Deterministic components in the CVAR

A graduate course in the Cointegrated VAR model: Special topics in Rome

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November 2011

A constant and a trend in the CVAR

We consider the simple CVAR without short-run dynamics:

$$\Delta x_t = \alpha\beta' x_{t-1} + \mu_0 + \mu_1 t + \varepsilon_t \qquad (1)$$

$$\varepsilon_t \sim N_p(0, \Omega\ ), t = 1, ..., T \qquad (2)$$

and in the MA form:

$$x_t = C\sum_{i=1}^{t}(\varepsilon_i + \mu_0 + i\mu_1) + \sum_{i=0}^{\infty} C_i^*(\varepsilon_{t-i} + \mu_0 + (t-i)\mu_1) \qquad (3)$$

Because Δx_t and $\beta' x_{t-1}$ are stationary around their mean, we can express (1) as:

$$\Delta x_t - E\Delta x_t = \alpha(\beta' x_{t-1} - E(\beta' x_{t-1})) + \varepsilon_t \qquad (4)$$

where

$$E\Delta x_t = \alpha E(\beta' x_{t-1}) + \mu_0 + \mu_1 t \qquad (5)$$

$$E(\beta' x_t) = (I + \beta'\alpha)E(\beta' x_{t-1}) + \beta'\mu_0 + \beta'\mu_1 t \qquad (6)$$

Decomposing μ_0 and μ_1 into the space spanned by α and $\beta_\perp$

$$\begin{aligned} \mu_0 &= \alpha\beta_0 + \gamma_0 \\ \mu_1 &= \alpha\beta_1 + \gamma_1 \end{aligned} \tag{7}$$

By substituting (7) in (1) we get:

$$\Delta x_t = \alpha\beta' x_{t-1} + \alpha\beta_0 + \alpha\beta_1 t + \gamma_0 + \gamma_1 t + \varepsilon_t, \tag{8}$$

and by rearranging, (8) can be written as:

$$\Delta x_t = \alpha \left[\boldsymbol{\beta}', \boldsymbol{\beta}_0, \boldsymbol{\beta}_1\right] \begin{bmatrix} \mathbf{x}_{t-1} \\ 1 \\ t \end{bmatrix} + \gamma_0 + \gamma_1 t + \varepsilon_t.$$

Thus, (1) can be reformulated as:

$$\Delta x_t = \alpha\tilde{\beta}' \tilde{x}_{t-1} + \gamma_0 + \gamma_1 t + \varepsilon_t, \tag{9}$$

where $\tilde{\beta}' = [\beta', \beta_0, \beta_1]$ and $\tilde{x}_{t-1} = (x_{t-1}, 1, t)'$.

Interpretation

The γ components:

$$E(\Delta \mathbf{x}_t) = \gamma_0 + \gamma_1 t, \tag{10}$$

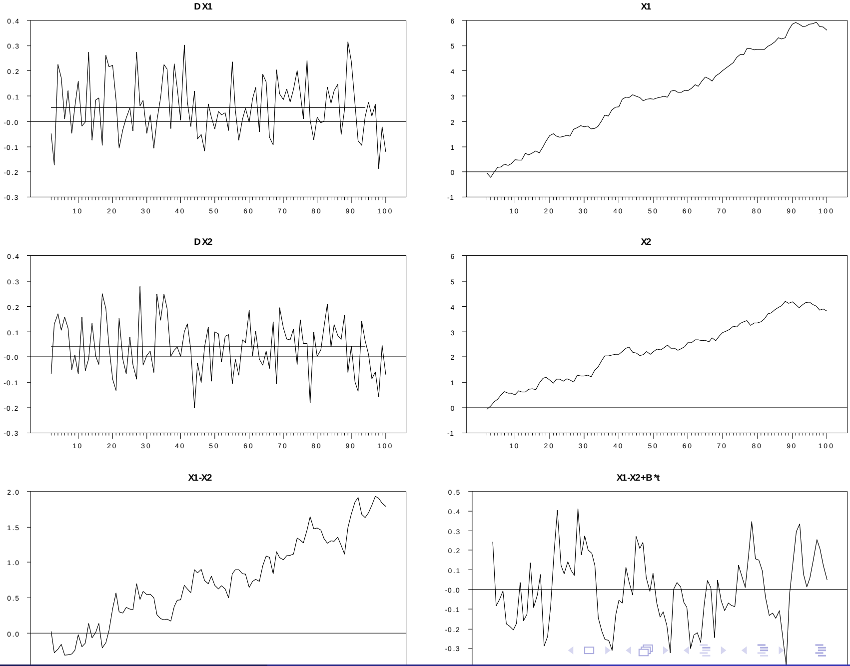
i.e. $\gamma_0 \neq 0$, implies linear growth in at least some of the variables and $\gamma_1 \neq 0$ implies quadratic trends in the variables.
The β components.

$$E\left(\boldsymbol{\beta}' \mathbf{x}_t\right) = \beta_0 + \beta_1 t$$

The role of the constant term and the trend: an illustrative example

$$
\begin{aligned}
&\Delta x_{1,t} = a_{10} - \alpha \left\{ x_{1,t-1} - x_{2,t-1} + (a_{10} - a_{20})t \right\} + \varepsilon_{1,t} \\
&\Delta x_{2,t} = a_{20} + \varepsilon_{2,t} \\
&t = 1, ..., 100 \\
&a_{10} = 0.01,\ a_{20} = 0.03,\ \alpha = 0.2,\ \ \sigma_1 = \sigma_2 = 0.1.
\end{aligned}
$$

The two variables, $x_{1,t}$ and $x_{2,t}$ generated from this model are nonstationary with a common stochastic trend, but with different linear deterministic trends.


DX1
X1
DX2
X2
X1-X2
X1-X2+B*t

Five cases

Case 1. μ_1, $\mu_0 = 0$. This case corresponds to a model with no deterministic components in the data, i.e. $E(\Delta\mathbf{x}_t) = 0$ and $E(\beta' x_t) = 0$, implying that the intercept of every cointegrating relation is zero. An intercept is, however, often needed to account for the initial level of measurements, X_0.

Case 2. $\mu_1 = 0, \mu_0 \neq 0$ with $\gamma_0 = 0$ but $\beta_0 \neq 0$, i.e. the constant term is restricted to be in the cointegrating relations. In this case, there are no linear trends in the data, consistent with $E(\Delta x_t) = 0$. The only deterministic component in the model is the intercept of the cointegrating relations, implying that the equilibrium mean is different from zero.

Case 3. $\mu_1 = 0$, so that $(\beta_1, \gamma_1) = 0$. The constant term μ_0 is unrestricted, i.e. no linear trends in the CVAR model, but there are linear trends in the data. In this case, there is no trend in the cointegration relations, but $E(\Delta x_t) = \gamma_0 \neq 0$, is consistent with linear trends in the variables. Since $\beta_1 = 0$, these trends cancel in the cointegrating relations and $\mu_0 \neq 0$ implies both linear trends in the data and a non-zero mean of the cointegration relations.

Case 4. $\mu_1 \neq 0$ with $\beta_1 \neq 0$, but $\gamma_1 = 0$, i.e. $(\gamma_0, \beta_0, \beta_1) \neq 0$. Thus, the trend is restricted to be only in the cointegrating relations, whereas the constant is unrestricted in the model. $E(\Delta x_t) = \gamma_0 \neq 0$, implies a linear trend in the level of x_t, and $\gamma_1 = 0$ implies no quadratic trends in the data. $\beta_1 \neq 0$ implies that the linear trends in the variables do not cancel in the cointegrating relations, i.e. our VAR model allows for 'trend-stationary' variables or trend-stationary cointegrating relations. The hypothesis that a variable is trend-stationary, say the output gap, can be tested in this model.

Case 5. No restrictions on μ_0, μ_1, i.e. the trend and the constant are unrestricted in the VAR model. In this case, the model is consistent with linear trends in the differenced series Δx_t and thus quadratic trends in x_t. Although quadratic trends may sometimes improve the fit within the sample, forecasting outside the sample is likely to produce implausible results. Instead, it seems preferable to find out what has caused this approximate quadratic growth, and if possible include more appropriate information in the model (for example, population growth or the proportion of old/young people in a population).

The MA representation

$$\triangle x_t = \alpha\beta' x_{t-1} + \mu_0 + \mu_1 t + \varepsilon_t$$

can be inverted:

$$\begin{aligned} \triangle x_t &= C(L)(\varepsilon_t + \mu_0 + \mu_1 t) \\ &= [C(1) + C^*(L)(1-L)](\varepsilon_t + \mu_0 + \mu_1 t) \end{aligned} \tag{11}$$

and summed to yield the common trends representation

$$x_t = C(1)\frac{(\boldsymbol{\varepsilon}_t + \boldsymbol{\mu}_0 + \boldsymbol{\mu}_1 t)}{(1-L)} + C^*(L)(\varepsilon_t + \mu_0 + \mu_1 t) \tag{12}$$

$$= C(1)\sum_{i=1}^{\infty}(\varepsilon_i + \mu_0 + \mu_1 i) + C^*(L)(\varepsilon_t + \mu_0 + \mu_1 t). \tag{13}$$

$$C(1) = C = \beta_\perp(\alpha'_\perp \Gamma \beta_\perp)^{-1}\alpha'_\perp. \tag{14}$$

By summing and rearranging deterministic and stochastic components:

$$x_t = \underbrace{C\mu_0 t + \frac{1}{2}C\mu_1 t^2 + \frac{1}{2}C\mu_1 t + C^*(L)\mu_1 t + C^*(1)\mu_0}_{\text{determ. comp.}} + \qquad (15)$$

$$+\underbrace{C\sum_{i=1}^{t}\varepsilon_i + C^*(L)\varepsilon_t}_{\text{stoch. comp.}} + \tilde{X}_0, \text{ for } t = 1, ..., T.$$

where $\tilde{X}_0$ contains the effect of the initial values defined so that $\beta'\tilde{X}_0 = 0$ and $C^*(L) = C_0^* + C_1^*L + \cdots + C_t^*L^t$. Substituting the expression for C in (15):

$$x_t = \beta_\perp(\alpha'_\perp\Gamma\beta_\perp)^{-1}\alpha'_\perp \underbrace{\left[\boldsymbol{\mu}_0 t + \frac{1}{2}\boldsymbol{\mu}_1 t + \frac{1}{2}\boldsymbol{\mu}_1 t^2\right]} + \underbrace{[\mathbf{C}^*(L)\boldsymbol{\mu}_1 t]} +$$

$$+C\sum_{i=1}^{t}\varepsilon_i + C^*(L)\varepsilon_t + C^*(1)\mu_0 + \tilde{X}_0. \qquad (16)$$

Focusing on the linear and quadratic trend components:

$$\alpha'_{\perp}\mu_0 t = \underbrace{\boldsymbol{\alpha}'_{\perp}\boldsymbol{\alpha}\boldsymbol{\beta}_0 t}_{0} + \alpha'_{\perp}\gamma_0 t$$

$$\alpha'_{\perp}\frac{1}{2}\mu_1 t = \frac{1}{2}(\underbrace{\boldsymbol{\alpha}'_{\perp}\boldsymbol{\alpha}\boldsymbol{\beta}_1 t}_{0} + \alpha'_{\perp}\gamma_1 t)$$

and

$$\alpha'_{\perp}\frac{1}{2}\mu_1 t^2 = \frac{1}{2}(\underbrace{\boldsymbol{\alpha}'_{\perp}\boldsymbol{\alpha}\boldsymbol{\beta}_1 t^2}_{0} + \alpha'_{\perp}\gamma_1 t^2),$$

The MA representation with a trend in the eqations

$$x_t = \beta_\perp(\alpha'_\perp\Gamma\beta_\perp)^{-1}\alpha'_\perp\{\gamma_0 t + \tfrac{1}{2}\gamma_1 t + \tfrac{1}{2}\gamma_1 t^2\} + C^*(L)\mu_1 t + \\ + C^*(1)\mu_0 + C\sum_{i=1}^{t}\varepsilon_i + C^*(L)\varepsilon_t + \tilde{X}_0. \tag{17}$$

Thus, (17) shows that linear trends in the variables can originate from three different sources in the VAR model:

1. the α component $(C^*(L)\mu_1 t)$ of the unrestricted linear trend $\mu_1 t$
2. the $\beta_\perp$ component $(\gamma_1 t)$ of the unrestricted linear trend $\mu_1 t$
3. the $\beta_\perp$ component $(\gamma_0 t)$ of unrestricted constant term μ_0

(17) in a more compact form:

$$x_t = C\{\underbrace{\boldsymbol{\tau}_1 t + \tau_2 t^2}_{\text{det. components}}\} + \underbrace{C\Sigma\varepsilon_i + C^*(L)\varepsilon_t}_{\text{stoch. comp.}} + \tilde{X}_0, \tag{18}$$