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# A Theory of Bayesian Decision Making

by

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**Abstract**

This paper presents a complete, choice-based, axiomatic Bayesian decision theory. It introduces a new choice set consisting of information-contingent plans for choosing actions and bets and subjective expected utility model with effect-dependent utility functions and action-dependent subjective probabilities which, in conjunction with the updating of the probabilities using Bayes' rule, gives rise to a unique prior and a set of action-dependent posterior probabilities representing the decision maker's prior and posterior beliefs.

**Keywords:** Bayesian decision making, subjective probabilities, prior distributions, beliefs

**JEL classification numbers**: D80, D81, D82

# 1  Introduction

A choice-based theory of Bayesian decision making blends five key ideas. First, the patterns revealed by choice are the sole evidence by which the underlying theoretical concepts may be refuted.[1] Second, the decision-maker's evaluation of the objects of choice – payoffs contingent on the realization of events – reflects his tastes as well as his beliefs regarding the likelihoods of the relevant events. Third, the decision maker's beliefs, both prior and posterior, are measurable cognitive phenomena representable by probabilities. Forth, new information affects the decision maker's preferences, or choice behavior, through its effect on the decision maker's beliefs rather than his tastes. Fifth, the posterior probabilities representing the decision maker's posterior beliefs are obtained by updating the prior probabilities representing his prior beliefs using Bayes' rule. By themselves these ideas do not imply that Bayesian decision makers are expected utility maximizers.

In the wake of the seminal work of Savage (1954), it is commonplace to depict the alternatives in the choice set as mappings from a state space, whose elements represent resolutions of uncertainty, to a set of consequences. The objects of choice have the interpretation of alternative courses of action and are referred to as acts. The two most commonly used specifications of the choice set in the literature are those of Savage (1954), in whose formulation the set of states is infinite and the set of consequences arbitrary, and Anscombe and Aumann (1963), in whose formulation the set of states is finite and the set of consequences are

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[1]This is an application of the revealed-preference methodology.

lotteries with finite sets of arbitrary prizes.

The literature abounds with axiomatic theories specifying preference relations on these choice sets whose representations involve unique subjective probabilities, interpreted as the Bayesian prior.[2] However, in all the models that invoke Savage's analytical framework, the uniqueness of the probabilities is due to the use of a convention maintaining that *constant acts are constant-utility acts.* This convention lacks choice-theoretic meaning and, as a result, is not refutable in the context of the revealed-preference methodology.

To grasp this claim, let $S$ denote the set of states, $C$ the set of consequences and $F$ the set of acts. Decision makers are characterized by their preference relations on $F$. In Savage's subjective expected utility theory, the structure of a preference relation, $\succeq$, on $F$, depicted axiomatically, allows its representation by an expected utility functional, that is, for all $f \in F$,

$$f \mapsto \int_S u\left(f(s)\right) d\pi\left(s\right), \tag{1}$$

where $u$ is a real-valued (utility) function defined on the consequences and $\pi$ is a finitely additive, nonatomic probability measure on $S$. Moreover, the utility function $u$ is unique up to positive linear transformation, and, given $u$, the subjective probability measure $\pi$ is unique.

The uniqueness of $\pi$, however, is predicated on the implicit assumption that constant

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[2] Prominent among these theories are the expected utility models of Savage (1954), Anscombe and Aumann (1963), and Wakker (1989), as well as the probability sophisticated choice models of Machina and Schmeidler (1992, 1995).

acts are constant utility acts (that is, the utility function is state-independent). As already mentioned, this assumption is not implied by the axioms and is, therefore, devoid of behavioral content. In fact, there are infinitely many prior probability measures consistent with a decision maker's prior preferences. Put differently, even if a decision maker's beliefs constitute a psychological phenomenon quantifiable by a probability measure and his choice behavior is consistent with the axiomatic structure of expected utility theory, the proposition that the subjective probabilities ascribed to him by Savage's model represent the decision maker's beliefs is untestable. To prove this assertion, let $\gamma$ be a strictly positive, bounded, real-valued function on $S$, and let $\bar{\gamma} = \int_S \gamma(s) \, d\pi(s)$. Then the prior preference relation, depicted by the representation (1), is also represented by

$$f \mapsto \int_S \hat{u}(f(s), s) \, d\hat{\pi}(s), \tag{2}$$

where $\hat{u}(\cdot, s) = u(\cdot) / \gamma(s)$ and $\hat{\pi}$ is a finitely additive, nonatomic probability measure on $S$ defined by $\hat{\pi}(E) = \int_E \pi(s) \gamma(s) \, ds / \bar{\gamma}$, for all $E \in 2^S$.[3]

The fact that the uniqueness of the subjective probabilities in Savage's theory, and in other theories that invoke Savage's (1954) analytical framework, is not a choice-based property of the model means that these subjective probabilities do not constitute a behavioral foundations of Bayesian statistics.

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[3]This point was recognized by Drèze (1987); Schervish, Seidenfeldt, and Kadane (1990); Karni (1996, 2003); Karni and Schmeidler (1993); and Nau (1995). Note also that, even if a decision maker is Bayesian (that is, updates his preferences using Bayes' rule), neither his prior nor his posterior beliefs, as defined by the representing probabilities, are unique.

The popularity of Savage's notion of subjective probabilities among economists and decision theorists is due, in part, to the elegance of the representation it affords, both in terms of its mathematical formulation and the linguistic ease of describing its ingredients.[4] However, this elegance comes at a cost. To attain the separation of subjective probabilities from utilities it is necessary to assume that the preference relation exhibits state independence, which entails substantial loss of generality and limits the applicability of the model. For instance, Anscombe and Aumann (1963) impose state-independence to decompose the terms of a separately additive representation into a product of utility and probability.[5] The imposition of substantive restrictions to attain mathematical elegance is inconsistent with good scholarship. Furthermore, Karni (2008) gives an example involving the design of optimal insurance in the presence of moral hazard, in which the insurer knows the insured's prior preferences and assumes, correctly, that the insured is Bayesian. The example shows that, failure to ascribe to the insured his true prior probabilities and utilities may result in attributing to him the wrong posterior preferences. In such case, when new information (for instance, a study indicating a decline in the incidence of theft in the neighborhood in which the insured resides) necessitates changing the terms of the insurance policy, the insurer may offer the insured a policy that is individually rational and incentive incompatible. More generally, in the presence of moral hazard, correct prediction of an agent's changing behavior by the

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[4]Quite often in the literature, the term subjective probability is used interchangeably with the term beliefs. Yet, as just demonstarted, this usage is hardly justified if not outright misleading. It does, however, serve the purpose of lending the theory intuitive meaning that readers apparently find compelling.

[5]The analogous axioms in Savage's model are P3 and P4 (see Hill (2008)).

application of Bayes rule requires that the agent be ascribed a prior that faithfully represent his beliefs. A more meaningful notion of subjective probability, one that is a measurement of subjective beliefs when these beliefs have structure that allows their representation by probability measure, is developed in this paper.

Building on Karni (2006), this paper introduces a new analytical framework that includes actions, effects, bets, observations, and strategies. Actions are initiatives by which decision makers believe they can affect the likely realization of effects. Effects are observable realizations of eventualities on which decision makers can place bets, and which might also be of direct concern to them. Bets are real-valued mapping on the set of effects. Observations correspond to information that the decision maker may receive before choosing his action and bet. Strategies are maps from the set of observations to the set of action-bet pairs. In this model, decision makers are characterized by preference relations on the set of all strategies whose axiomatic structure lends the notion of constant utility bets choice-theoretic meaning. In other words, unlike in models that use Savage's analytical framework, in this model constant utility bets leave their unique signature, or trace, in the pattern of choice. Because the constant utility bets are identifiable, it possible to define a unique family of action-dependent, joint subjective probability distributions on the product set of effects and observations. Moreover, the prior probabilities are the unconditional marginal probabilities on the set of effects and the posterior probabilities are the distributions on the effects conditional on the observations. To my knowledge, this is the only complete choice-based Bayesian decision theory available.

The work which is closest to this paper is Karni (2006). In that paper, some of the elements of the present analytical framework already appear. However, that work only characterizes the notion of *constant valuation bets* involving compensating variations between the direct utility cost associated with the actions and their impact on the probabilities of the effects. Hence, constant valuation bets are not constant utility bets. Consequently, the uniqueness of the probability, in that work, must rely on an arbitrary normalization of the utility functions. The two ingredients of the present theory that make it possible to identify the constant utility bets, namely, observations and strategies, are new. The significant of this can hardly be overstated. It allows a choice-based definition of subjective probabilities whose uniqueness does not rely on an arbitrary choice of a utility function, thus resolving an, almost century old, issue first raised by Ramsey (1931).

The model presented here accommodates effect-dependent preferences, lending itself to natural interpretations in the context of medical decision making and the analysis of life insurance, health insurance, as well as standard portfolio and property insurance problems. The fact that the probabilities are action dependent means that the model furnishes an axiomatic foundation for the behavior of the principal and agent depicted in the parametrized distribution formulation of agency theory introduced by Mirrlees (1974, 1976).

The pioneering attempt to extend the subjective expected utility model to include moral hazard and state-dependent preferences is due to Drèze (1961,1987). Invoking the analytical framework of Anscombe and Aumann (1963), he departed from their "reversal of order" axiom, assuming instead that decision makers may strictly prefer knowing the outcome of

a lottery before the state of nature becomes known. According to Drèze, this suggests that the decision maker believes that he can influence the probabilities of the states. How this influence is produced is not made explicit. The representation entails the maximization of subjective expected utility over a convex set of subjective probability measures.[6]

The next section introduces the theory and the main results. Concluding remarks appear in section 3. The proof of the main representation theorem appears in section 4.

# 2    The Theory

## 2.1    The analytical framework

Let $\Theta$ be a finite set of *effects*, $X$ a finite set of *observations* or *signals,* and $A$ a connected separable topological space whose elements are referred to as *actions*. Actions correspond to initiatives (e.g., time and effort) that decision makers may take to influence the likely realization of effects.

A *bet* is a real-valued mapping on $\Theta$ interpreted as monetary payoffs contingent on the realization of the effects. Let $B$ denote the set of all bets on $\Theta$ and assume that it is endowed with the $\mathbb{R}^{|\Theta|}$ topology. Denote by $(b_{-\theta}r)$ the bet obtained from $b \in B$ by replacing the $\theta$

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[6]The model in this paper differs from that of Drèze in several important respects, including the specification of the means by which a decision maker thinks he may influence the likelihood of the alternative effects. For more deatils see Karni (2006).

coordinate of $b$ (that is, $b(\theta)$) with $r$. Effects are analogous to Savage's (1954) states in the sense that they resolve the uncertainty associated with the payoff of the bets. Unlike states, however, the likely realization of effects may conceivably be affected by the decision maker's actions.[7]

Observations may be obtained before the choice of bets and actions, in which case they affect these choices. For example, upon learning the result of a new study concerning the effect of cholesterol level in blood on the likelihood of a heart attack, a decision maker may adopt an exercise and diet regimen to reduce the risk of heart attack and, at the same time, take out health insurance and life insurance policies. In this instance the new findings correspond to observations, the diet and exercise regimens correspond to actions, the states of health are effects, and the financial terms of an insurance policy constitute a bet on $\Theta$.[8]

To model this "dynamic" aspect of the decision making process, I assume that a decision maker formulates a strategy, or contingent plan, specifying the action-bet pairs to be implemented contingent on the observations. Formally, denote by $o$ the event "no new information" and let $\bar{X} = X \cup \{o\}$, then *strategy* is a function $I : \bar{X} \to A \times B$ that has the interpretation of a set of instructions specifying, for each $x \in \bar{X}$, an action-bet pair to be implemented if $x$ is observed.[9] Let $\mathcal{I}$ be the set of all strategies.

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[7] It is sufficient, for my purpose, that the decision maker believes that he may affect the likely realization of the effects by his choice of action.

[8] Clearly, the information afforded by the new observation is conditioned by the existing regimen. The decision problem is how to modify the existing regimen in light of the new information.

[9] Alternatively stated, $o$ is a non-informative observation (that is, anticipating the representaion below,

A decision maker is characterized by a preference relation $\succsim$ on $\mathcal{I}$. The strict preference relation, $\succ$, and the indifference relation, $\sim$, are the asymmetric and symmetric parts of $\succsim$, respectively. Denote by $I_{-x}(a, b) \in \mathcal{I}$ the strategy in which the $x$ coordinate of $I$ is replaced by $(a, b)$. An observation, $x$, is *essential* if $I_{-x}(a, b) \succ I_{-x}(a', b')$ for some $(a, b), (a', b') \in A \times B$ and $I \in \mathcal{I}$. I assume throughout that all elements of $\bar{X}$ are essential.

In the terminology of Savage (1954), $\bar{X}$ may be interpreted as a set of states and contingent plans as acts. However, because the decision maker's beliefs about the likelihoods of the effects depend on both the actions and the observations, the preferences on action-bet pairs are inherently observation dependent. Thus applying Savage's state-independent axioms, P3 and P4, to $\succsim$ on $\mathcal{I}$, makes no sense.

To grasp the role of the various ingredients of the model and set the stage for the statement of the axioms, it is useful, at this junction, to look ahead at the representation of $\succsim$ on $\mathcal{I}$. The representation involves an array of continuous, effect-dependent utility functions $\{u(\cdot, \theta) : \mathbb{R} \to \mathbb{R}\}_{\theta \in \Theta}$ and a utility of actions function $v : A \to \mathbb{R}$ unique up to common positive linear transformation, and a unique family of action-dependent joint probability measures, $\{\pi(\cdot, \cdot \mid a)\}_{a \in A}$ on $\bar{X} \times \Theta$ such that $\succsim$ on $\mathcal{I}$ is represented by

$$I \mapsto \sum_{x \in \bar{X}} \sum_{\theta \in \Theta} \pi\left(x, \theta \mid a_{I(x)}\right) u\left(b_{I(x)}(\theta), \theta\right) + v\left(a_{I(x)}\right), \tag{3}$$

where $b_{I(x)}$ and $a_{I(x)}$ are the bet and action assigned by the strategy $I$ to the observation $x$. Furthermore, for all $x \in \bar{X}$, $\mu(x) := \sum_{\theta \in \Theta} \pi(x, \theta \mid a)$ is independent of $a$. Hence the

the subjective probability distribution on the effects conditional on $o$ is the same as that under the current information).

representation (3) may be written as

$$I \mapsto \sum_{x \in \bar{X}} \mu(x) \left[ \sum_{\theta \in \Theta} \pi\left(\theta \mid x, a_{I(x)}\right) u\left(b_{I(x)}(\theta), \theta\right) + v\left(a_{I(x)}\right) \right], \tag{4}$$

where, for all $x \in X$, $\pi(\theta \mid x, a) = \pi(x, \theta \mid a) / \mu(x)$ is the posterior probability of $\theta$ conditional on $x$ and $a$, and for each $a \in A$, $\pi(\theta \mid o, a) = \frac{1}{1-\mu(o)} \sum_{x \in X} \pi(x, \theta \mid a)$ is the prior probability of $\theta$ conditional $a$.[10]

In either representation the choice of strategy entails evaluation of the bets by their expected utility. Actions enter this representation as a direct source of (dis)utility as well as instrument by which the decision maker believes he may affect the likely realizations of the effects.

## 2.2   Axioms and additive representation of $\succcurlyeq$ on $\mathcal{I}$

The first axiom is standard:

(A.1) (**Weak order**) $\succcurlyeq$ *is a complete and transitive binary relation.*

A topology on $\mathcal{I}$ is needed to define continuity of the preference relation $\succcurlyeq$. Recall that $\mathcal{I} = (A \times B)^{\bar{X}}$ and let $\mathcal{I}$ be endowed with the product topology.[11]

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[10]Describing $\pi(\cdot \mid o, a)$ as the prior distribution is appropriate because conditioning on $o$ is means that not information is obtained before a decision is taken.

[11]That is, the topology on $\mathcal{I}$ is the product topology on the Cartesian product $(A \times B)^{|\bar{X}|}$.

(A.2) (**Continuity**) *For all $I \in \mathcal{I}$, the sets $\{I' \in \mathcal{I} \mid I' \succcurlyeq I\}$ and $\{I' \in \mathcal{I} \mid I \succcurlyeq I'\}$ are*

*closed.*

The next axiom, coordinate independence, is analogous to but weaker than Savage's (1954) sure thing principle.[12]

(A.3) (**Coordinate independence**) *For all $x \in \bar{X}$, $I, I' \in \mathcal{I}$, and $(a, b), (a', b') \in A \times B$,*

$$I_{-x}(a, b) \succcurlyeq I'_{-x}(a, b) \text{ if and only if } I_{-x}(a', b') \succcurlyeq I'_{-x}(a', b').$$

An array of real-valued functions $(v_s)_{s \in S}$ is said to be a *jointly cardinal additive represen-tation* for a binary relation $\succeq$ on a product set $D = \Pi_{s \in S} D_s$ if, for all $d, d' \in D$, $d \succeq d'$ if and only if $\sum_{s \in S} v_s(d_s) \geq \sum_{s \in S} v_s(d'_s)$, and the class of all functions that constitute an additive representation of $\succeq$ consists of those arrays of functions, $(\hat{v}_s)_{s \in S}$, for which $\hat{v}_s = \eta v_s + \zeta_s$, $\eta > 0$ for all $s \in S$. The representation is continuous if the functions $v_s, s \in S$ are continuous.

The following theorem is an application of Theorem III.4.1 in Wakker (1989):[13]

**Theorem 1** *Let $\mathcal{I}$ be endowed with the product topology and $\mid \bar{X} \mid \geq 3$. Then a preference relation $\succcurlyeq$ on $\mathcal{I}$ satisfies (A.1)–(A.3) if and only if there exist an array of real-valued func-tions $\{w(\cdot, \cdot, x) \mid x \in \bar{X}\}$ on $A \times B$ that constitute a jointly cardinal, continuous, additive representation for $\succcurlyeq$ .*

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[12]See Wakker (1989) for details.

[13]To simplify the exposition I state the theorem for the case in which $\bar{X}$ contains at least three essential coordinates. Additive representation when there are only two essential coordinates requires the imposition of the hexagon condition (see Wakker [1989] theorem III.4.1).

## 2.3  Independent betting preferences

For every given $x \in \bar{X}$, denote by $\succcurlyeq^x$ the induced preference relation on $A \times B$ defined by $(a, b) \succcurlyeq^x (a', b')$ if and only if $I_{-x}(a, b) \succcurlyeq I_{-x}(a', b')$. The induced strict preference relation, denoted by $\succ^x$, and the induced indifference relation, denoted by $\sim^x$, are the asymmetric and symmetric parts of $\succcurlyeq^x$, respectively.[14] The induced preference relation $\succcurlyeq^o$ is referred to as the *prior* preference relation; the preference relations $\succcurlyeq^x, x \in X$, are the *posterior* preference relations. For each $a \in A$ the preference relation $\succcurlyeq^x$ induces a conditional preference relation on $B$ defined as follows: for all $b, b' \in B$, $b \succcurlyeq_a^x b'$ if and only if $(a, b) \succcurlyeq^x (a, b')$. The asymmetric and symmetric part of $\succcurlyeq_a^x$ are denoted by $\succ_a^x$ and $\sim_a^x$, respectively.

An effect, $\theta$, is said to be *nonnull given the observation-action pair* $(x, a)$ if $(b_{-\theta}r) \succ_a^x (b_{-\theta}r')$, for some $b \in B$ and $r, r' \in \mathbb{R}$; it is *null given the observation-action pair* $(x, a)$ otherwise. Given a preference relation, $\succcurlyeq$, denote by $\Theta(a, x)$ the subset of effects that are nonnull given the observation-action pair $(x, a)$. Assume that $\Theta(a, o) = \Theta$, for all $a \in A$.

Two effects, $\theta$ and $\theta'$, are said to be *elementarily linked* if there are actions $a, a' \in A$ and observations $x, x' \in \bar{X}$ such that $\theta, \theta' \in \Theta(a, x) \cap \Theta(a', x')$. Two effects are said to be *linked* if there exists a sequence of effects $\theta = \theta_0, ..., \theta_n = \theta'$ such that $\theta_j$ and $\theta_{j+1}$ are elementarily linked, $j = 0, ..., n-1$. The set of effects, $\Theta$, is linked if every pair of its elements is linked.

The next axiom requires that the "intensity of preferences" for monetary payoffs contingent on any given effect be independent of the action and the observation:

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[14] For preference relations satisfaying (A.1) - (A.3), these relation are well-defined.

(A.4) (**Independent betting preferences**) *For all* $(a, x), (a', x') \in A \times \bar{X}, b, b', b'', b''' \in B,$

$\theta \in \Theta(a, x) \cap \Theta(a', x'),$ *and* $r, r', r'', r''' \in \mathbb{R},$ *if* $(b_{-\theta} r) \succsim_a^x (b'_{-\theta} r'), (b'_{-\theta} r'') \succsim_a^x (b_{-\theta} r'''),$

*and* $(b''_{-\theta} r') \succsim_{a'}^{x'} (b'''_{-\theta} r)$ *then* $(b''_{-\theta} r'') \succsim_{a'}^{x'} (b'''_{-\theta} r''').$

To grasp the meaning of independent betting preferences, think of the preferences $(b_{-\theta}, r) \succsim_a^x$ $(b'_{-\theta}, r')$ and $(b'_{-\theta}, r'') \succsim_a^x (b_{-\theta}, r''')$ as indicating that given the action $a$, the observation $x$, and the effect $\theta$, the intensity of the preferences of $r''$ over $r'''$ is sufficiently larger than that of $r$ over $r'$ as to reverse the preference ordering of the effect-contingent payoffs $b_{-\theta}$ and $b'_{-\theta}$. The axiom requires that these intensities not be contradicted when the action is $a'$ instead of $a$ and the observation is $x'$ instead of $x$.

The idea may be easier to grasp by considering a specific instance in which $(b_{-\theta}, r) \sim_a^x$ $(b'_{-\theta}, r'), (b_{-\theta} r'') \sim_a^x (b'_{-\theta} r''')$ and $(b''_{-\theta} r') \sim_{a'}^{x'} (b'''_{-\theta} r).$ The first pair of indifferences indicates that, given $a$ and $x$, the difference in the payoffs $b$ and $b'$ contingent on the effects other than $\theta$ measures the intensity of preferences between the payoffs $r$ and $r'$ and between $r''$ and $r''',$ contingent on $\theta$. The indifference $(b''_{-\theta} r') \sim_{a'}^{x'} (b'''_{-\theta} r)$ then indicates that given another action-observation pair, $a'$ and $x',$ the intensity of preferences between the payoffs $r$ and $r'$ contingent on $\theta$ is measured by the difference in the payoffs the bets $b''$ and $b'''$ contingent on the effects other than $\theta$. The axiom requires that, in this case, the difference in the payoffs $b''$ and $b'''$ contingent on the effects other than $\theta$ is also a measure of the intensity of the payoffs $r''$ and $r'''$ contingent on $\theta$. Thus the intensity of preferences between two payoffs given $\theta$ is independent of the actions and the observations.

## 2.4  Belief consistency

To link the decision maker's prior and posterior probabilities, the next axiom asserts that for every $a \in A$ and $\theta \in \Theta$, the prior probability of $\theta$ given $a$ is the sum over $X$ of the joint probability distribution on $X \times \Theta$ conditional on $\theta$ and $a$ (that is, the prior is the marginal probability on $\Theta$).

Let $I^{-o}(a, b)$ denote the strategy that assigns the action-bet pair $(a, b)$ to every observation other than $o$ (that is, $I^{-o}(a, b)$ is a strategy such that $I(x) = (a, b)$ for all $x \in X$).

(A.5) (**Belief consistency**) *For every* $a \in A$, $I \in \mathcal{I}$ *and* $b, b' \in B$, $I_{-o}(a, b) \sim I_{-o}(a, b')$ *if and only if* $I^{-o}(a, b) \sim I^{-o}(a, b')$.

The interpretation of axiom (A.5) is as follows. The decision maker is indifferent between two strategies that agree on $X$ and, in the event that no new information becomes available, call for the implementation of the alternative action-bet pairs $(a, b)$ or $(a, b')$ if and only if he is indifferent between two strategies that agree on $o$ and call for the implementation of the same action-bet pairs $(a, b)$ or $(a, b')$ regardless of the observation. Put differently, given any action, the preferences on bets conditional on there being no new information is the same as that when new information may not be used to select the bet. Hence, in and of itself, information is worthless.

## 2.5 Constant utility bets

Constant utility bets are bets whose payoffs offset the direct impact of the effects. Formally

**Definition 2** *A bet $\bar{b} \in B$ is a constant utility bet according to $\succcurlyeq$ if, for all $I, I', I'', I''' \in \mathcal{I}$, $a, a', a'', a''' \in A$ and $x, x' \in \bar{X}$, $I_{-x}\left(a, \bar{b}\right) \sim I'_{-x}\left(a', \bar{b}\right)$, $I_{-x}\left(a'', \bar{b}\right) \sim I'_{-x}\left(a''', \bar{b}\right)$ and $I''_{-x'}\left(a, \bar{b}\right) \sim I'''_{-x'}\left(a', \bar{b}\right)$ imply $I''_{-x'}\left(a'', \bar{b}\right) \sim I'''_{-x'}\left(a''', \bar{b}\right)$ and $\cap_{(x,a) \in X \times A}\{b \in B \mid b \sim^x_a \bar{b}\} = \{\bar{b}\}$.*

To render the definition meaningful it is assumed that, given $\bar{b}$, for all $a, a', a'', a''' \in A$ and $x, x' \in \bar{X}$ there are $I, I', I'', I''' \in \mathcal{I}$ such that the indifferences $I_{-x}\left(a, \bar{b}\right) \sim I'_{-x}\left(a', \bar{b}\right)$, $I_{-x}\left(a'', \bar{b}\right) \sim I'_{-x}\left(a''', \bar{b}\right)$ and $I''_{-x'}\left(a, \bar{b}\right) \sim I'''_{-x'}\left(a', \bar{b}\right)$ hold.

As in the interpretation of axiom (A.4), to understand the definition of constant utility bets it is useful to think of the preferences $I_{-x}\left(a, \bar{b}\right) \sim I'_{-x}\left(a', \bar{b}\right)$ and $I_{-x}\left(a'', \bar{b}\right) \sim I'_{-x}\left(a''', \bar{b}\right)$ as indicating that, given $\bar{b}$ and $x$, the preferential difference between the substrategies $I_{-x}$ and $I'_{-x}$ measure the intensity of preference of $a$ over $a'$ and that of $a''$ over $a'''$. The indifference $I''_{-x'}\left(a, \bar{b}\right) \sim I'''_{-x'}\left(a', \bar{b}\right)$ implies that, given $\bar{b}$, and another observation $x'$, the preferential difference between the substrategies $I''_{-x'}$ and $I'''_{-x'}$ is another measure the intensity of preference of $a$ over $a'$. Then it must be true that it also measure the intensity of preference of $a''$ over $a'''$.

The requirement that $\cap_{(x,a) \in X \times A}\{b \in B \mid b \sim^x_a \bar{b}\} = \{\bar{b}\}$ implicitly asserts that actions and observations affect the probabilities of the effects, and that these actions and observations are

sufficiently rich so that $\bar{b}$ is well-defined. It is worth emphasizing that the axiomatic structure does not rule out that the decision maker believes that his choice of action does not affect the likelihoods of the effects. However, the uniqueness part of definition 2, by excluding the existence of distinct bets constant utility bets belonging to the same equivalence classes, for all $(a, x) \in A \times X$, implies that, not only does the decision maker believe in his ability to affect the likely realization of the effects by his choice of action, but also that these likelihoods depend on the observations.

To understand why this implies that $\bar{b}$ is a constant utility bet recall that, in general, actions affect decision makers in two ways: directly through their utility cost and indirectly by altering the probabilities of the effects. Moreover, only the indirect impact depends on the observations. The definition requires that, given $\bar{b}$, the intensity of the preferences over the actions be observation-independent. This means that the indirect influence of the actions is neutralized, which can happen only if the utility associated with $\bar{b}$ is invariable across the effects.

Let $B^{cu}(\succsim)$ be a subset of all constant utility bets according to $\succsim$. In general, this set may be empty. This is the case if the range of the utility of the monetary payoffs across effects do not overlap. Here I am concerned with the case in which $B^{cu}(\succsim)$ is nonempty. The set $B^{cu}(\succsim)$ is said to be *inclusive* if for every $(x, a) \in X \times A$ and $b \in B$ there is $\bar{b} \in B^{cu}(\succsim)$ such that $b \sim_a^x \bar{b}$.[15]

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[15]Inclusiveness of $B^{cu}(\succsim)$ simplifies the exposition. For existence and uniqueness of the probabilities in the main result below it is enough that for every given $x$ and $a$, $B^{cu}(\succsim)$ contains at least two bets.

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The next axiom requires that the trade-offs between the actions and the substrategies that figure in definition 2 are independent of the constant utility bets.

(A.6) (**Trade-off independence**) For all $I, I' \in \mathcal{I}$, $x \in \bar{X}$, $a, a' \in A$ and $\bar{b}, \bar{b}' \in B^{cu}(\succcurlyeq)$, $I_{-x}\left(a, \bar{b}\right) \succcurlyeq I'_{-x}\left(a', \bar{b}\right)$ if and only if $I_{-x}\left(a, \bar{b}'\right) \succcurlyeq I'_{-x}\left(a', \bar{b}'\right)$.

Finally, it is also required that the direct effect (that is, cost) of actions, measured by the preferential difference between $\bar{b}$ and $\bar{b}'$ in $B^{cu}(\succcurlyeq)$, be independent of the observation.

(A.7) (**Conditional monotonicity**) For all $\bar{b}, \bar{b}' \in B^{cu}(\succcurlyeq)$, $x, x' \in \bar{X}$, and $a, a' \in A$, $\left(a, \bar{b}\right) \succcurlyeq^{x} \left(a', \bar{b}'\right)$ if and only if $\left(a, \bar{b}\right) \succcurlyeq^{x'} \left(a', \bar{b}'\right)$.

## 2.6    The main representation theorem

The next theorem asserts the existence of subjective expected utility representation of the preference relation $\succcurlyeq$ on $\mathcal{I}$, and characterizes the uniqueness properties of its constituent utilities and the probabilities. For each $I \in \mathcal{I}$ let $\left(a_{I(x)}, b_{I(x)}\right)$ denote the action-bet pair corresponding to the $x$ coordinate of $I$ – that is, $I(x) = \left(a_{I(x)}, b_{I(x)}\right)$.

**Theorem 3** *Let $\succcurlyeq$ be a preference relation on $\mathcal{I}$ and suppose that $B^{cu}(\succcurlyeq)$ is inclusive, then:*

*(a) The following two conditions are equivalent:*

*(a.i) $\succcurlyeq$ satisfies (A.1)–(A.7)*

*(a.ii) there exist continuous, real-valued functions $\{u(\cdot,\theta) \mid \theta \in \Theta\}$ on $R$ and $v \in R^A$, and a family, $\{\pi(\cdot, \cdot \mid a) \mid a \in A\}$, of joint probability measures on $\bar{X} \times \Theta$ such that $\succsim$ on $I$ is represented by*

$$I \mapsto \sum_{x \in \bar{X}} \mu(x) \left[ \sum_{\theta \in \Theta} \pi\left(\theta \mid x, a_{I(x)}\right) u\left(b_{I(x)}(\theta), \theta\right) + v\left(a_{I(x)}\right) \right], \tag{5}$$

*where $\mu(x) = \sum_{\theta \in \Theta} \pi(x, \theta \mid a)$ for all $x \in \bar{X}$ is independent of $a$, $\pi(\theta \mid x, a) = \pi(x, \theta \mid a) / \mu(x)$ for all $(x, a) \in X \times A$, $\pi(\theta \mid o, a) = \frac{1}{1-\mu(o)} \sum_{x \in X} \pi(x, \theta \mid a)$ for all $a \in A$, and, for every $\bar{b} \in B^{cu}(\succsim)$, $u(\bar{b}(\theta), \theta) = u(\bar{b})$, for all $\theta \in \Theta$.*

*(b) If $\{\hat{u}(\cdot, \theta) \mid \theta \in \Theta\}$, $\hat{v} \in R^A$ and $\{\hat{\pi}(\cdot, \cdot \mid a) \mid a \in A\}$ is another set of utilities and a family of joint probability measures representing $\succsim$ in the sense of (5), then $\hat{\pi}(\cdot, \cdot \mid a) = \pi(\cdot, \cdot \mid a)$ for every $a \in A$ and there are numbers $m > 0$ and $k, k'$ such that $\hat{u}(\cdot, \theta) = m\hat{u}(\cdot, \theta) + k$, $\theta \in \Theta$ and $\hat{v} = mv + k'$.*

Although the joint probability distributions $\pi(\cdot, \cdot \mid a), a \in A$ depend on the actions, the distribution $\mu$ is independent of $a$. This is consistent with the formulation of the decision problem according to which the choice of actions is contingent on the observations. In other words, if new information in the form of an observation becomes available, it precedes the choice of action. Consequently, the dependence of the joint probability distributions $\pi(\cdot, \cdot \mid a)$ on $a$ captures solely the decision maker's beliefs about his ability to influence the likelihood of the effects by his choice of action.[16]

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[16] If an action-effect pair are already "in effect" when new information arrives, they constitute a default course of action. In such instance, the interpretation of the decision at hand is possible choice of new action

The key to obtaining the uniqueness of the joint probability distributions $\pi\left(\cdot,\cdot\mid a\right),a\in A$ is the existence and uniqueness of constant utility bets. The definition of these bets requires, in turn, that the decision maker perceives the likelihoods of the effects to depends on both his actions and the observations. It is worth underscoring that, neither actions nor observations can be dispense with and still obtain a choice-based definition of if constant utility bets.

Unlike the subjective probability in the theory of Savage (1954) (and in all other theories that invoke Savage's analytical framework) whose uniqueness is predicated on an arbitrary specification of the utility function, the uniqueness of the probabilities in this theory is entirely choice based. In particular, the theory of this paper is immune to the critique of Savage's theory in the introduction (that is, it does not allow the multiplication of the joint probabilities by positive numbers and dividing the utility functions by the same numbers and renormalizing to obtain an equivalent representation).

and bet. For example, a modification of a diet regimen coupled with a possible change of life insurance policy.

# 3 Concluding Remarks

## 3.1 Effect-independent preferences and effect-independent utility functions

The choice-based Bayesian decision theory presented in this paper includes, as a special case, effect-independent preferences. In particular, following Karni (2006), effect independent preferences is captured by the following axiom:

(A.8) **Effect-independent betting preferences** *For all $x \in \bar{X}$, $a \in A$, $b, b', b'', b''' \in B$, $\theta, \theta' \in \Theta$, and $r, r', r'' r''' \in \mathbb{R}$, if $(b'_{-\theta}, r) \succcurlyeq_a^x (a, (b_{-\theta}, r'))$, $(b_{-\theta}, r'') \succcurlyeq_a^x (b'_{-\theta}, r''')$, and $(b''_{-\theta'}, r') \succcurlyeq_a^x (b'''_{-\theta'}, r)$ then $(b''_{-\theta'}, r'') \succcurlyeq_a^x (b'''_{-\theta'}, r''')$.*

The interpretation of this axiom is analogous to that of action-independent betting preferences. The preferences $(b'_{-\theta}, r') \succcurlyeq_a^x (b_{-\theta}, r)$ and $(b_{-\theta}, r'') \succcurlyeq_a^x (b'_{-\theta}, r''')$ indicate that, for every given $(a, x)$, the "intensity" of the preference for $r''$ over $r'''$ given the effect $\theta$ is sufficiently greater than that of $r$ over $r'$ as to reverse the order of preference between the payoffs $b'_{-\theta}$ and $b_{-\theta}$. Effect independence requires that these intensities not be contradicted by the preferences between the same payoffs given any other effect $\theta$.

Adding axiom (A.8) to the hypothesis of Theorem 3 implies that the utility function that figures in the representation takes the form $u(b(\theta), \theta) = t(\theta) u(b(\theta)) + s(\theta)$, where $t(\theta) > 0$. In other words, even if the preference relation exhibits *effect-independence* over

bets, the *utility function* may still display effect dependence, in the form of the additive and multiplicative coefficient. Thus, effects may impact the decision maker's well-being without necessarily affecting his risk preferences.

Let $B^c$ be the subset of constant bets (that is, trivial bets with the same payoff regardless of the effect that obtains). If the set of constant utility bets coincides with the set of constant bets (that is, $B^c = B^{cu}(\succsim)$), then the utility function is effect independent (that is, $u(b(\theta),\theta) = u(b(\theta))$ for all $\theta \in \Theta$). The implicit assumption that the set of constant utility bets coincides with the set of constant bets is the convention invoked by the standard subjective utility models. Unlike in those models, however, in the theory of this paper, this assumption is a testable hypothesis.

## 3.2   Conditional preferences and dynamic consistency

The specification of the decision problem implies that, before the decision maker chooses an action-bet pair, either no informative signal arrives (that is, the observation is $o$) or new informative signal arrives in the form of an observation $x \in X$. One way or another, given the information at his disposal, the decision maker must choose among action-bet pairs. Let $\left(\hat{\succsim}^x\right)_{x\in\bar{X}}$ be binary relations on $A \times B$ depicting the decision maker's choice behavior conditional on observing $x$. I refer to $\left(\hat{\succsim}^x\right)_{x\in\bar{X}}$ by the name *ex-post preference relations*.

Dynamic consistency requires that at each $x \in \bar{X}$, the decision maker implements his plan of action envisioned for that contingency by the original strategy. Formally,

**Definition 4** *A preference relation $\succcurlyeq$ on $\mathcal{I}$ is dynamically consistent with the ex-post pref-erence relations $\left(\hat{\succcurlyeq}^{x}\right)_{x\in\bar{X}}$ on $A\times B$ if the posterior preference relations $\left(\succcurlyeq^{x}\right)_{x\in\bar{X}}$ satisfy $\succcurlyeq^{x}=\hat{\succcurlyeq}^{x}$ for all $x\in\bar{X}$.*

The following is an immediate implication of Theorem 3.

**Corollary 5** *Let $\succcurlyeq$ be preference relation on $\mathcal{I}$ satisfying (A.1)–(A.7) and suppose that $B^{cu}\left(\succcurlyeq\right)$ is inclusive. Then $\succcurlyeq$ is dynamically consistent with the ex-post preference relations $\left(\hat{\succcurlyeq}^{x}\right)_{x\in\bar{X}}$ on $A\times B$ if and only if, for all $x\in\bar{X}$, $\hat{\succcurlyeq}^{x}$ is represented by*

$$(a,b)\mapsto\sum_{\theta\in\Theta}\pi\left(\theta\mid x,a\right)u\left(b\left(\theta\right),\theta\right)+v\left(a\right),\tag{6}$$

*where $\{u\left(\cdot,\theta\right)\mid\theta\in\Theta\}$ and $\{\pi\left(\cdot\mid x,a\right)\mid x\in\bar{X},a\in A\}$ are the utility functions and conditional subjective probabilities that appear in the representation (5).*

For every $a\in A$ the subjective action-contingent prior on $\Theta$ is $\pi\left(\cdot\mid o,a\right)$ and the sub-jective action-contingent posteriors on $\Theta$ are $\pi\left(\cdot\mid x,a\right)$, $x\in X$. The subjective action-dependent prior is the marginal distribution on $\Theta$ induced by the distribution on $X\times\Theta$, and the subjective action-dependent posteriors are obtained from the action-contingent joint distribution on $X\times\Theta$ by conditioning on the observation.

# 4   Proof of Theorem 3

For expository convenience, I write $B^{cu}$ instead of $B^{cu}\left(\succcurlyeq\right)$.

(a) $(a.i) \Rightarrow (a.ii)$. Suppose that $\succcurlyeq$ on $\mathcal{I}$ satisfies (A.1)–(A.7) and $B^{cu}$ is inclusive. By Theorem 1, $\succcurlyeq$ is represented by

$$I \mapsto \sum_{x \in \bar{X}} w\left(a_{I(x)}, b_{I(x)}, x\right). \tag{7}$$

where $w\left(., ., x\right), x \in \bar{X}$ are jointly cardinal, continuous, real-valued functions.

Since $\succcurlyeq$ satisfies (A.4), Lemmas 4 and 5 in Karni (2006) applied to $\succcurlyeq^x$, $x \in \bar{X}$, and Theorem III.4.1 in Wakker (1989) imply that for every $(a, x) \in A \times \bar{X}$ such that $\Theta(a, x)$ contains at least two effects, there exist array of functions $\{v_{(a,x)}\left(\cdot; \theta\right) : \mathbb{R} \to \mathbb{R} \mid \theta \in \Theta\}$ that constitute a jointly cardinal, continuous additive representation of $\succcurlyeq^x_a$ on $B$. Moreover, by the proof of Lemma 6 in Karni (2006), $\succcurlyeq$ satisfies (A.1)–(A.4) if and only if, for every $(a, x), (a', x') \in A \times \bar{X}$ such that $\Theta(a, x) \cap \Theta(a', x') \neq \varnothing$ and $\theta \in \Theta(a, x) \cap \Theta(a', x')$, there exist $\beta_{((a',x'),(a,x),\theta)} > 0$ and $\alpha_{((a',x'),(a,x),\theta)}$ satisfying $v_{(a',x')}\left(\cdot, \theta\right) = \beta_{((a',x'),(a,x),\theta)} v_{(a,x)}\left(\cdot, \theta\right) + \alpha_{((a',x'),(a,x),\theta)}$.[17]

Fix $\hat{a} \in A$ and define $u\left(\cdot, \theta\right) = v_{(\hat{a},o)}\left(\cdot, \theta\right)$, $\lambda\left(a, x; \theta\right) = \beta_{((a,x),(\hat{a},o),\theta)}$ and $\alpha\left(a, x, \theta\right) = \alpha_{((a,x),(\hat{a},o),\theta)}$ for all $a \in A$, $x \in \bar{X}$, and $\theta \in \Theta$. For every given $(a, x) \in A \times \bar{X}$, $w\left(a, b, x\right)$ represents $\succcurlyeq^x_a$ on $B$. Hence

$$w\left(a, b, x\right) = H\left(\sum_{\theta \in \Theta}\left(\lambda\left(a, x, \theta\right) u\left(b\left(\theta\right); \theta\right) + \alpha\left(a, x, \theta\right)\right), a, x\right), \tag{8}$$

where $H$ is a continuous, increasing function.

Consider next the restriction of $\succcurlyeq$ to $\left(A \times B^{cu}\right)^{\bar{X}}$.

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[17] By definition, for all $(a, x)$ and $\theta$, $\beta_{((a,x),(a,x),\theta)} = 1$ and $\alpha_{((a,x),(a,x),\theta)} = 0$.

**Lemma 6** *There exist a function $U : A \times B^{cu} \to \mathbb{R}$, $\xi \in \mathbb{R}_{++}^{|\bar{X}|}$ and $\zeta \in \mathbb{R}^{|\bar{X}|}$ such that, for all $(a, \bar{b}, x) \in A \times B^{cu} \times \bar{X}$,*

$$w(a, \bar{b}, x) = \xi(x) U(\bar{b}, a) + \zeta(x). \tag{9}$$

*Proof:* Let $I, I', I'', I''' \in \mathcal{I}$, $a, a', a'', a''' \in A$ and $\bar{b}$ be as in definition 2. Then, for all $x, x' \in \bar{X}$, $I_{-x}(a, \bar{b}) \sim I'_{-x}(a', \bar{b})$, $I_{-x}(a'', \bar{b}) \sim I'_{-x}(a''', \bar{b})$, $I''_{-x'}(a, \bar{b}) \sim I'''_{-x'}(a', \bar{b})$ and $I''_{-x'}(a'', \bar{b}) \sim I'''_{-x'}(a''', \bar{b})$. By the representation (7), $I_{-x}(a, \bar{b}) \sim I'_{-x}(a', \bar{b})$ implies that

$$\sum_{y \in \bar{X} - \{x\}} w(a_{I(y)}, b_{I(y)}, y) + w(a, \bar{b}, x) = \sum_{y \in \bar{X} - \{x\}} w(a_{I'(y)}, b_{I'(y)}, y) + w(a', \bar{b}, x). \tag{10}$$

Similarly, $I_{-x}(a'', \bar{b}) \sim I'_{-x}(a''', \bar{b})$ implies that

$$\sum_{y \in \bar{X} - \{x\}} w(a_{I(y)}, b_{I(y)}, y) + w(a'', \bar{b}, x) = \sum_{y \in \bar{X} - \{x\}} w(a_{I'(y)}, b_{I'(y)}, y) + w(a''', \bar{b}, x), \tag{11}$$

$I''_{-x'}(a, \bar{b}) \sim I'''_{-x'}(a', \bar{b})$ implies that

$$\sum_{y \in \bar{X} - \{x'\}} w(a_{I''(y)}, b_{I''(y)}, y) + w(a, \bar{b}, x') = \sum_{y \in \bar{X} - \{x'\}} w(a_{I'''(y)}, b_{I'''(y)}, y) + w(a', \bar{b}, x'), \tag{12}$$

and $I''_{-x'}(a'', \bar{b}) \sim I'''_{-x'}(a''', \bar{b})$ implies that

$$\sum_{y \in \bar{X} - \{x'\}} w(a_{I''(y)}, b_{I''(y)}, y) + w(a'', \bar{b}, x') = \sum_{y \in \bar{X} - \{x'\}} w(a_{I'''(y)}, b_{I'''(y)}, y) + w(a''', \bar{b}, x'). \tag{13}$$

But (10) and (11) imply that

$$w(a, \bar{b}, x) - w(a', \bar{b}, x) = w(a'', \bar{b}, x) - w(a''', \bar{b}, x). \tag{14}$$

and (12) and (13) imply that

$$w(a, \bar{b}, x') - w(a', \bar{b}, x') = w(a'', \bar{b}, x') - w(a''', \bar{b}, x'). \tag{15}$$

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Define a function $\phi_{(x,x',\bar{b})}$ as follows: $w\left(\cdot, \bar{b}, x\right) = \phi_{(x,x',\bar{b})} \circ w\left(\cdot, \bar{b}, x'\right)$. Axiom (A.7) with $\bar{b} = \bar{b}'$ imply that it is monotonic increasing. Then $\phi_{(x,x',\bar{b})}$ is continuous. Moreover, equations (14) and (15) in conjunction with Lemma 4.4 in Wakker (1987) imply that $\phi_{(x,x',\bar{b})}$ is affine.

Let $\beta_{(x,o,\bar{b})} > 0$ and $\delta_{(x,o,\bar{b})}$ denote, respectively, the multiplicative and additive coefficients corresponding to $\phi_{(x,o,\bar{b})}$, where the inequality follows from the monotonicity of $\phi_{(x,o,\bar{b})}$. Observe that, by (A.6), $I_{-o}\left(a, \bar{b}\right) \sim I'_{-o}\left(a', \bar{b}\right)$ if and only if $I_{-o}\left(a, \bar{b}'\right) \sim I'_{-o}\left(a', \bar{b}'\right)$. Hence

$$\beta_{(x,o,\bar{b})}\left[w\left(a, \bar{b}, o\right) - w\left(a', \bar{b}, o\right)\right] = \beta_{(x,o,\bar{b}')}\left[w\left(a, \bar{b}', o\right) - w\left(a', \bar{b}', o\right)\right] \tag{16}$$

for all $\bar{b}, \bar{b}' \in B^{cu}$. Thus, for all $x \in \bar{X}$ and $\bar{b}, \bar{b}' \in B^{cu}$, $\beta_{(x,o,\bar{b})} = \beta_{(x,o,\bar{b}')} := \xi(x) > 0$.

Let $a, a' \in A$ and $\bar{b}, \bar{b}' \in B^{cu}$ satisfy $\left(a, \bar{b}\right) \sim^o \left(a', \bar{b}'\right)$. By axiom (A.7) $\left(a, \bar{b}\right) \sim^o \left(a', \bar{b}'\right)$ if and only if $\left(a, \bar{b}\right) \sim^o \left(a', \bar{b}'\right)$. By the representation this equivalence implies that

$$w\left(a, \bar{b}, o\right) = w\left(a', \bar{b}', o\right). \tag{17}$$

if and only if,

$$\xi(x) w\left(a, \bar{b}, o\right) + \delta_{(x,o,\bar{b})} = \xi(x) w\left(a', \bar{b}', o\right) + \delta_{(x,o,\bar{b}')}. \tag{18}$$

Thus $\delta_{(x,o,\bar{b})} = \delta_{(x,o,\bar{b}')}$.

By this argument and continuity (A.2) the conclusion can be extended to $B^{cu}$. Let $\delta_{(x,o,\bar{b})} := \zeta(x)$ for all $\bar{b} \in B^{cu}$.

For every given $\bar{b} \in B^{cu}$ and all $a \in A$, define $U\left(\bar{b}, a\right) = w\left(a, \bar{b}, o\right)$. Then, for all $x \in \bar{X}$,

$$w\left(a, \bar{b}, x\right) = \xi(x) U\left(\bar{b}, a\right) + \zeta(x), \ \xi(x) > 0. \tag{19}$$

This completes the proof of Lemma 6. ♣

Equations (8) and (9) imply that for every $x \in \bar{X}$, $\bar{b} \in B^{cu}$ and $a \in A$,

$$\xi(x) U(\bar{b}, a) + \zeta(x) = H\left(\sum_{\theta \in \Theta} \lambda(a, x, \theta) u(\bar{b}(\theta), \theta) + \hat{\alpha}(a, x), a, x\right). \tag{20}$$

**Lemma 7** *The identity (20) holds if and only if* $u(\bar{b}(\theta), \theta) = u(\bar{b})$ *for all* $\theta \in \Theta$, $\sum_{\theta \in \Theta} \frac{\lambda(a, x, \theta)}{\xi(x)} = \varphi(a)$, $\frac{\hat{\alpha}(a, x)}{\xi(x)} = v(a)$ *for all* $a \in A$,

$$H\left(\sum_{\theta \in \Theta} \lambda(a, x, \theta) u(\bar{b}(\theta), \theta) + \hat{\alpha}(a, x), a, x\right) = \xi(x)\left[u(\bar{b}) + v(a)\right] + \zeta(x), \tag{21}$$

*and there is* $\kappa(a) > 0$ *such that*

$$\kappa(a) \sum_{\theta \in \Theta} \frac{\lambda(a, x, \theta)}{\xi(x)} u(\bar{b}(\theta), \theta) + \frac{\hat{\alpha}(a, x)}{\xi(x)} = U(\bar{b}, a). \tag{22}$$

*Proof:* (Sufficiency) Let $u(\bar{b}(\theta), \theta) := u(\bar{b})$ for all $\theta \in \Theta$, $\sum_{\theta \in \Theta} \frac{\lambda(a, x, \theta)}{\xi(x)} := \varphi(a)$ and $c(a) := \kappa(a) \varphi(a)$ for all $a \in A$ and suppose that (22) holds.

But axiom (A.6) and the representation imply that, for all $\bar{b}, \bar{b}' \in B^{cu}$,

$$c(a) u(\bar{b}) + v(a) = c(a') u(\bar{b}) + v(a')$$

if and only if

$$c(a) u(\bar{b}') + v(a) = c(a') u(\bar{b}') + v(a').$$

Hence $c(a) = c(a') = c$ for all $a, a' \in A$.

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Normalize $u$ so that $c = 1$. Then equation (20) follows from equations (21) and (22).

(Necessity) Multiply and divide the first argument of $H$ by $\xi(x) > 0$. Equation (20) may be written as follows:

$$\xi(x) U(\bar{b}, a) + \zeta(x) = H\left(\xi(x)\left[\sum_{\theta \in \Theta} \frac{\lambda(a, x, \theta)}{\xi(x)} u(\bar{b}(\theta), \theta) + \frac{\hat{\alpha}(a, x)}{\xi(x)}\right], a, x\right). \qquad (23)$$

Define $V(a, \bar{b}, x) = \sum_{\theta \in \Theta} \frac{\lambda(a,x,\theta)}{\xi(x)} u(\bar{b}(\theta), \theta) + \frac{\hat{\alpha}(a,x)}{\xi(x)}$ then, for every given $(a, x) \in A \times X$ and all $\bar{b}, \bar{b}' \in B^{cu}$,

$$U(\bar{b}', a) - U(\bar{b}, a) = \left[H\left(\xi(x) V(a, \bar{b}', x), a, x\right) - H\left(\xi(x) V(a, \bar{b}, x), a, x\right)\right] / \xi(x). \qquad (24)$$

Hence $H(\cdot, a, x)$ is a linear function whose intercept is $\zeta(x)$ and the slope

$$\left[U(\bar{b}', a) - U(\bar{b}, a)\right] / \left[V(a, \bar{b}', x) - V(a, \bar{b}, x)\right] := \kappa(a),$$

is independent of $x$. Thus

$$\xi(x) U(\bar{b}, a) + \zeta(x) = \kappa(a) \xi(x)\left[\sum_{\theta \in \Theta} \frac{\lambda(a, x, \theta)}{\xi(x)} u(\bar{b}(\theta), \theta) + \frac{\hat{\alpha}(a, x)}{\xi(x)}\right] + \zeta(x). \qquad (25)$$

Hence

$$U(\bar{b}, a) / \kappa(a) = \sum_{\theta \in \Theta} \frac{\lambda(a, x, \theta)}{\xi(x)} u(\bar{b}(\theta), \theta) + \frac{\hat{\alpha}(a, x)}{\xi(x)} \qquad (26)$$

is independent of $x$. However, because $\succsim_a^x \neq \succsim_a^{x'}$ for all $a$ and some $x, x' \in \bar{X}$, in general, $\lambda(a, x, \theta) / \xi(x)$ is not independent of $\theta$. Moreover, because $\hat{\alpha}(a, x) / \xi(x)$ is independent of $b$, the first term on the right-hand side of (26) must be independent of $x$. For this to be true $u(\bar{b}(\theta), \theta)$ must be independent of $\theta$ and $\sum_{\theta \in \Theta} \lambda(a, x, \theta) / \xi(x) := \varphi(a)$ be independent of $x$. Moreover, because the first term on the right-hand side of (26) is independent of $x$,

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$\hat{\alpha}\left(a,x\right)/\xi\left(x\right)$ must also be independent of $x$. Finally, by definition, $\bar{b}$ the unique element in its equivalence class that has the property that $u\left(\bar{b}\left(\theta\right),\theta\right)$ is independent of $\theta$.

Define $v\left(a\right):=\hat{\alpha}\left(a,x\right)/\xi\left(x\right)$, $u\left(\bar{b}\left(\theta\right),\theta\right)=u\left(\bar{b}\right)$, for all $\theta\in\Theta$, and $U\left(\bar{b},a\right)=u\left(\bar{b}\right)+v\left(a\right)$ and $\kappa\left(a\right)\varphi\left(a\right)=1$. Thus

$$U\left(\bar{b},a\right)=\kappa\left(a\right)\sum_{\theta\in\Theta}\frac{\lambda\left(a,x,\theta\right)}{\xi\left(x\right)}u\left(\bar{b}\left(\theta\right);\theta\right)+\frac{\hat{\alpha}\left(a,x\right)}{\xi\left(x\right)}. \tag{27}$$

This completes the proof of Lemma 7. ♣

Note that

$$U\left(\bar{b},a\right)=\sum_{\theta\in\Theta}\frac{\lambda\left(a,x,\theta\right)}{\xi\left(x\right)\varphi\left(a\right)}u\left(\bar{b}\left(\theta\right);\theta\right)+\frac{\hat{\alpha}\left(a,x\right)}{\xi\left(x\right)}=u\left(\bar{b}\right)+v\left(a\right). \tag{28}$$

But, by Lemma 7, $\sum_{\theta\in\Theta}\lambda\left(a,x,\theta\right)=\xi\left(x\right)\varphi\left(a\right)$. Hence, by the inclusivity of $B^{cu}$, the representation (7) is equivalent to

$$I\mapsto\sum_{x\in\bar{X}}\left[\sum_{\theta\in\Theta}\frac{\lambda\left(a_{I(x)},x,\theta\right)}{\sum_{\theta\in\Theta}\lambda\left(a_{I(x)},x,\theta\right)}u\left(b_{I(x)}\left(\theta\right);\theta\right)+\frac{\hat{\alpha}\left(a_{I(x)},x\right)}{\xi\left(x\right)}\right]. \tag{29}$$

For all $x\in X, a\in A$ and $\theta\in\Theta$, define the joint subjective probability distribution on $\Theta\times\bar{X}$ by

$$\pi\left(x,\theta\mid a\right)=\frac{\lambda\left(a,x,\theta\right)}{\sum_{x'\in\bar{X}}\sum_{\theta'\in\Theta}\lambda\left(a,x',\theta'\right)}. \tag{30}$$

Since $\sum_{\theta\in\Theta}\lambda\left(a,x,\theta\right)=\xi\left(x\right)\varphi\left(a\right)$, for all $x\in\bar{X}$,

$$\sum_{\theta\in\Theta}\pi\left(x,\theta\mid a\right)=\frac{\xi\left(x\right)\varphi\left(a\right)}{\sum_{x'\in\bar{X}}\xi\left(x'\right)\varphi\left(a\right)}=\frac{\xi\left(x\right)}{\sum_{x'\in\bar{X}}\xi\left(x'\right)}. \tag{31}$$

Define the subjective probability of $x\in\bar{X}$ as follows:

$$\mu\left(x\right)=\frac{\xi\left(x\right)}{\sum_{x'\in\bar{X}}\xi\left(x'\right)}. \tag{32}$$

Then the subjective probability of $x$ is given by the marginal distribution on $X$ induced by the joint distributions $\pi\left(\cdot,\cdot\mid a\right)$ on $X\times\Theta$ and is independent of $a$.

For all $\theta\in\Theta$, define the subjective posterior and prior probability of $\theta$, respectively, by

$$\pi\left(\theta\mid x,a\right)=\frac{\pi\left(x,\theta\mid a\right)}{\mu\left(x\right)}=\frac{\lambda\left(a,x,\theta\right)}{\sum_{\theta\in\Theta}\lambda\left(a,x,\theta\right)}\tag{33}$$

and

$$\pi\left(\theta\mid o,a\right)=\frac{\lambda\left(a,o,\theta\right)}{\sum_{\theta\in\Theta}\lambda\left(a,o,\theta\right)}.\tag{34}$$

Substitute in (29) to obtain the representation (5),

$$I\mapsto\sum_{x\in\bar{X}}\mu\left(x\right)\left[\sum_{\theta\in\Theta}\pi\left(\theta\mid x,a_{I(x)}\right)u\left(b_{I(x)}\left(\theta\right),\theta\right)+v\left(a_{I(x)}\right)\right].\tag{35}$$

Let $a\in A$, $I\in\mathcal{I}$ and $b,b'\in B$, satisfy $I_{-o}\left(a,b\right)\sim I_{-o}\left(a,b'\right)$. Then, by (35),

$$\sum_{\theta\in\Theta}\pi\left(\theta\mid o,a\right)u\left(b\left(\theta\right),\theta\right)=\sum_{\theta\in\Theta}\pi\left(\theta\mid o,a\right)u\left(b'\left(\theta\right),\theta\right)\tag{36}$$

and, by axiom (A.5) and (35)

$$\sum_{x\in X}\frac{\mu\left(x\right)}{1-\mu\left(0\right)}\sum_{\theta\in\Theta}\pi\left(\theta\mid x,a\right)u\left(b\left(\theta\right),\theta\right)=\sum_{x\in X}\frac{\mu\left(x\right)}{1-\mu\left(0\right)}\sum_{\theta\in\Theta}\pi\left(\theta\mid x,a\right)u\left(b'\left(\theta\right),\theta\right).\tag{37}$$

Thus

$$\sum_{\theta\in\Theta}\left[u\left(b\left(\theta\right),\theta\right)-u\left(b'\left(\theta\right),\theta\right)\right]\left[\pi\left(\theta\mid o,a\right)-\sum_{x\in X}\frac{\mu\left(x\right)}{1-\mu\left(0\right)}\pi\left(\theta\mid x,a\right)\right]=0.\tag{38}$$

This implies that $\pi\left(\theta\mid o,a\right)=\sum_{x\in X}\mu\left(x\right)\pi\left(\theta\mid x,a\right)/\left[1-\mu\left(0\right)\right]$.

(If $\pi\left(\theta\mid o,a\right)>\sum_{x\in X}\mu\left(x\right)\pi\left(\theta\mid x,a\right)/\left[1-\mu\left(0\right)\right]$ for some $\theta$ and $\mu\left(o\right)\pi\left(\theta'\mid o,a\right)<\sum_{x\in X}\mu\left(x\right)\pi\left(\theta'\mid x,a\right)/\left[1-\mu\left(0\right)\right]$ for some $\theta'$, let $\hat{b},\hat{b}'\in B$ be such that $\hat{b}\left(\theta\right)>b\left(\theta\right)$ and

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$\hat{b}\left(\hat{\theta}\right) = b\left(\hat{\theta}\right)$ for all $\hat{\theta} \in \Theta - \{\theta\}$, $\hat{b}'\left(\theta'\right) > b'\left(\theta'\right)$ and $\hat{b}'\left(\hat{\theta}\right) = b'\left(\hat{\theta}\right)$ for all $\hat{\theta} \in \Theta - \{\theta'\}$

and $I_{-o}\left(a, \hat{b}\right) \sim I_{-o}\left(a, \hat{b}'\right)$. Then

$$\sum_{\theta \in \Theta} \left[u\left(\hat{b}\left(\theta\right), \theta\right) - u\left(\hat{b}'\left(\theta\right), \theta\right)\right] \left[\pi\left(\theta \mid o, a\right) - \sum_{x \in X} \frac{\mu\left(x\right)}{1 - \mu\left(0\right)} \pi\left(\theta \mid x, a\right)\right] > 0. \qquad (39)$$

But this contradicts (A.5).)

$(a.ii) \Rightarrow (a.i)$. The necessity of (A.1), (A.2) and (A.3) follows from Theorem 1. To

see the necessity of (A.4), suppose that $I_{-x}\left(a, b_{-\theta}r\right) \succcurlyeq I_{-x}\left(a, b'_{-\theta}r'\right)$, $I_{-x}\left(a, b''_{-\theta}r''\right) \succcurlyeq I_{-x}\left(a, b_{-\theta}r'''\right)$, and $I_{-x'}\left(a', b''_{-\theta}r'\right) \succcurlyeq I_{-x'}\left(a', b'''_{-\theta}r\right)$. By representation (8)

$$\sum_{\theta' \in \Theta - \{\theta\}} \lambda\left(a, x, \theta'\right) u\left(b\left(\theta'\right), \theta'\right) + \lambda\left(a, x, \theta\right) u\left(r, \theta\right) \geq \sum_{\theta' \in \Theta - \{\theta\}} \lambda\left(a, x, \theta'\right) u\left(b'\left(\theta'\right), \theta'\right) + \lambda\left(a, x, \theta\right) u\left(r', \theta\right),$$
$$(40)$$

$$\sum_{\theta' \in \Theta - \{\theta\}} \lambda\left(a, x, \theta'\right) u\left(b'\left(\theta'\right), \theta'\right) + \lambda\left(a, x, \theta\right) u\left(r'', \theta\right) \geq \sum_{\theta' \in \Theta - \{\theta\}} \lambda\left(a, x, \theta'\right) u\left(b\left(\theta'\right), \theta'\right) + \lambda\left(a, x, \theta\right) u\left(r''', \theta\right),$$
$$(41)$$

and

$$\sum_{\theta' \in \Theta - \{\theta\}} \lambda\left(a', x', \theta'\right) u\left(b''\left(\theta'\right), \theta'\right) + \lambda\left(a', x', \theta\right) u\left(r', \theta\right) \geq \sum_{\theta' \in \Theta - \{\theta\}} \lambda\left(a', x', \theta'\right) u\left(b'''\left(\theta'\right), \theta'\right) + \lambda\left(a', x', \theta\right) u\left(r, \theta\right)$$
$$(42)$$

But (40) and (41) imply that

$$u\left(r'', \theta\right) - u\left(r''', \theta\right) \geq \frac{\sum_{\theta' \in \Theta - \{\theta\}} \lambda\left(a, x, \theta'\right) \left[u\left(b\left(\theta'\right), \theta'\right) - u\left(b'\left(\theta'\right), \theta'\right)\right]}{\lambda\left(a, x, \theta\right)} \geq u\left(r', \theta\right) - u\left(r, \theta\right). \qquad (43)$$

Inequality (42) implies

$$u\left(r', \theta\right) - u\left(r, \theta\right) \geq \frac{\sum_{\theta' \in \Theta - \{\theta\}} \lambda\left(a', x', \theta'\right) \left[u\left(b'''\left(\theta'\right), \theta'\right) - u\left(b''\left(\theta'\right), \theta'\right)\right]}{\lambda\left(a', x', \theta\right)}. \qquad (44)$$

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But (43) and (44) imply that

$$u\left(r'',\theta\right)-u\left(r''',\theta\right)\geq\frac{\sum_{\theta'\in\Theta-\{\theta\}}\lambda\left(a',x',\theta'\right)\left[u\left(b'''\left(\theta'\right),\theta'\right)-u\left(b''\left(\theta'\right),\theta'\right)\right]}{\lambda\left(a',x',\theta\right)}. \tag{45}$$

Hence

$$\sum_{\theta'\in\Theta-\{\theta\}}\lambda\left(a',x',\theta'\right)\left[u\left(b''\left(\theta'\right),\theta'\right)-u\left(b'''\left(\theta'\right),\theta'\right)\right]+\lambda\left(a',x',\theta\right)\left[u\left(r'',\theta\right)-u\left(r''',\theta\right)\right]\geq 0. \tag{46}$$

Thus, $I_{-x'}\left(a',b''_{-\theta}r''\right)\succsim I_{-x'}\left(a',b'''_{-\theta}r'''\right).$

Next I show that if $\bar{b}\in B$ satisfies $u\left(\bar{b}\left(\theta\right),\theta\right)=u\left(\bar{b}\right)$ for all $\theta\in\Theta$ then $\bar{b}\in B^{cu}$. Suppose

that representation (5) holds and let $I,I',I'',I'''\in\mathcal{I}$, $a,a',a'',a'''\in A$ and $x,x'\in\bar{X}$, such

that $I_{-x}\left(a,\bar{b}\right)\sim I'_{-x}\left(a',\bar{b}\right)$, $I'_{-x}\left(a'',\bar{b}\right)\sim I_{-x}\left(a''',\bar{b}\right)$ and $I''_{-x'}\left(a',\bar{b}\right)\sim I'''_{-x'}\left(a,\bar{b}\right)$. Then the

representation (7) implies that

$$\sum_{\hat{x}\in\bar{X}-\{x\}}w\left(a_{I(\hat{x})},b_{I(\hat{x})},\hat{x}\right)+\mu\left(x\right)\left[u\left(\bar{b}\right)+v\left(a\right)\right]=\sum_{\hat{x}\in\bar{X}-\{x\}}w\left(a_{I'(\hat{x})},b_{I'(\hat{x})},\hat{x}\right)+\mu\left(x\right)\left[u\left(\bar{b}\right)+v\left(a'\right)\right] \tag{47}$$

$$\sum_{\hat{x}\in\bar{X}-\{x\}}w\left(a_{I'(\hat{x})},b_{I'(\hat{x})},\hat{x}\right)+\mu\left(x\right)\left[u\left(\bar{b}\right)+v\left(a''\right)\right]=\sum_{\hat{x}\in\bar{X}-\{x\}}w\left(a_{I(\hat{x})},b_{I(\hat{x})},\hat{x}\right)+\mu\left(x\right)\left[u\left(\bar{b}\right)+v\left(a'''\right)\right] \tag{48}$$

and

$$\sum_{\hat{x}\in\bar{X}-\{x'\}}w\left(a_{I''(\hat{x})},b_{I''(\hat{x})},\hat{x}\right)+\mu\left(x'\right)\left[u\left(\bar{b}\right)+v\left(a'\right)\right]=\sum_{\hat{x}\in\bar{X}-\{x'\}}w\left(a_{I'''(\hat{x})},b_{I'''(\hat{x})},\hat{x}\right)+\mu\left(x'\right)\left[u\left(\bar{b}\right)+v\left(a\right)\right]. \tag{49}$$

But (47) and (48) imply that

$$v\left(a\right)-v\left(a'\right)=v\left(a''\right)-v\left(a'''\right). \tag{50}$$

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Equality (49) implies

$$\frac{\sum_{\hat{x} \in \bar{X} - \{x'\}} \left[ w \left( a_{I''(\hat{x})}, b_{I''(\hat{x})}, \hat{x} \right) - w \left( a_{I'''(\hat{x})}, b_{I'''(\hat{x})}, \hat{x} \right) \right]}{\mu(x')} = v(a) - v(a'). \tag{51}$$

Thus

$$\sum_{\hat{x} \in \bar{X} - \{x'\}} w \left( a_{I''(\hat{x})}, b_{I''(\hat{x})}, \hat{x} \right) + u(\bar{b}) + v(a''') = \sum_{\hat{x} \in \bar{X} - \{x'\}} w \left( a_{I'''(\hat{x})}, b_{I'''(\hat{x})}, \hat{x} \right) + u(\bar{b}) + v(a'') \tag{52}$$

Hence $I''_{-x'} (a''', \bar{b}) \sim I'''_{-x'} (a''', \bar{b})$ and $\bar{b} \in B^{cu}$.

To show the necessity of (A.5) let $a \in A$, $I \in \mathcal{I}$ and $b, b' \in B$, by the representation $I_{-o}(a, b) \sim I_{-o}(a, b')$ if and only if

$$\sum_{\theta \in \Theta} \pi(\theta \mid o, a) u(b(\theta), \theta) = \sum_{\theta \in \Theta} \pi(\theta \mid o, a) u(b'(\theta), \theta). \tag{53}$$

But $\pi(\theta \mid o, a) = \sum_{x \in X} \mu(x) \pi(\theta \mid x, a) / [1 - \mu(0)]$. Thus (53) holds if and only if

$$\sum_{x \in X} \mu(x) \sum_{\theta \in \Theta} \pi(\theta \mid x, a) u(b(\theta), \theta) = \sum_{x \in X} \mu(x) \sum_{\theta \in \Theta} \pi(\theta \mid x, a) u(b'(\theta), \theta). \tag{54}$$

But (54) is valid if and only if $I^{-o}(a, b) \sim I^{-o}(a, b')$.

For all $I$ and $x$, let $K(I, x) = \sum_{y \in X - \{x\}} \mu(y) \left[ \sum_{\theta \in \Theta} \pi(\theta \mid x, a) u\left(b_{I(y)}(\theta)\right) + v\left(a_{I(y)}\right) \right]$.
To show the necessity of (A.6) Then $I_{-x}(a, \bar{b}) \succcurlyeq I'_{-x}(a', \bar{b})$ if and only if

$$K(I, x) + u(\bar{b}) + v(a) \geq K(I', x) + u(\bar{b}) + v(a') \tag{55}$$

if and only if

$$K(I, x) + u(\bar{b}') + v(a) \geq K(I', x) + u(\bar{b}') + v(a') \tag{56}$$

if and only if $I_{-x} \left( a, \bar{b}' \right) \succcurlyeq I'_{-x} \left( a', \bar{b}' \right).$

To show that axiom (A.7) is implied, not that $I_{-x} \left( a, \bar{b} \right) \succcurlyeq I'_{-x} \left( a', \bar{b}' \right)$ if and only if

$$K(I, x) + u \left( \bar{b} \right) + v \left( a \right) \geq K(I, x) + u \left( \bar{b}' \right) + v \left( a' \right) \tag{57}$$

if and only if

$$K(I, x') + u \left( \bar{b} \right) + v \left( a \right) \geq K(I, x') + u \left( \bar{b}' \right) + v \left( a' \right) \tag{58}$$

if and only if $I_{-x'} \left( a, \bar{b} \right) \succcurlyeq I'_{-x'} \left( a', \bar{b}' \right).$ This completes the proof of part (a).

(b) Suppose, by way of negation, that there exist continuous, real-valued functions $\{ \tilde{u} \left( \cdot, \theta \right) \mid \theta \in \Theta \}$ on $\mathbb{R}$, $\tilde{v} \in \mathbb{R}^A$ and, for every $a \in A$, there is a joint probability measure $\tilde{\pi} \left( \cdot, \cdot \mid a \right)$ on $\bar{X} \times \Theta$, distinct from those that figure in the representation (5), such that $\succcurlyeq$ on $\mathcal{I}$ is represented by

$$I \mapsto \sum_{x \in \bar{X}} \tilde{\mu} \left( x \right) \left[ \sum_{\theta \in \Theta} \tilde{\pi} \left( \theta \mid x, a_{I(x)} \right) \tilde{u} \left( b_{I(x)} \left( \theta \right), \theta \right) + \tilde{v} \left( a_{I(x)} \right) \right], \tag{59}$$

where $\tilde{\mu} \left( x \right) = \sum_{\theta \in \Theta} \tilde{\pi} \left( x, \theta \mid a \right)$ for all $x \in \bar{X}$, and $\tilde{\pi} \left( \theta \mid x, a \right) = \tilde{\pi} \left( x, \theta \mid a \right) / \tilde{\mu} \left( x \right)$ for all $\left( \theta, x, a \right) \in \Theta \times X \times A$.

Define $\kappa \left( x \right) = \tilde{\mu} \left( x \right) / \mu \left( x \right)$, for all $x \in \bar{X}$. Then the representation (59) may be written as

$$I \mapsto \sum_{x \in \bar{X}} \mu \left( x \right) \left[ \sum_{\theta \in \Theta} \pi \left( \theta \mid x, a_{I(x)} \right) \gamma \left( \theta, x, a_{I(x)} \right) \kappa \left( x \right) \tilde{u} \left( b_{I(x)} \left( \theta \right), \theta \right) + \kappa \left( x \right) \tilde{v} \left( a_{I(x)} \right) \right]. \tag{60}$$

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Hence, by (5), $\tilde{u}\left(b\left(\theta\right),\theta\right)=u\left(b\left(\theta\right),\theta\right)/\tilde{\gamma}\left(\theta,x,a\right)\kappa\left(x\right)$ and $\tilde{v}\left(a\right)=v\left(a\right)/\kappa\left(x\right)$. The second

equality implies that $\kappa\left(x\right)=\kappa$ for all $x\in\bar{X}$. Consequently, the first inequality implies that

$\tilde{\gamma}\left(\theta,x,a\right)=\gamma\left(\theta\right)$ for all $\left(x,a\right)\in\bar{X}\times A$. Thus, for $\bar{b}\in B^{cu}$,

$$I\mapsto\sum_{x\in\bar{X}}\mu\left(x\right)\left[\sum_{\theta\in\Theta}\pi\left(\theta\mid x,a_{I\left(x\right)}\right)\frac{u\left(\bar{b}\right)}{\gamma\left(\theta\right)}+v\left(a_{I\left(x\right)}\right)\right].\tag{61}$$

Let $\hat{b}\in B$ be defined by $u\left(\hat{b}\left(\theta\right),\theta\right)=u\left(\bar{b}\right)/\gamma\left(\theta\right)$ for all $\theta\in\Theta$. Then, $\hat{b}\sim_{a}^{x}\bar{b}$ for all

$\left(x,a\right)\in\bar{X}\times A$, and, by definition 2, $\hat{b}\in B^{cu}$. Moreover, if $\gamma\left(\cdot\right)$ is not a constant function

then $\hat{b}\neq\bar{b}$. This contradicts the uniqueness of $\bar{b}$ in definition 2. Thus $\gamma\left(\theta\right)=\gamma$ for all $\theta\in\Theta$.

But

$$1=\sum_{x\in\bar{X}}\sum_{\theta\in\Theta}\tilde{\pi}\left(\theta,x\mid a_{I\left(x\right)}\right)=\gamma\sum_{x\in\bar{X}}\sum_{\theta\in\Theta}\pi\left(\theta,x\mid a\right)=\gamma.\tag{62}$$

Hence, $\tilde{\pi}\left(\theta,x\mid a\right)=\pi\left(\theta,x\mid a\right)$ for all $\left(\theta,x\right)\in\Theta\times\bar{X}$ and $a\in A$.

Next consider the uniqueness of the utility functions. The representations (5) and (7)

imply that

$$w\left(a,b,x\right)=\mu\left(x\right)\left[\sum_{\theta\in\Theta}\pi\left(\theta\mid x,a\right)u\left(b\left(\theta\right),\theta\right)+v\left(a\right)\right].\tag{63}$$

Hence, by the uniqueness part of Theorem 1, $\{\tilde{u}\left(\cdot,\theta\right)\}_{\theta\in\Theta}$ and $\tilde{v}\in R^{A}$ must satisfy

$$\sum_{\theta\in\Theta}\pi\left(\theta\mid x,a\right)v\left(b\left(\theta\right),\theta\right)+\tilde{v}\left(a\right)=m\left[\sum_{\theta\in\Theta}\pi\left(\theta\mid x,a\right)u\left(b\left(\theta\right),\theta\right)+v\left(a\right)\right]+K\left(x\right),\tag{64}$$

where $m>0$. Clearly, this is the case if $\tilde{u}\left(\cdot,\theta\right)=mu\left(\cdot,\theta\right)+k$ and $\tilde{v}=mv+k'$.

Suppose that $\tilde{u}\left(\cdot,\theta\right)=mu\left(\cdot,\theta\right)+k$, $\tilde{v}=m'v+k'$ and, without loss of generality, let

$m>m'>0$. Take $a,a'\in A$ and $\bar{b},\bar{b}'\in B^{cu}$ such that $\left(a,\bar{b}\right)\sim^{x}\left(a',\bar{b}'\right)$. Then, by the

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representation (5),

$$u\left(\bar{b}\right) - u\left(\bar{b}'\right) = v\left(a\right) - v\left(a'\right).\tag{65}$$

But

$$\tilde{u}\left(\bar{b}\right) - \tilde{u}\left(\bar{b}'\right) = m\left[u\left(\bar{b}\right) - u\left(\bar{b}'\right)\right] > m'\left[v\left(a\right) - v\left(a'\right)\right] = \tilde{v}\left(a\right) - \tilde{v}\left(a'\right).\tag{66}$$

Hence $\tilde{u}\left(\cdot,\theta\right)$ and $\tilde{v}$ do not represent $\succcurlyeq$ .

Consider next $\tilde{u}\left(\cdot,\theta\right) = mu\left(\cdot,\theta\right) + k\left(\theta\right)$, where $k\left(\cdot\right)$ is not a constant function. Let $\bar{k}\left(x,a\right) = \sum_{\theta\in\Theta}\pi\left(\theta\mid x,a\right)k\left(\theta\right)$. Take $a, a' \in A$ and $\bar{b}, \bar{b}' \in B^{cu}$ such that $\left(a,\bar{b}\right) \sim^{x} \left(a',\bar{b}'\right)$ and $\left[\bar{k}\left(x,a\right) - \bar{k}\left(x,a'\right)\right] \neq 0$ for some $x$. Then

$$\tilde{u}\left(\bar{b}\right) - \tilde{u}\left(\bar{b}'\right) = m\left[u\left(\bar{b}\right) - u\left(\bar{b}'\right)\right] + \left[\bar{k}\left(x,a\right) - \bar{k}\left(x,a'\right)\right] \neq m\left[v\left(a\right) - v\left(a'\right)\right] = \tilde{v}\left(a\right) - \tilde{v}\left(a'\right).$$

Hence $\tilde{u}\left(\cdot,\theta\right)$ and $\tilde{v}$ do not represent $\succcurlyeq$ .

If $\tilde{v}\left(a\right) = mv\left(a\right) + k'\left(a\right)$, where $k\prime\left(\cdot\right)$ is not a constant function then, by a similar argument, $\tilde{u}\left(\cdot,\theta\right)$ and $\tilde{v}$ do not represent $\succcurlyeq$ . ∎

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