market maker is not able to distinguish u_1 from fundamental demand. Thus the arbitrageur may back-run - partially offset the shock in period 2, $b_3 < 0$. Thus second period price still moves in the direction of the shock, but less and the arbitrageur may profit. When informed are present but the arbitrageur has good private signal, σ_ε^2 is low, situation is the same, - learn u_1 and offset. However, when the arbitrageur does not have good private signal, he may also learn information about θ from first period outcomes. And so when arbitrageur's private information is bad enough, he actually trades in the direction of first period combined demand of informed and uninformed, $b_3 > 0$, and he does so in order to trade based on better fundamental signal about θ.

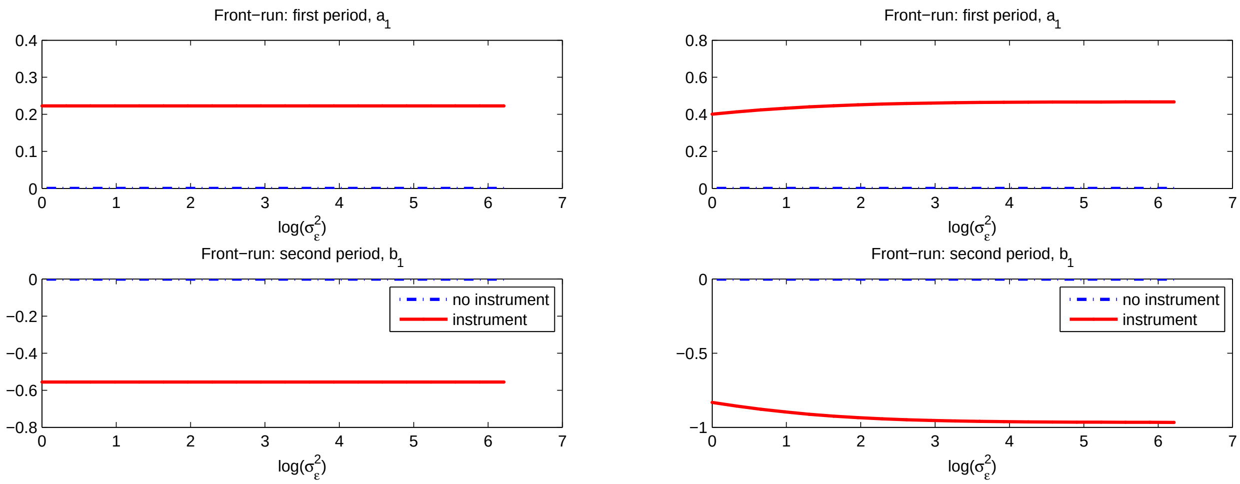


Figure 24: Arbitrageur's strategy: front-run. Left box - no informed traders, right box- with informed traders. Blue line - no instrument, red line - with new instrument. X axis represents logarithm of σ_ε^2- noise in arbitrageur's private signal. Arbitrageur's demand is $x_1 = a_1 u_2 + a_2 s$ and $x_2 = b_1 u_2 + b_2 s + b_3 s_p$, where $s = \theta + \varepsilon$,and $y_f = \varkappa_{u2} u_2$, and $s_p = g_1 \theta + u_2$.

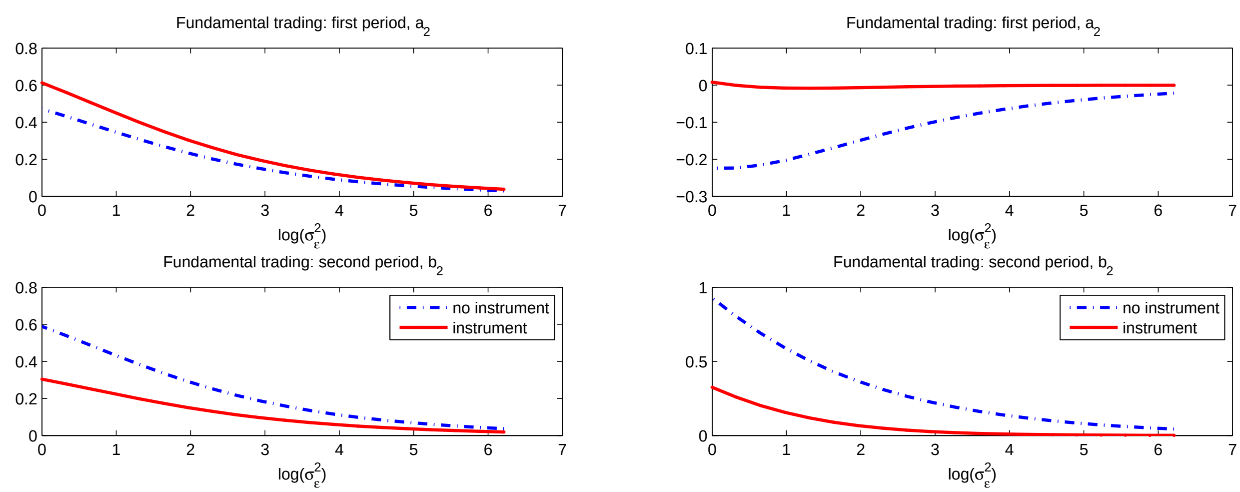


Figure 25: Arbitrageur's strategy: fundamental trading. Left box - no informed traders, right box- with informed traders. Blue line - no instrument, red line - with new instrument. X axis represents logarithm of σ_ε^2- noise in arbitrageur's private signal. Arbitrageur's demand is $x_1 = a_1 u_2 + a_2 s$ and $x_2 = b_1 u_2 + b_2 s + b_3 s_p$, where $s = \theta + \varepsilon$,and $y_f = \varkappa_{u2} u_2$, and $s_p = g_1 \theta + u_2$.

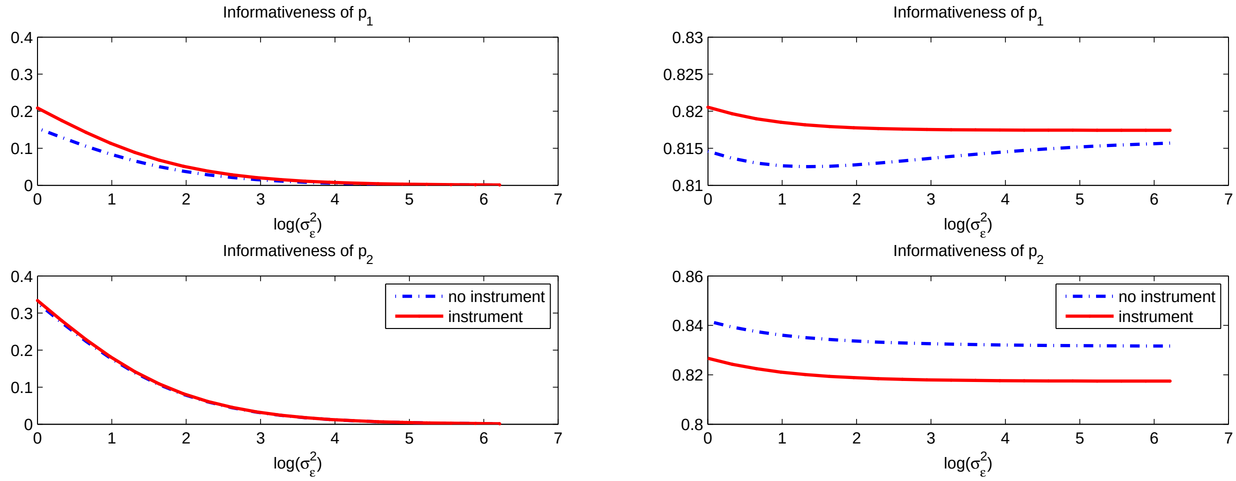


Figure 26: Informativeness of prices. Left box - no informed traders, right box- with informed traders.

Figures 24, 25, and display arbitrageur's fundamental trading and front-running strategies: without informed traders on the left, and with informed traders on the right. Blue dashed line represents no instrument case, red line represents case with the new instrument. Comparison with figures 2 and 3 shows that ability to observe first period information only quantitatively changes arbitrageur's strategy, however main pattern stays the same. Moreover, the arbitrageur front-runs somewhat more now that he observes first period price or order flow and in the case with no informed traders, as left box in figure 24 shows that a_1 is greater than 0.2, whereas left box in figure 2 shows that $a_1 = 0.15$. When informed traders are present, the arbitrageur now trades a little bit more on fundamentals in period 2, b_2 is larger.

Not surprisingly, given the arbitrageur's front-running strategy is almost unchanged, main results concerning informativeness of prices and total losses of uninformed continue to be true. Figures 26 and 27 are directly compared to figures 7 and 8. One can see very similar pattern: uninformed gain from the introduction of the new instrument when there are no informed traders, and lose when there are. Informativeness of second period price decreases when new instrument is introduced and informed traders are present. Magnitudes are almost the same.

Thus, ability to observe first period price only slightly changes the arbitrageur's behavior and not at all crucial for our results. The case with informed in period 2 provides the same results and is not presented here to save space.

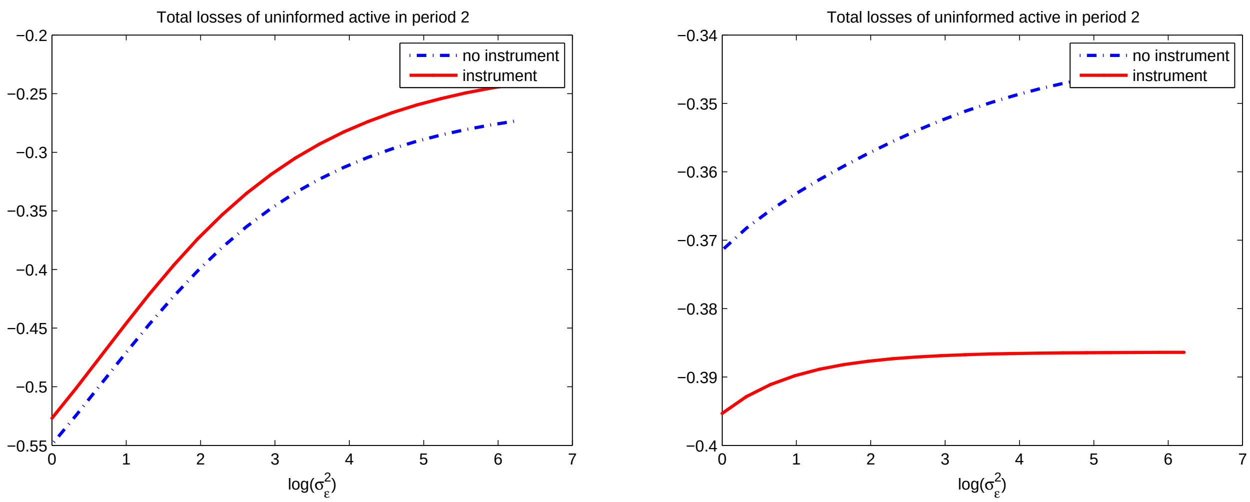


Figure 27: Total losses of uninformed active in period 2. Left box - no informed traders, right box-with informed traders.