Learning from Prices: Amplification and Sentiments*

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Abstract

We provide a new theory of expectations-driven business cycles in an economy in which all shocks are fundamental and consumers learn from the prices they observe. Learning from prices causes small changes in aggregate productivity to induce large shifts in aggregate beliefs, generating positive price-quantity comovement. Amplification may be strengthened when fundamental shocks become smaller. In the limit of arbitrarily small shocks, aggregate fluctuations appear to be driven by "sentiments." Providing noisy public information reinforces amplification rather than dampening it, while weakening price-quantity correlation. In extreme cases, quantities respond exclusively to noise, while prices respond almost exclusively to fundamentals. Finally, we analyze equilibrium stability and find that equilibria with high amplification may be stable.

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1 Introduction

We propose a new mechanism—based on learning from prices—that delivers expectations-driven economic fluctuations without relying on any source of extrinsic noise. We show that when consumers learn from the prices of the goods they consume, higher prices can lead consumers to become unduly optimistic about their economic prospects. Initial optimism causes consumers to demand more goods, further increasing prices beyond their full-information level. The self-reinforcing nature of this feedback loop leads to equilibria in which small shocks to supply drive large changes in beliefs and induce the positive price-quantity comovement typically associated with demand shocks. We show that such equilibria can occur in a standard economic environment and provide a rich set of testable implications for the study of business cycles.

The mechanism of this paper offers a resolution to a longstanding challenge for macroeconomic theory: how to rationalize large fluctuations in economic outcomes with the small measured volatility of total factor productivity and other aggregate fundamentals that may drive these outcomes. Our analysis unifies two competing approaches to resolving this problem. First, it shares the insight of the recent noise-shock and sentiment literature, which shows that fluctuations may be driven by expectational errors that are correlated across agents (Lorenzoni, 2009; Angeletos and La'O, 2013; Benhabib et al., 2015). Second, it shares the focus on amplification with studies of aggregate transmission mechanisms that lead otherwise modest economic shocks to have large aggregate consequences (Kiyotaki and Moore, 1997; Bernanke et al., 1999; Brunnermeier and Sannikov, 2014). In our environment, expectational errors originate with fundamental shocks and are amplified by agents' inferences using endogenous price signals.

We begin our analysis with a static microfounded economy inspired by the large family metaphor introduced by Lucas (1980). The economy is divided into islands, each of which is inhabited by three types of agents—producers, workers, and shoppers—who belong to the same representative family. The utility of the family is perturbed by island-specific preference shocks that shift the relative importance of each island's contribution to family well-being, while producers are subject to a common (aggregate) productivity shock. Producers and workers behave as if they had full information. Producers produce a local consumption good using local labor and a tradable endowment good. Workers supply labor according to the marginal disutility of labor, embedding the local preference shock in the equilibrium local wage. The choices of producers and workers combine to generate a price for the local good that reflects both exogenous shocks—one idiosyncratic, one aggregate—and the price of the endowment good, which is endogenous to the equilibrium actions of all agents in the economy.

For their part, shoppers do not observe the preference shock on their own island when they shop for the local consumption good. They are therefore uncertain about the marginal utility contributed to the household by consuming the local good and seek to infer this utility from the good's equilibrium price. A high local price could be a signal of high marginal utility of the local good or it could reflect a change in aggregate productivity, which plays the role of noise in the shopper's inference. If productivity shocks do not move prices much, rational shoppers attribute price increases primarily to high marginal utility of their variety, leading price increases to drive demand *up*. Since the average price level reflects productivity conditions, however, aggregate productivity shocks shift the average beliefs of shoppers about local conditions. The aggregate supply shock can thus coordinate an expectations-driven increase (or decrease) in demand across islands. In general equilibrium, the aggregate increase in demand raises the price of the endowment good, which in turn further pushes up the price of local goods, reinforcing shoppers' initial mistaken inference. In short, learning through prices leads to productivity-driven shifts in demand, while the feedback of aggregate conditions to local prices offers the potential for a strong amplification mechanism.

We next characterize equilibrium in the economy, describing cases with both unique and multiple equilibria. Under most circumstances, the informational feedbacks described above are *reinforcing*: The volatility of beliefs relative to fundamentals grows as fundamental volatility falls. Feedbacks are reinforcing because lowering the variance of aggregate shocks leads, *ceteris paribus*, to an increase in the precision of local price signals, which in turn causes agents to increase the inference weights they place on those signals. Higher weights lead to larger fluctuations in beliefs for similarly sized realizations of the productivity shock.

When the local consumption price depends strongly enough on aggregate conditions, the feedback of actions into beliefs leads some equilibria to exhibit nontrivial aggregate fluctuations, even in the limit of arbitrarily small aggregate productivity shocks. To an econometrician, the fluctuations emerging at the limit of no aggregate shocks would appear as shocks to sentiments, but the origin of sentiment is different from that described by Angeletos and La'O (2013) or Benhabib et al. (2015).[1] First, sentiments emerge in our model as a case of extreme sensitivity to fundamental shocks, rather than as a response to extrinsic randomness. Second, our model demonstrates how the price system leads markets to endogenously coordinate on this particular shock to drive sentiments, rather than assuming coordination on the shock from the outset.

In addition to providing a novel foundation for sentiment shocks, our model exhibits several qualitative features that are attractive for business cycle analysis. In particular, we show that *all* the equilibria in our economy—not only limit cases with sentiment-like equilibria—feature positive price-quantity comovement in response to sufficiently small productivity shocks. When the aggregate shock in the price signal is small, the informational role of prices dominates their allocative role, so that agents react to prices more for what they mean than for the costs they impose: Higher prices lead to higher expected marginal utility, increasing quantity demanded by more than higher costs reduce it. When aggregate productivity shocks are small, the balance of these forces causes the aggregate demand schedule to become upward sloping, generating positive price-quantity comovements. Our static mechanism provides one alternative to the dynamic mechanism of Lorenzoni (2009), which also links productivity shocks with changes in expectations and demand-driven fluctuations, but relies on extrinsic shocks to information.

After establishing the key features of our model of price-driven amplification, we go on to demonstrate several important implications regarding the effects of noisy public information. We show that public information, which might be expected to prevent agents from making the correlated errors associated with sentiments, actually facilitates the coordination failure that generates these fluctuations. In particular, adding an exogenous public signal exacerbates the informational feedback channel, making demand-driven fluctuations more likely rather than less. Moreover, whereas in the baseline model aggregate expectations are perfectly correlated

[1] Indeed, we show that these limit equilibria have *exactly* the same stochastic properties as the equilibrium documented by Benhabib et al. (2015).

with aggregate fundamentals, the additional public signal shifts the coordination of beliefs toward the noise in the signal. In this case, the economy appears to be driven by two types of shocks: one that reflects supply-side conditions, originating in the predicted component of productivity, and one that reflects demand conditions, originating in the combination of noise and unpredicted productivity shocks. Overall volatility in this case can rise or fall with an increase in the precision of public information, and the comovement of prices and quantities takes on intermediate values.

We discuss several extensions that demonstrate the robustness of the basic insight. First, we show how preferences—specifically, concavity in the disutility of labor—can serve to increase the likelihood that strong informational amplification will arise in equilibrium. Second, we show that a similar characterization of equilibria obtains when, instead of a common productivity shock, we introduce correlation in preference shocks. Indeed, the aggregate consequences of the price-feedback mechanism arise in our economy whenever any fundamental shock has an aggregate component. Lastly, we show that while high prices do indeed spur total demand, the model need not imply the existence of a positive price-quantity relationship at the good level.

One argument favoring recent models of sentiments, rather than traditional sunspot models of "animal spirits," is that the existence of sunspot equilibria typically relies on non-convexities in the payoff structure of private agents, which often lead these equilibria to fail standard tests of equilibrium stability.[2] We therefore conclude by examining the stability properties of the equilibria we have emphasized. To do this, we consider two equilibrium selection techniques, higher-order belief stability and adaptive learnability. We show that the limiting sentiment equilibria do not survive either of these tests, while the limiting sentiment-free equilibria do. Outside of the limit, however, equilibria with strong informational feedbacks, and positive comovement of price and quantity, can be stable.

In addition to the sentiment literature cited above, this paper is related to a long strand of work studying sunspot fluctuations (see Azariadis, 1981; Cass and Shell, 1983; Cooper and

[2]See Guesnerie (2005) and Evans and Honkapohja (2001). A notable exception is Woodford (1990), who shows the existence of adaptively learnable sunspots; for a comprehensive discussion, see also Evans and McGough (2011). Examples of stability under higher-order belief dynamics are found by Desgranges and Negroni (2003).

John, 1988; and Benhabib and Farmer, 1994, among others.) Particularly related is Manuelli and Peck (1992), who describe cases where small changes in exogenous endowments lead to sunspot-like fluctuations in overlapping generations models. This paper also belongs to a long literature that studies coordination games with incomplete information. Amador and Weill (2010), Manzano and Vives (2011), and Vives (2012) all consider imperfect information models in which the endogeneity of price signals plays an important role, including in generating multiple equilibria.[3] Gaballo (2015) shows that information transmitted by prices can generate learnable dispersed-information equilibria in the limit of zero cross-sectional variance of fundamentals, for cases in which a distinct non-learnable perfect-information equilibrium also exists. Recent work by Bergemann and Morris (2013) characterizes the full set of incomplete-information equilibria in similar coordination games. Related work by Bergemann et al. (2015) studies the exogenous information structures that give rise to maximal aggregate volatility, and the extrema they find are typically achieved by the endogenous signal structures considered here. Recent studies by Hassan and Mertens (2011, 2014) have shown that arbitrarily small deviations from rational expectations can generate nontrivial aggregate consequences, in a manner that resembles the multiplier effect that we find.

2 Amplification Through Learning

2.1 A microfounded model

In this section, we develop a microfounded economy that endogenously generates the information structure we wish to study. In the microfounded economy, all shocks are fundamental and all signals are derived as endogenous outcomes of competitive markets.

Preferences and technology

To model heterogeneity of information, we employ the "family" metaphor first introduced by Lucas (1980) and more recently adopted by Amador and Weill (2010) and Angeletos and La'O (2010), among others. The economy is inhabited by a representative price-taking household

[3] The literature on price revelation in auction markets following Milgrom (1981) also features a dual informational/allocative role for prices. For recent examples, see Rostek and Weretka (2012); Lauermann et al. (2012); Atakan and Ekmekci (2014).

composed of a continuum of members. Each member can be a "shopper," a "producer," or a "worker." Members are evenly distributed across islands indexed by $i \in [0,1]$. On each island i, a representative worker chooses how much labor of type i to supply; a representative producer transforms labor of type i along with a tradable endowment input good into a local consumption good of variety i; and a representative shopper uses household budget resources to buy consumption goods of variety i, which are finally consumed by the household.

The utility function of the family is:

$$\int e^{\mu_i} \left(\log C_i - \phi N_i\right) di, \tag{1}$$

where C_i and N_i denote, respectively, consumption and labor of variety i, ϕ is a positive constant, and e^{μ_i} is an island-specific preference shock with $\mu_i \sim N(0, \sigma_\mu)$ independently distributed across islands. The preference shock is meant to capture heterogeneity in the value of each island business (variety) in terms of overall utility of the household. The household is subject to the following budget constraint:

$$\int P_i C_i di = QZ + \int W_i N_i di + \int \Pi_i di, \tag{2}$$

where P_i is the price of the good i, W_i is the nominal wage of labor of type i, Π_i is the profit in island i, and Q is the price of the endowment good used in production, which is available in a fixed supply Z and trades freely across islands.[4]

The tradable endowment good is combined with island-specific labor to produce the final good, C_i, according to the technology,

$$C_i = N_i^{\gamma} \left(e^{-\zeta} Z_{(i)}\right)^{1-\gamma}, \tag{3}$$

with $\gamma \in (0,1)$, where $Z_{(i)}$ denotes the quantity of the endowment good used in production of the consumption good i and $e^{-\zeta}$ is an aggregate productivity shock distributed according to $\zeta \sim N(0, \sigma_\zeta)$. Note that the sign convention we employ implies that a positive value for

[4]The endowment good would naturally map to the capital stock in a dynamic model. Our mechanism requires only the existence of some portion of inputs whose aggregate supply is predetermined within the period.

ζ corresponds to a negative productivity shock. Finally, market clearing requires

$$\int Z_{(i)} di = Z.$$

For ease of exposition, we assume that workers and producers have full information but shoppers do not. In particular, shoppers must infer local conditions—the local preference shock—based on their observation of the price of the local consumption good.[5] Moreover, as in Angeletos and La'O (2013), we normalize the value of the Lagrange multiplier associated with the budget constraint of the household to serve as a numeraire.[6] With these assumptions, we can write the maximization problems of the worker, the producer, and the shopper in island i as follows:

$$\text{worker} \quad : \quad \max_{N_i} \; W_i N_i - e^{\mu_i} \phi N_i, \tag{4}$$

$$\text{producer} \quad : \quad \max_{N_i, Z_{(i)}} \; P_i N_i^{\gamma} \left(e^{-\zeta} Z_{(i)} \right)^{1-\gamma} - W_i N_i - Q Z_{(i)}, \tag{5}$$

$$\text{shopper} \quad : \quad \max_{C_i} \; E[e^{\mu_i} | P_i] \log C_i - P_i C_i. \tag{6}$$

subject to the budget constraint (2).

The shopper does not know μ_i, which is an exogenous island-specific disturbance. Meanwhile, the price of her good, P_i, depends on both local conditions and the price of the aggregate tradable input, Q. This price, in turn, depends on the total demand for the endowment and is the only market link across islands: All other prices and quantities are island-specific.

Equilibria with learning from prices

The definition of equilibrium is formally given by the following.

Definition 1. *For a given realization of* $\{\mu_i\}_0^1$ *and* ζ*, a rational expectations equilibrium is a collection of prices* $\{\{P_i, W_i\}_0^1, Q\}$ *and quantities* $\{N_i, C_i, Z_{(i)}\}_0^1$ *such that agents' choices are optimal given the prices they observe, and markets clear.*

[5] In fact, if we assumed—as we do for shoppers—that producers and workers observe only the actual or shadow prices of the resources they use, then these agents would behave in equilibrium as if they held perfect information. We show this in the appendix.

[6] In this case, we could have equivalently fixed the average wage to one, as do Benhabib et al. (2015). In the appendix, we show that our normalization is equivalent to fixing a monetary numeraire, which is the typical approach in the DSGE literature: Our economy can be seen as the "cashless" limit of a monetary economy. Alternatively, we could have obtained the same result in an i.i.d. dynamic economy by allowing the household to trade a nominal bond in zero net supply and ruling out bubbles.

The first-order conditions of the family members' problems are:

$$E[e^{\mu_i}|P_i] = C_i P_i, \tag{7}$$

$$W_i = e^{\mu_i}\phi, \tag{8}$$

$$Q = (1-\gamma) P_i N_i^{\gamma} Z_{(i)}^{-\gamma} e^{-(1-\gamma)\zeta}, \tag{9}$$

$$W_i = \gamma P_i N_i^{\gamma-1} Z_{(i)}^{1-\gamma} e^{-(1-\gamma)\zeta}, \tag{10}$$

Letting $x \equiv \log(X/\bar{X})$ for any level variable X, the full set of equilibrium conditions of the economy can be written in terms of log-deviations of each variable from its steady-state value $\bar{X}$. Combining the log-linear version of (9) and (10), we obtain the standard result,

$$p_i = \gamma w_i + (1-\gamma)(q+\zeta), \tag{11}$$

which states that the equilibrium price of the local good is a linear combination of the costs of factor inputs, corrected for productivity, with weights according to the share of that input in production.

From equation (8), it follows that $w_i = \mu_i$, i.e., the wage is a direct measure of the island-specific preference shock. Combining the optimality condition for z in (9) with the production function in (3), it is possible to show that $q = p_i + c_i - z_{(i)}$. Then, using the log-linear version of shopper optimality in (7) and exploiting the market-clearing condition, $\int z_{(i)} di = 0$, we have

$$q = \int E[\mu_i|p_i]di. \tag{12}$$

Equation (12) states that fluctuations in the price of the endowment are driven only by the correlated component of shoppers' expectations about their own local conditions.

We can therefore rewrite the marginal cost expression in (11) as

$$p_i = \gamma\mu_i + (1-\gamma)\left(\int E[\mu_i|p_i]di + \zeta\right). \tag{13}$$

The signal structure implied by this final equation captures the endogenous feedback effect of inference *from* prices back *into* prices, and it is on this structure that we focus our subsequent analysis.

Before proceeding to an analytical characterization, it is helpful to spell out the economic intuition behind the inference problem being solved by shoppers. When shoppers see the equilibrium price of their good fluctuating, they cannot determine the extent to which the change is due to island-specific rather than economy-wide factors. From equation (13), it is clear that an increase in price can be triggered by local factors—that is, by an increase in the local wage—in which case the higher price indicates an increase in the marginal value of the local variety. Nevertheless, the same increase in price also could be driven by aggregate factors, either an increase in the price of the endowment or a decrease in aggregate productivity, that are not related to local conditions. Shoppers' confusion about these sources of price fluctuations means that a price increase driven by a small negative productivity shock is at least partially interpreted by shoppers on each island as a positive local preference shock, thereby potentially triggering an increase in demand for all local final goods. Higher demand for final goods leads to higher demand for the inelastically supplied input good, raising its price, which then feeds back and is reflected again in final good prices. The fact that shoppers extract information from local prices thus amplifies the volatility of the endowment good's price, making shoppers' *equilibrium* inference worse.

The following proposition provides a characterization of equilibrium in terms of the profile of expectations, so that it will be easy to map the outcomes of the inference problem to the equilibria of the economy.

Characterization of the equilibrium. *An equilibrium is characterized by a profile of shoppers' expectation* $\{E[\mu_i|p_i]\}_{i=0}^{1}$ *so that, given (12), in each island* $i \in (0,1)$ *we have*

$$\begin{aligned} p_i &= \gamma\mu_i + (1-\gamma)(q+\zeta), && (14)\\ c_i &= E[\mu_i|p_i] - \gamma\mu_i - (1-\gamma)(q+\zeta), && (15)\\ w_i &= \mu_i && (16)\\ n_i &= E[\mu_i|p_i] - \mu_i, && (17)\\ z_{(i)} &= E[\mu_i|p_i] - q. && (18) \end{aligned}$$

A rational expectations equilibrium is one for which shoppers' expectations, $E[\mu_i|p_i]$*, are rational.*

Proof. Derivations are provided in Appendix A.2. ∎

It is easy to check that, when shoppers have perfect information, price and quantity move in opposite directions.[7] In particular, a positive productivity shock—by our convention, a negative value for ζ—produces a typical-looking supply-driven fluctuation: Total production goes up and the average price level falls.

2.2 Inference with endogenous signals

In this section, we analyze the abstract signal extraction problem created by the information structure microfounded above. We show how to solve the shoppers' inference problem, highlighting the strategic interaction engendered by the endogeneity of the price signal. In particular, we demonstrate that informational feedback can generate amplification of fundamental shocks, which in some cases is strong enough to deliver nontrivial responses to vanishingly small shocks.

Best individual weight function

Given her price signal, p_i, which depends on the aggregate expectation, shopper i must infer μ_i, the marginal utility of her consumption type. The key feature of the resulting signal extraction problem is that the precision of the signal depends on the nature of average actions across the population and, therefore, on the average reaction of shoppers to their own price signals. A rational expectations equilibrium is therefore a situation in which the individual reaction to the signal is consistent with its actual precision, i.e., is an optimal response to the average reaction of others.

We now characterize the equilibria of the economy. Since we assume that all stochastic elements are normal, the optimal forecasting strategy is linear. As a consequence, the individual expectation is linear in p_i and can be written as

$$E[\mu_i|p_i] = a_i\left(\gamma\epsilon_i + (1-\gamma)\left(\int E[\mu_i|p_i]di + \zeta\right)\right), \tag{19}$$

where a_i is the coefficient, determined prior to the realization of shocks, which measures the strength of the reaction of shopper i's beliefs to the signal she will receive. Since the signal is *ex ante* identical for all shoppers, each uses a similar strategy, and we can recover the average

[7] Substitute $E[\mu_i|p_i]$ with μ_i, substitute (14) into (15), and take the integral on both sides to get $c = -p$.

expectation by integrating across the population:

$$\int E[\mu_i|p_i]di = a\,(1-\gamma)\left(\int E[\mu_i|p_i]di + \zeta\right), \tag{20}$$

with $a \equiv \int a_i di$ denoting the average weight applied to the signal. Solving the expression above for the average expectation yields

$$\int E[\mu_i|p_i]di = \frac{a\,(1-\gamma)}{1-a\,(1-\gamma)}\zeta, \tag{21}$$

which is a nonlinear function of the average weight. Importantly, this function features a singularity at the point $1/(1-\gamma)$. When $a < 1/(1-\gamma)$, the average expectation comoves with the productivity shock and the opposite holds when $a > 1/(1-\gamma)$.

The variance of the aggregate expectation—equivalently, of the endowment price—is given by

$$\sigma_q^2(a) = \left(\frac{a\,(1-\gamma)}{1-a\,(1-\gamma)}\right)^2 \sigma^2, \tag{22}$$

where $\sigma_q^2 \equiv var(q)/\sigma_\epsilon^2$ and $\sigma^2 \equiv \sigma_\zeta^2/\sigma_\epsilon^2$ are the variances of the aggregate expectation and the aggregate shock, respectively, once each is normalized by the variance of the idiosyncratic fundamental.

Substituting the average expectation in (21) into the price signal described in equation (13), we get an expression for the local price exclusively in terms of the idiosyncratic and aggregate shocks:

$$p_i = \gamma\epsilon_i + \frac{1-\gamma}{1-a\,(1-\gamma)}\zeta, \tag{23}$$

whose precision with regard to ϵ_i is given by

$$\tau(a) = \left(\frac{\gamma\,(1-a\,(1-\gamma))}{(1-\gamma)\sigma}\right)^2. \tag{24}$$

We are now ready to compute the shopper's optimal inference, taking the average weight of other shoppers as given. We seek an a_i such that $\mathrm{E}[p_i(\epsilon_i - a_i p_i)] = 0$, i.e., the covariance between the signal and forecast error is zero in expectation. This condition implies that information is used optimally. The best individual weight is given by

$$a_i(a) = \frac{1}{\gamma}\left(\frac{\tau(a)}{1+\tau(a)}\right). \tag{25}$$

Given the linear-quadratic environment, we can interpret $a_i(a)$ in a game-theoretic fashion as an individual best reply to the profile of others' actions summarized by a sufficient statistic a. To be precise, every a_i is associated with one and only one contingent shopping strategy that describes the conditional expectation $E[\mu_i|p_i] = a_i p_i$ of shopper i, where p_i identifies a set of states of the world indistinguishable to the shopper i.

Equilibria

Given that agents face an information structure with the same stochastic properties, a rational expectations equilibrium must be symmetric. This last requirement completes our notion of equilibrium, which is formally stated below.

Definition 2. *A noisy rational expectations equilibrium is characterized by a profile of shoppers' expectations* $\{E[\mu_i|p_i]\}_{i=0}^{1}$ *such that* $E[\mu_i|p_i] = \hat{a}p_i$ *with* $a_i(\hat{a}) = \hat{a}$, *for each* $i \in (0,1)$.

Our game-theoretic interpretation of the optimal coefficient makes clear the equivalence between a rational expectations equilibrium and a Nash equilibrium: No one has any individual incentive to deviate when everybody else conforms to the equilibrium prescriptions.

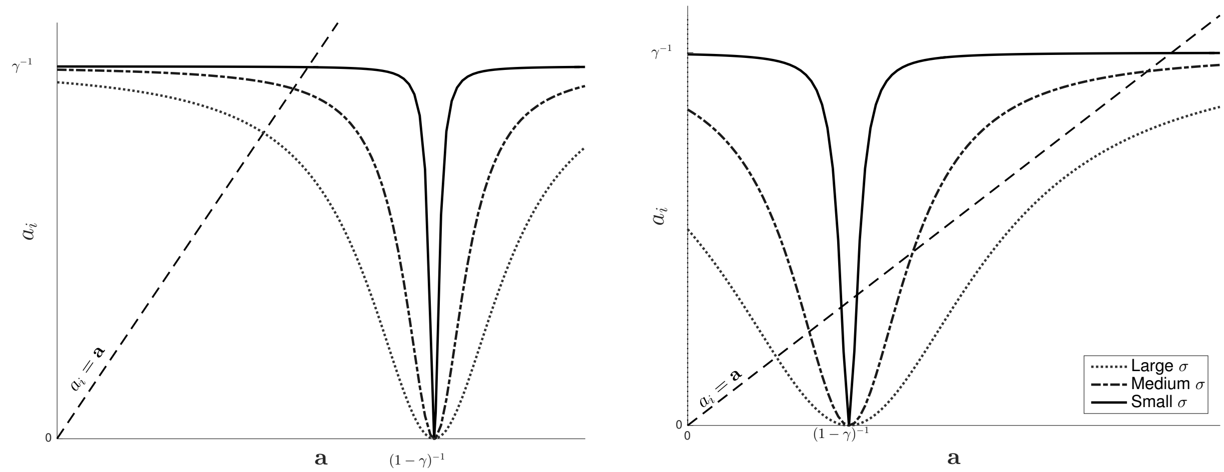
An equilibrium of the model is a fixed point of the individual best-weight mapping given by equation (25). In practice, there are as many equilibria as intersections between $a_i(a)$ and the bisector. The fixed-point relation delivers a cubic equation, which may have one or three real roots. The following proposition characterizes these equilibrium points.

Proposition 1. *For* $\gamma \geq 1/2$, *there always exists a unique REE equilibrium for* $\hat{a} = a_u \in (0, \gamma^{-1})$.

For $\gamma < 1/2$, *there always exists a low REE equilibrium for* $\hat{a} = a_- \in (0, (1-\gamma)^{-1})$. *In addition, there exists a threshold* $\bar{\sigma}^2$ *such that, for any* $\sigma^2 \in (0, \bar{\sigma}^2)$, *a middle and a high REE equilibrium also exist for* $\hat{a} = a_\circ$ *and* $\hat{a} = a_+$, *respectively, both lying in the range* $((1-\gamma)^{-1}, \gamma^{-1})$.

Proof. Given in Appendix A.1. ∎

Proposition 1 states that when the aggregate component receives relatively high weight in the signal, the model may exhibit multiplicity. In particular, there are three equilibria whenever $\gamma < 1/2$ and the variance of the productivity shock is small enough; otherwise, a unique equilibrium exists. While an analytical characterization of these equilibria is possible,



(a) $\gamma = 0.75$ (b) $\gamma = 0.25$

Figure 1: The figure illustrates four properties of $a_i(a)$ for given γ and σ: (i) $a_i(0) > 0$; (ii) $a_i'(a) < 0$ for $a \in (0, (1-\gamma)^{-1})$, and $a_i((1-\gamma)^{-1}) = 0$; (iii) $a_i'(a) > 0$ for for $a \in ((1-\gamma)^{-1}, \gamma^{-1})$ and $\lim_{a\to\infty} = \gamma^{-1}$; (iv) $\partial a_i(a)/\partial\sigma \geq 0$.

the expressions are rather complicated. Nevertheless, the relevant properties can be grasped from the reaction functions plotted in Figure 1 (see figure caption).

The slope of the $a_i(a)$ curve at the intersection with the bisector determines the nature of the strategic incentives underlying each equilibrium. Equilibria a_u and a_- are characterized by *substitutability in information*, as the optimal individual weight is decreasing in the average weight, i.e., $a_i'(\hat{a}) < 0$.[8] In contrast, the equilibria $a_\circ$ and a_+ are characterized by *complementarity in information* since $a_i'(\hat{a}) > 0$. In fact, as soon as $a > (1-\gamma)^{-1}$, the higher the a the higher the precision of the signal, which further pushes up the optimal weight. The emergence of complementarity explains the upward-sloping part of the best-weight function and is key for the existence of multiple equilibria.

While complementarity is essential for generating multiple equilibria, it is neither necessary nor sufficient to imply a strong informational multiplier. To see this, define the multiplier, $\Gamma(\hat{a}) \equiv \sigma_q^2(\hat{a})/\sigma^2$, as the volatility of beliefs relative to the volatility of the shock ζ for some equilibrium point $\hat{a}$. We will say that the economy exhibits *amplifying* informational feedback whenever a fall in the volatility of the exogenous shock leads to an increase in $\Gamma(\hat{a})$, i.e., $\partial\Gamma(\hat{a})/\partial\sigma < 0$, and *dampening* feedback otherwise. The following proposition classifies the equilibria in Proposition 1 according to the type of feedback they generate.

[8] See equation (65) in appendix A.1.

Proposition 2. *The equilibria a_u, a_-, and $a_\circ$ all exhibit amplifying feedback, while the equilibrium a_+ exhibits dampening feedback.*

Proof. Given in Appendix A.1. ∎

The characterization of informational feedbacks as either amplifying or dampening depends on whether the equilibrium value of a gets closer to $(1-\gamma)^{-1}$ as σ shrinks. From Figure 1, it is clear that $a_u, a_\circ$, and a_- feature amplifying feedback, whereas a_+ features dampening feedback. Nevertheless, the feedback effects in $a_\circ$ and a_- are distinct from that in a_u for reasons we discuss in the following section.

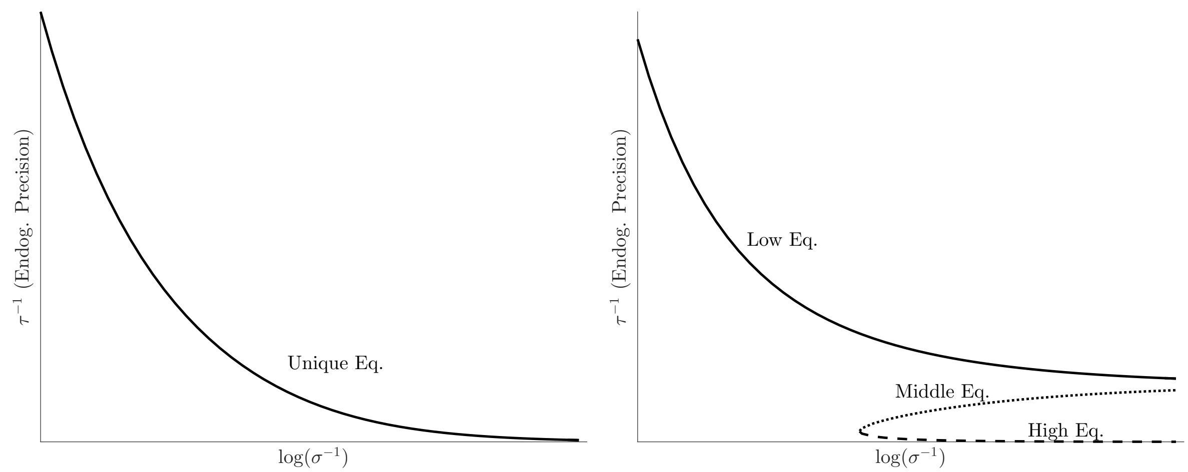
2.3 Low variance in productivity shocks

In this section we analyze the properties of equilibrium when the variance of the productivity shock becomes small. This case is particularly interesting because, as σ^2 decreases, the volatility of the consumption price decreases, leading its informational content about local conditions to rise. Hence, as aggregate volatility shrinks, the informational role of prices dominates their allocative role and agents react to prices more for their informational content than for the costs they impose. This mechanism provides the key insight for understanding the strong amplification we document in this section.

Sentiment equilibria as limit case of strong amplification

Here we show that learning from prices can generate such high amplification of fundamental shocks that the economy can sustain sizable aggregate fluctuations, even in the limit $\sigma^2 \to 0$. We see this as a new characterization of sentiments-driven fluctuations, which have recently received growing attention in the literature. The intuition for this result is captured by Figures 2 and 3, which plot, for each equilibrium, the evolution of signal precision and the variance of the average expectation as a function of the volatility of productivity shocks. As σ shrinks, the unique and the high equilibria, namely a_u and a_+, approach infinite precision and no aggregate volatility. In contrast, the middle and the low equilibria $a_\circ$ and a_- converge to finite precision and sizable aggregate volatility.

The plots numerically demonstrate that, as σ goes to zero, the informational feedbacks in the middle and low equilibria grow at a speed that makes the product of the two achieve



(a) $\gamma = 0.75$ (b) $\gamma = 0.25$

Figure 2: (Inverse) signal precision as a function of exogenous shock volatility.

a finite limit. The following proposition establishes the result formally.

Proposition 3. *In the limit $\sigma^2 \to 0$,*

i. the unique equilibrium (for $\gamma \geq 1/2$) and the high equilibrium (for $\gamma < 1/2$) converge to a point with no aggregate volatility:

$$\lim_{\sigma^2 \to 0} a_{u,+} = \max\left(\frac{1}{\gamma}, \frac{1}{1-\gamma}\right) \qquad \lim_{\sigma^2 \to 0} \sigma_q^2(a_{u,+}) = 0. \tag{26}$$

ii. the low and middle equilibria (for $\gamma < 1/2$) converge to the same point and exhibit non-trivial aggregate volatility:

$$\lim_{\sigma^2 \to 0} a_{-,\circ} = (1-\gamma)^{-1} \qquad \lim_{\sigma^2 \to 0} \sigma_q^2(a_{\circ,-}) = \frac{\gamma(1-2\gamma)}{(1-\gamma)^2}. \tag{27}$$

Proof. Given in Appendix A.1. ∎

In the limit of $\sigma \to 0$, the middle and low equilibria have the same stochastic properties as the sentiment equilibria described by Benhabib et al. (2015). This means that, despite the infinitesimal size of the fundamental shock, its realization is able to coordinate sizable fluctuations in agents' expectations.

The limiting result suggests that a strict dichotomy between fundamental and non-fundamental fluctuations is misleading. Since endogenous signal structures can generate strong multiplier effects on small shocks, they can deliver fluctuations that effectively span

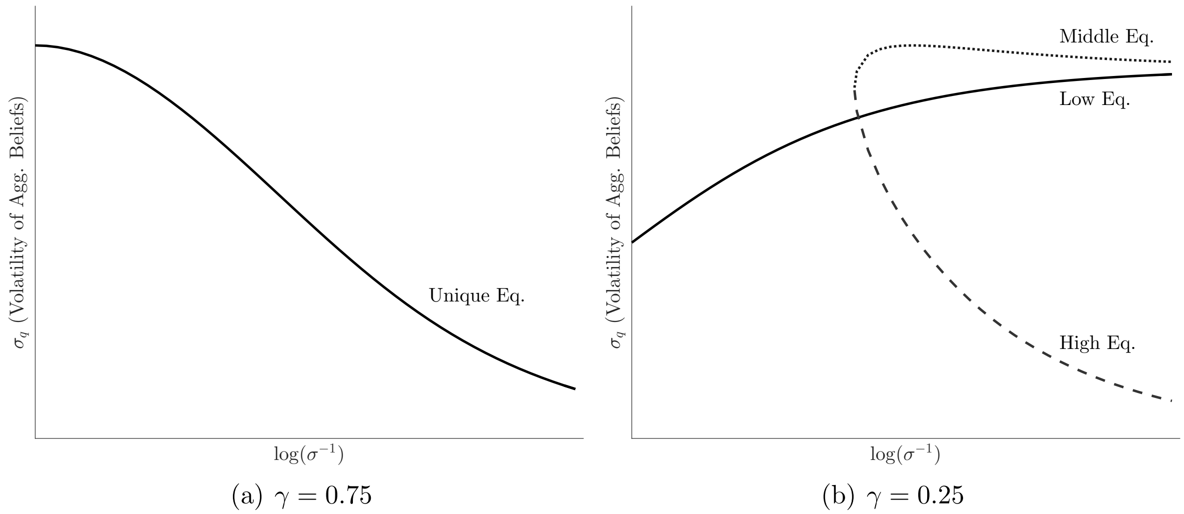


Figure 3: Belief volatility approaching the limit.

a continuum from purely fundamental-driven to purely sentiment-driven. Of course, this possibility does not preclude the existence of fluctuations that originate from truly payoff-irrelevant shocks, but the possibility of fundamental-based sentiments may appeal to those who find such fluctuations implausible.

Moreover, in our economy, agents' coordination on small fundamental shocks as the drivers of beliefs arises endogenously through the competitive price system, rather than being assumed from the outset. The analysis of Benhabib et al. (2015) occurs at the limit point rather than approaching it, so it cannot explain the origins of coordination on a particular sentiment shock.

A special feature of our account of sentiments is that, as the economy approaches the limit, expectations and aggregate outcomes are *perfectly* correlated with fundamentals, although the fluctuations in fundamentals themselves become progressively more difficult for the econometrician to measure. In Section 3, we show how the addition of a noisy public signal to the economy can generate expectations-driven fluctuations that are imperfectly correlated with economic fundamentals.

A final implication of our basic analysis here is that the addition of a small amount of *aggregate* noise in the signal—in this case, captured by the effect of productivity on the price signal—can sustain additional equilibria that do not arise under full information. A previous literature has demonstrated cases in which adding *idiosyncratic* noise to signals can either eliminate (Morris and Shin, 1998) or generate (Gaballo, 2015) additional equilibria. But this is the first time it has been observed, to our knowledge, that adding aggregate noise can cause equilibria to proliferate.

Supply shocks generate demand-driven fluctuations

In this section, we show that limiting sentiment-like equilibria share an important feature with all the equilibria of our economy: When aggregate shocks are not too large, final good prices, total output, the price of the endowment, and total employment positively comove. This happens in *all* equilibria because, as aggregate volatility falls, the informational value of the price signal rises, leading agents' beliefs about their local conditions to respond more strongly to it. Stronger aggregate effects on beliefs eventually lead the informational channel of prices to dominate, so that consumption increases in response to higher prices. Learning from prices thus provides a new mechanism for generating expectations-driven demand shocks in an economy hit only by fundamental shocks to productivity. In this respect, the fluctuations in our economy are similar to those studied by Lorenzoni (2009), although our mechanism is static and does not require the presence of exogenous shocks to information.

The consequences of endogenous information for business cycle comovements are most intuitively seen by analyzing the aggregate demand and aggregate supply schedules in our economy. Given (3), (7), and (17), we can express aggregate demand and supply as

$$AD \quad : \quad c = q - p, \tag{28}$$

$$AS \quad : \quad c = \gamma q - (1-\gamma)\zeta. \tag{29}$$

When the endowment price q has no effect on shoppers' beliefs, this relationship implies a standard downward-sloping aggregate demand relation. However, once we account for the equilibrium feedback of prices into shoppers' inference, the aggregate demand and the aggregate supply relations become

$$AD \quad : \quad p = \frac{1}{a-1}c \tag{30}$$

$$AS \quad : \quad p = \frac{1}{\gamma a}c + \frac{1-\gamma}{\gamma a}\zeta \tag{31}$$

where a denotes the average weight put on the price by shoppers who form their expectations according to $E[\mu_i|p_i] = ap_i$, so that $q = \int E[\mu_i|p_i]di = ap$.

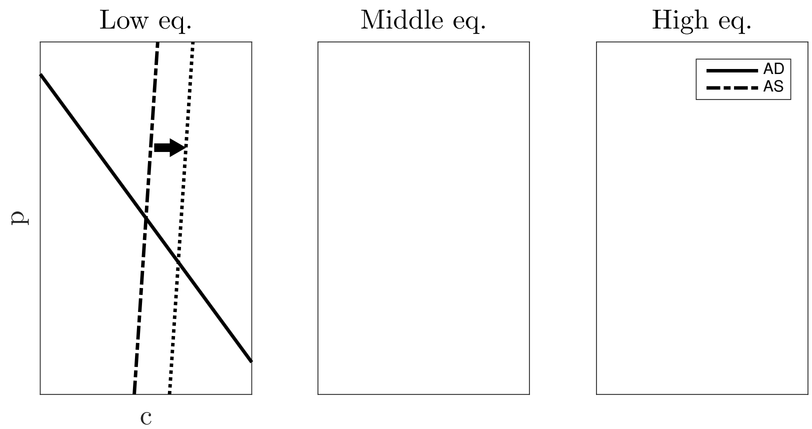
Crucially, the relation in (30) implies that aggregate demand is upward sloping for any a larger than unity. That is, price and quantity will move together, despite the nature of the shock hitting the economy! As the relative variance σ decreases, this will be true *for all equilibria* in the economy. Even the equilibrium a_h, which displays no fluctuations in the limit $\sigma \to 0$, exhibits comovements in prices and quantities away from that limit, *as if* the economy is hit by a common preference shock. In fact, the equilibrium condition $a > 1$ always entails a situation in which the informational content of prices is more important than their allocative effect, that is, movements in expected marginal utility of a good more than compensate for a change in its price. In the model driven by aggregate productivity shocks, the consequences for aggregate demand have immediate implications for the comovement of prices and quantities in the economy.

Proposition 4. *For σ^2 sufficiently small, all equilibria exhibit comovement of aggregate output, employment, the price level, and the price of the endowment.*

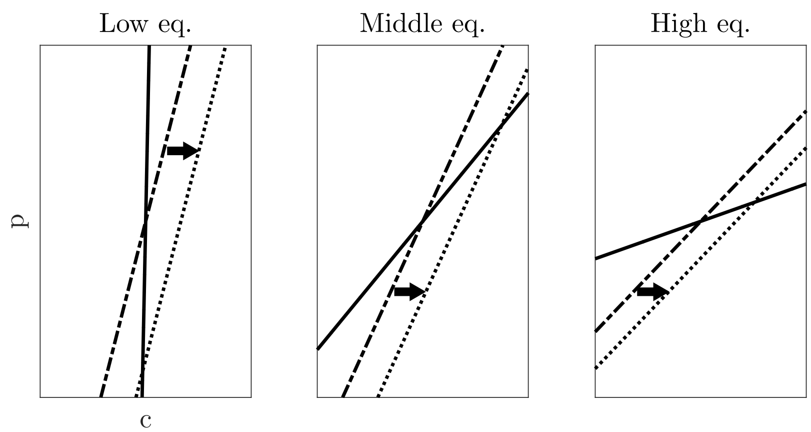
Proof. The results follows from continuity of the best-response function, and the observation that all limit equilibria entail $\hat{a} > 1$. ∎

As the proposition shows, with sufficiently informative prices, the economy will always demonstrate positive comovement between prices and quantities, even though the corresponding full-information equilibrium will exhibit covariance of the opposite sign. The learning-through-prices mechanism can thus easily reverse the typical assumptions made regarding the identification of demand and supply shocks, and it does not require being at the limit or a particular equilibrium selection to do so.

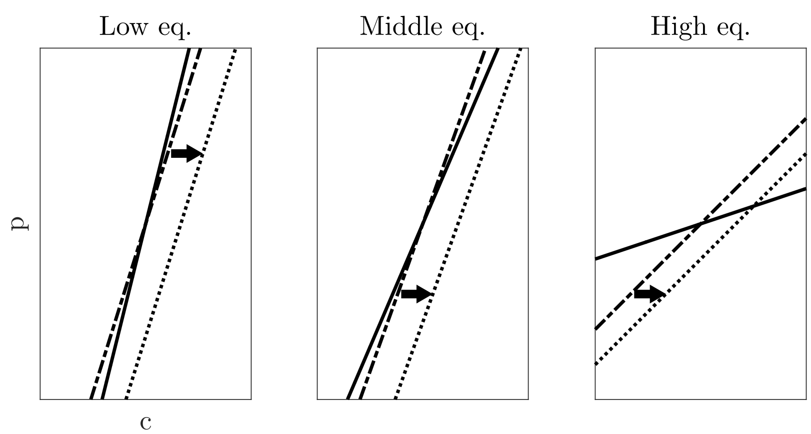
Figure 4 plots aggregate supply and demand relations for different values of the relative volatility σ, for a case in which a multiplicity is possible ($\gamma = 0.25$). As σ shrinks, the slope of aggregate demand in the low equilibrium first switches signs and then, as σ approaches the limit, it becomes nearly parallel with the aggregate supply curve of the economy. In



(a) Large σ



(b) Medium σ



(c) Small σ

Figure 4: Aggregate supply and demand in the microfounded model.

particular, the limit situation corresponds to

$$AD \quad : \quad p = \frac{\gamma}{1-\gamma}c \tag{32}$$

$$AS \quad : \quad p = c + \frac{1-\gamma}{\zeta} \tag{33}$$

when considering a_u and a_+, for which $\lim_{\sigma \to 0} a_i(a_{u,+}) = \gamma^{-1}$, and

$$AD \quad : \quad p = \frac{1-\gamma}{\gamma}c \tag{34}$$

$$AS \quad : \quad p = \frac{1-\gamma}{\gamma}c + \frac{(1-\gamma)^2}{\gamma}\zeta \tag{35}$$

when considering $a_\circ$ and a_- for which $\lim_{\sigma \to 0} a_i(a_{\circ,-}) = (1-\gamma)^{-1}$.

The figure provides an easy intuition for the extremely large informational multiplier implied by our sentiment-like equilibria, as even small shifts in aggregate supply imply large changes in the equilibrium quantity of consumption. Moreover, because all equilibria with sufficiently small σ demonstrate upward-sloping aggregate demand, the same information mechanism that delivers sentiments as a special case more generally offers the potential to be an important amplification mechanism for nontrivial aggregate shocks, which then show up as expectations-driven fluctuations.

3 Noisy Public Information

Given that agents generally observe some indicators of aggregate conditions, a natural question is whether the results above generalize to a situation in which agents observe some signal regarding the fundamental shock. In this section, we therefore consider expanding agents' information to include a second signal of the form

$$g = \zeta + \vartheta,$$

where $\vartheta \sim N(0, \sigma_\vartheta^2)$ is a common aggregate noise term. In this case, agents form expectations with weights on both the price signal and the public signal. In particular, the public signal is useful for refining the information of the price signal, allowing agents to partial out a portion of the aggregate productivity shock, which blurs the inference of shoppers.

Signal extraction with public information

Let us consider the following linear forecasting rule:

$$E[\mu_i|p_i, g] = a_i p_i - b_i a_i (1-\gamma) g, \tag{36}$$

where a_i is the same as before and b_i represents the weight that shopper i puts on the public signal, which we re-scale for convenience by a_i and $(1-\gamma)$. Therefore, we can rewrite shoppers' expectation as

$$E[\mu_i|\tilde{p}_i] = a_i \tilde{p}_i, \tag{37}$$

where

$$\tilde{p}_i \equiv \gamma \mu_i + (1-\gamma)\left(\int E[\mu_i|\tilde{p}_i]di + \zeta - b_i g\right), \tag{38}$$

represents a new signal embodying the information available to the individual shopper.[9] In particular, the highest precision of the new signal $\tilde{p}_i$ is obtained when b_i is set to minimize the variance of the correlated component, which depends on ζ and ϑ, taking the average weight b as fixed. To recover the best weight function $b_i(b)$, we again use the conjectured strategies of other agents to substitute out the average expectation in (38) to arrive at

$$\tilde{p}_i = \gamma \mu_i + (1-\gamma)\left(\frac{a(1-\gamma)}{1-a(1-\gamma)}(\zeta - bg) + \zeta - b_i g\right), \tag{39}$$

where $a = \int a_i di$ and $b = \int b_i di$. Therefore $b_i(b)$ is the value that minimizes the variance of the term multiplied by $(1-\gamma)$ in (39). It is easy to observe that

$$\hat{b} = \frac{\sigma_\vartheta^{-2}}{\sigma_\zeta^{-2} + \sigma_\vartheta^{-2}} \tag{40}$$

minimizes $E[(\zeta - \hat{b}g)^2|g]$ and we can conclude that $b_i(\hat{b}) = \hat{b}$ holds in any equilibrium.

Proposition 5. *Suppose that agents' information consists of $\{p_i, g\}$. Then, the equilibrium of the economy is characterized by $E[\mu_i|p_i, g] = a_i \tilde{p}_i$, where*

$$\tilde{p}_i = \gamma \mu_i + (1-\gamma)(q + \tilde{\zeta}) \tag{41}$$

[9] By the Frisch-Waugh theorem, the projection of μ_i on $\{p_i, g\}$ is equivalent to the projection of μ_i on $\tilde{p}_i \equiv p_i - E[p_i|g]$, where $E[p_i|g] = b_i(1-\gamma)g$.

and $\tilde{\zeta}$ *is given by*

$$\tilde{\zeta} \equiv (1 - \hat{b})\zeta - \hat{b}\vartheta. \tag{42}$$

The equilibrium values $\{a_u, a_-, a_\circ, a_+\}$ *and the conditions for their existence are isomorphic to the ones in the baseline economy once* $\tilde{\sigma}_\zeta^2$ *takes the place of* σ_ζ^2*.*

Proof. Given in appendix A.1. ∎

Equation (42) has several important implications. First, the one- and two-signal models coincide as $\sigma_\vartheta^2 \to 0$, and the aggregate public signal become uninformative. Moreover, to the extent that the aggregate signal is informative, its effect corresponds to a decrease in the volatility of the shock, $\tilde{\zeta}$, thereby pushing the economy toward a situation of multiple equilibria. For the low equilibrium, this implies an *increase* in the variance of the average expectation of shoppers.

This result makes clear that the addition of the aggregate signal does not prevent agents' endogenous price signals from coordinating errors in their inference. To the contrary, whenever the economy without the public signal exhibits reinforcing informational feedback, increasing the precision of the public signal strengthens the feedback generated by agents' endogenous signals. When $\gamma < 1/2$, this pushes the economy closer to the limit point in which there are multiple equilibria, including one that exhibits large aggregate fluctuations driven by beliefs.

Economic implications

While the signal extraction problem of the model with an aggregate signal is isomorphic to the baseline economy, several important differences emerge with respect to the qualitative business cycle implications for consumption and prices. To see these, notice that in equilibrium the average endowment price q is now a function of both the average price in the economy and the public signal,

$$q = a(p - (1 - \gamma)bg).$$

Substituting into the aggregate demand and aggregate supply expressions in equations (28) and (29) yields

$$AD \quad : \quad c = (a-1)p - a(1-\gamma)bg \tag{43}$$

$$AS \quad : \quad c = \gamma\left(ap - a(1-\gamma)bg\right) - (1-\gamma)\zeta. \tag{44}$$

Notice that *both* aggregate demand and aggregate supply are implicitly shifted by the productivity shock ζ and by the noise in the aggregate signal ϑ, via its appearance in the public signal g. Moreover, since the coefficients on these two shocks are different, they will have potentially different implications for aggregate supply and demand.

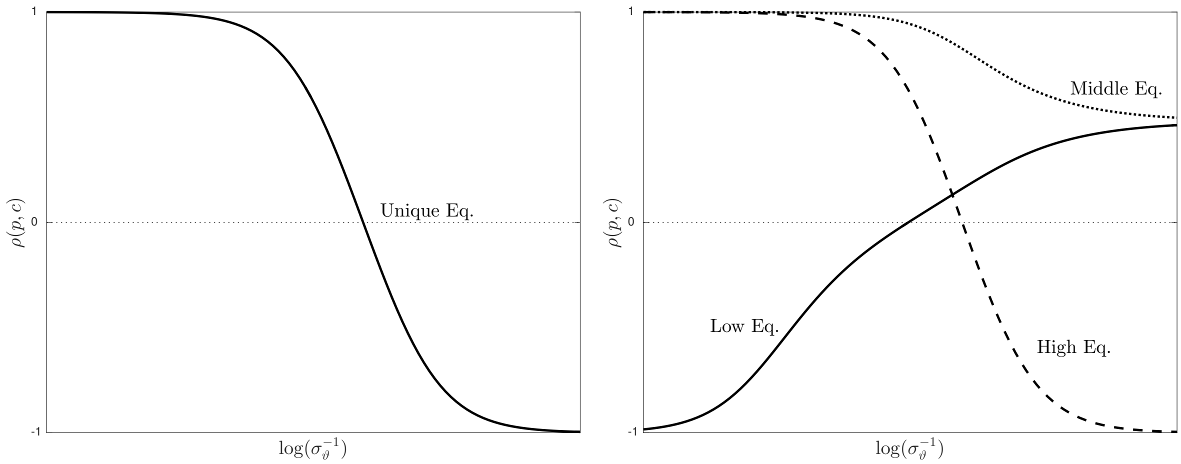
Solving for the full equilibrium of the economy yields the following expressions for price and consumption,

$$p = (1-\gamma)\left(\frac{1-\hat{a}(1-\gamma)\hat{b}}{1-\hat{a}(1-\gamma)}\zeta - \frac{\hat{a}(1-\gamma)\hat{b}}{1-\hat{a}(1-\gamma)}\vartheta\right) \tag{45}$$

$$c = -(1-\gamma)\left(\frac{1-\hat{a}+\hat{a}\gamma\hat{b}}{1-\hat{a}(1-\gamma)}\zeta + \frac{\hat{a}\gamma\hat{b}}{1-\hat{a}(1-\gamma)}\vartheta\right). \tag{46}$$

Comparing (45) and (46), it is clear that the coefficients on the fundamental ζ will take opposite signs whenever $\hat{b}$ is sufficiently close to one. That is, productivity shocks can induce negative comovements when the public signal is sufficiently precise. This result contrasts with the case in the previous section without a public signal, when the same productivity shock induced perfect positive comovement. On the other hand, it is immediate to see that noise shocks in the public signal will *always* induce positive comovement in the economy, regardless of the precision of the information. Unconditional comovement in the economy will therefore depend on the contribution of each shock to overall correlations and, in general, will fall somewhere strictly between zero and one.

In Figure 5, we document how the correlation between the average consumption price and total output may be affected by the variance of the noise in the public signal, with the case of $\gamma = 0.75$ in the left panel and the case of $\gamma = 0.25$ in the right panel. In particular, we fix a value of σ for which multiple equilibria exist when $\gamma = 0.25$. When the noise in the signal is large, the price-quantity correlation approaches either 1 or -1. Perfect negative



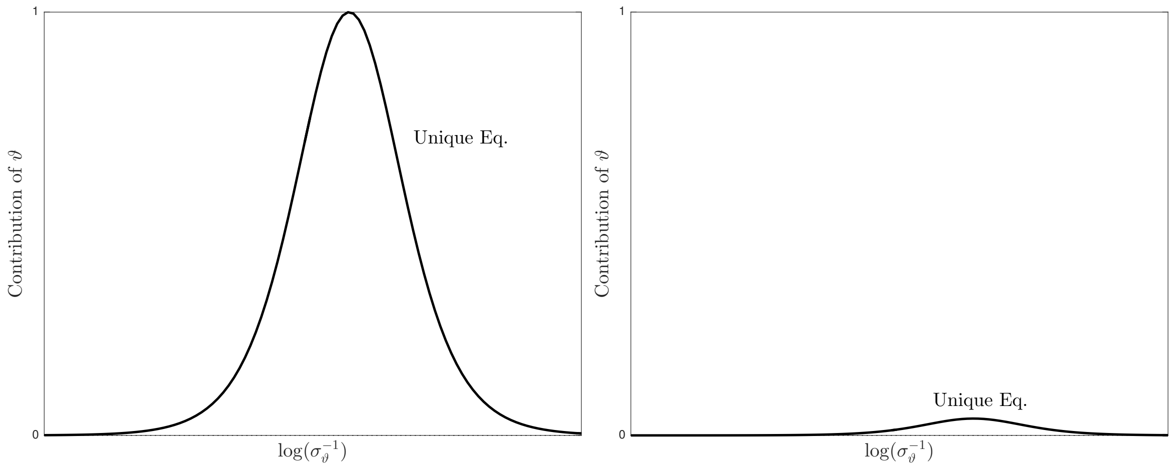
(a) $\gamma = 0.75$ (b) $\gamma = 0.25$

Figure 5: Correlation between average price and quantity as a function of the variance of noise in the public signal.

correlation arises only for the low equilibrium when σ is sufficiently large; for smaller values of σ, all equilibria exhibit positive price-quantity correlation following the analysis of the previous section.

The different equilibria respond in distinct ways as the variance of the noise in the public signal shrinks. In the unique and high equilibrium, supply-driven fluctuations crowd-out demand-driven ones, leading to growing negative correlations. This occurs because the high and unique equilibrium converge to points with no fluctuations in expectations, leaving behind only the supply-side effects of productivity shocks. In contrast, in the low equilibrium, higher precision of the public signal *crowds-in* expectations-driven demand shocks and drives price-quantity correlation up. The low equilibrium converges to a limit in which expectations are maximally volatile, but price-quantity correlation remains less than unity due to the presence of still-sizable productivity shocks. Finally, the middle equilibrium achieves the same correlation value as the low equilibrium since both equilibria converge to the same limit outcome.

This analysis show that the addition of a public signal—and therefore of an alternative source of aggregate noise—is one possible approach to breaking the perfect correlation, either positive or negative, between p and c that is implied by the baseline model. Instead, as the aggregate signal becomes sufficiently precise, the economy behaves as if it is hit by



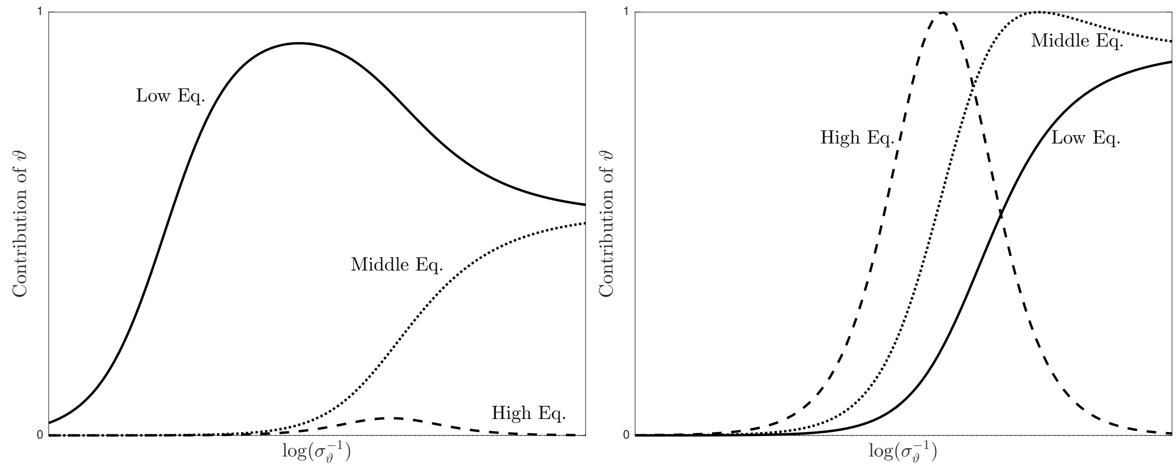
(a) Consumption (b) Price level

Figure 6: Variance decomposition when $\gamma = 0.75$: aggregate noise vs. fundamentals.

both supply- and demand-side disturbances, leading to imperfect but potentially positive correlation between price and quantity.

Figures 6 and 7 provide a variance decomposition for consumption and the price level, for the same cases illustrated above. Indeed, increasing the precision of the public signal generally increases the role of common noise in the low and middle equilibria, although some non-monotonicity does arise. On the other hand, the figures for the high and the unique equilibria are striking. It turns out that almost all of the variance in these equilibria is driven by productivity shocks when the variance of aggregate noise is either sufficiently small or sufficiently high. Nevertheless, in the unique equilibrium case, there exists an intermediate value of σ_ϑ for which consumption is driven entirely by common noise and prices nearly entirely by productivity shocks. A similar result holds for the high equilibrium when $\gamma = 0.25$ illustrated in panel (b) of Figure 7. Here, there exists an intermediate value of σ_ϑ for which consumption is driven entirely by productivity shocks, and prices almost entirely by common noise.

In particular, looking at (45) we can easily see that prices do not depend on fundamental shocks when $a = ((1-\gamma)b)^{-1}$, where recall that $b < 1$. Such a value can only be achieved by $a_\circ$ and a_+ for a single (different) value of σ, provided $b > \gamma/(1-\gamma)$, which ensures that $(1-\gamma)b \in ((1-\gamma)^{-1}, \gamma^{-1})$. On the other hand, consumption does not depend on fundamental shocks when $a = 1/(1-\gamma b)$. Such a value can only be achieved by a_u for a single value of σ,



(a) Consumption (b) Price level

Figure 7: Variance decomposition when $\gamma = 0.25$: aggregate noise vs. fundamentals.

provided $b > (1-\gamma)/\gamma$, which can only be for $\gamma > 1/2$ given that $b < 1$.

Summing up, the introduction of public information opens the way for aggregate beliefs to become disconnected from aggregate productivity shocks and allows for the simultaneous existence of "supply" and "demand" fluctuations in the economy, despite the presence of only a single aggregate fundamental shock. In the limit of arbitrarily small noise in the public signal, the model delivers fluctuations in aggregate beliefs that—although coordinated by noise in the public signal—could not be so-attributed by the econometrician who observes only imperceptibly small eventual revisions to the public data. Finally, we have shown that, even far from any limit, equilibria exist in which fluctuations in prices and quantities can be explained by different exogenous processes.

4 Extensions and Discussion

This section presents several extensions to the basic setup, showing that the insights of the main mechanism are robust to various modeling details. In the first, we show that convexity in the disutility of labor (i) expands the range of γ for which strong informational multipliers and equilibrium multiplicity may arise, and (ii) induces wages to comove positively along with prices and quantities for sufficiently small values of σ. We then show that allowing for correlation in the good-specific taste shocks μ_i does not materially affect the conclusions of the baseline model with productivity shocks. Finally, we allow for the disaggregation of goods at the island level to demonstrate that the existence of upward-sloping aggregate demand in our model does not require the existence of Giffen-type goods at the micro level.

4.1 Convexity in labor disutility

Here, we show how our framework easily extends to the case of convex disutility in labor. Let the household utility function be

$$\int e^{\mu_i} \left(\log C_i - \phi N_i^{1+\alpha} \right) di \tag{47}$$

where $\alpha > 0$ denotes the inverse of the Frisch elasticity of labor supply. In this case the local wage will not be a direct measure of the island-specific preference shock, but rather will depend on this shock and the aggregate quantity of labor supplied to the market. In Appendix A.2 we report the detailed derivation. Below we describe how this change affects the characterization of the equilibrium.

Characterization of the equilibrium, extended case. *An equilibrium is characterized by a profile of shoppers' expectation $\{E[\mu_i|p_i]\}_{i=0}^{1}$ so that, given (12), in each island $i \in (0,1)$ we have*

$$\begin{aligned}
p_i &= \frac{\gamma}{1+\alpha}\mu_i + \frac{\alpha\gamma}{1+\alpha}E[\mu_i|p_i] + (1-\gamma)(q+\zeta), && (48)\\
c_i &= \frac{1+\alpha(1-\gamma)}{1+\alpha}E[\mu_i|p_i] - \frac{\gamma}{1+\alpha}\mu_i - (1-\gamma)(q+\zeta), && (49)\\
w_i &= \mu_i + \alpha n_i && (50)\\
n_i &= \frac{1}{1+\alpha}E[\mu_i|p_i] - \mu_i, && (51)\\
z_{(i)} &= E[\mu_i|p_i] - q. && (52)
\end{aligned}$$

A rational expectations equilibrium is one for which shoppers' expectations, $E[\mu_i|p_i]$, are rational.

Proof. Derivations are provided in Appendix A.2. ∎

In this extension, the local price is affected by the individual expectation of the representative local shopper, as the equilibrium quantities of labor depend on shoppers' demand. One can easily show that our analysis of the baseline economy also applies also in this case, once the price signal is conveniently transformed. To arrive at a signal structure that is isomorphic to the baseline economy, subtract the individual expectation from (48) and rescale to obtain

$$\hat{p}_i = \frac{1+\alpha}{1+\alpha(1-\gamma)}\left(p_i - \frac{\alpha\gamma}{1+\alpha}E[\mu_i|p_i]\right) = \hat{\gamma}\mu_i + (1-\hat{\gamma})(q+\zeta),$$

where $\hat{\gamma} = \gamma/(1+\alpha(1-\gamma))$. The analysis of section 2.2 follows after substituting the original price signal p_i with the equivalent one $\hat{p}_i$.

The extension delivers two important additional insights. First, multiple equilibria exist whenever $\hat{\gamma} < 1/2$ which could well obtain even with $\gamma > 1/2$ for a sufficiently high α. This is desirable since typical estimate of the labor share pin $\gamma > 1/2$, and might have otherwise precluded the strongest informational multipliers from appearing in a realistic calibration of the model. Second, it follows from equation (50) that demand-driven fluctuations now also feature positive comovement of wages with the average consumption price, the price of the endowment, total output, and total employment. The economy thus generates a robust and realistic pattern of comovement across many variables.

4.2 Correlation in island-specific shocks

We now consider a version of the model in which preference shocks are correlated—that is $\mu_i = \mu + \epsilon_i$ where $\mu \sim N\left(0, \sigma_\mu^2\right)$—and there are no productivity shocks. Notice that previously, productivity shocks acted as noise in the signal, since shoppers were only interested in the forecast of μ_i. Now, the aggregate term μ represents a common objective in the signal extraction problem of shoppers.

Following the derivation of (13), the price signal is expressed as

$$p_i = \gamma(\mu+\epsilon_i) + (1-\gamma)\int E[\mu+\epsilon_i|p_i]di, \tag{53}$$

which no longer embeds a productivity shock. Nonetheless, correlated fundamentals generate confusion between the idiosyncratic and common components of the signal. As before, the individual expectation of a shopper of type i is formed according to the linear rule $E[\mu + \epsilon_i|p_i] = a_i p_i$. Hence, the signal embeds the average expectation, which again causes the precision of the signal to depend on the average weight a. Following the analysis of the earlier section, the realization of the price signal can be rewritten as

$$p_i = \gamma \epsilon_i + \frac{\gamma}{1 - a(1-\gamma)} \mu, \tag{54}$$

where a represents the average weight placed on the signal by other shoppers. The variance of the average expectation is given by

$$\sigma_q^2(a) = \left(\frac{\gamma a}{1 - a(1-\gamma)} \right)^2 \sigma^2, \tag{55}$$

which is slightly different from (22). The shopper's best response function is now given by

$$a_i(a) = \frac{1}{\gamma} \left(\frac{(1 - a(1-\gamma))^2 + (1 - a(1-\gamma))\, \sigma^2}{(1 - a(1-\gamma))^2 + \sigma^2} \right). \tag{56}$$

While the best-response function in equation (56) is slightly different than that of equation (25) for the case with productivity shocks, we can prove that the characterization of the limit equilibria is identical.

Proposition 6. *In the limit $\sigma_\mu^2 \to 0$, the equilibria of the economy converge to the same points as the baseline economy:*

$$\lim_{\sigma_\mu^2 \to 0} a_e^\mu = \lim_{\sigma^2 \to 0} a_e \qquad \lim_{\sigma_\mu^2 \to 0} \sigma^2(a_e^\mu) = \lim_{\sigma^2 \to 0} \sigma^2(a_e) \qquad \textit{for } e \in \{u, -, \circ, +\} \tag{57}$$

Proof. Given in Appendix A.1. ∎

More generally, it is possible to show that propositions 1 through 3 follow identically, and their proofs proceed in parallel with only the obvious algebraic substitutions.

4.3 A theory of Giffen goods?

One possible objection to the realism of our mechanism is the implication that the consumption of island-specific good C_i is rising in its price, i.e., that local consumption goods appear to be Giffen goods. Such behavior at the good level is not an essential aspect of our story. The most natural way to avoid this complication is to presume that, *within* islands, quantity-choosing firms produce a continuum of goods indexed by (i, j), which are then aggregated at the island-level good by a standard Dixit-Stiglitz aggregator, $C_i = \left(\int C_{i,j}^{1-\frac{1}{\theta}} \right)^{\frac{1}{1-\frac{1}{\theta}}} dj$ with $\theta > 1$.

Suppose now that each (i, j) producer is hit with an idiosyncratic, mean-zero productivity shock, $v_{i,j}$. In this case, the price of good $c_{i,j}$ in logs turns out to be

$$p_{i,j} = v_{i,j} + \gamma\mu_i + (1-\gamma)(q+\zeta).$$

Demand for good $c_{i,j}$ is governed by the standard formula

$$c_{i,j} = -\theta(p_{i,j} - p_i) + c_i,$$

which reflects a substitution effect governed by the standard elasticity parameter *at the good level:* An econometrician studying good-level prices would find no evidence that the typical good is Giffen. Nevertheless, the total price level on island i,

$$p_i = \int p_{i,j} dj = \gamma\mu_i + (1-\gamma)(q+\zeta),$$

is both (i) identical to its value in the baseline economy, and (ii) reflects the optimal (even) weighting of the signals $p_{i,j}$ that shoppers use in equilibrium to infer their local demand shock: subsequent analysis of the island-level and aggregate economy is not affected.

5 Equilibria Selection

This section exploits the characterization of the individual best-response function in the game implied by dispersed information to examine the stability of equilibria under two popular out-of-equilibrium beliefs dynamics: rationalizability and adaptive learning. We show that equilibria with reinforcing feedback may be stable, although sentiment-like equilibria are generally excluded by these tests.

5.1 Higher-order belief dynamics

In our economy, a "rational" expectation is characterized by a mapping $a_i(a) : \Re \to \Re$ which associates an individual best weight $a_i(a)$ with any value of the average weight a. A rational expectations equilibrium is an equilibrium weight $\hat{a}$ such that $a_i(\hat{a}) = \hat{a}$ for each $i \in (0, 1)$, and $\hat{a}$ reflects the precision of the endogenous signal at equilibrium. But how can people gain common knowledge that others will conform to the equilibrium prescription?

This is an old question on the epistemic foundations of Nash equilibrium with an important tradition in decision theory. A widely accepted concept is that of the rationalizable set (Bernheim, 1984; Pearce, 1984), defined as the strategy profile set that survives the iterated deletion of never-best replies. This criterion exploits implications from common knowledge of rationality in the model.

Guesnerie (1992) introduces the rationalizability argument to macroeconomics in the context of complete-information competitive economies. Here we adapt Guesnerie's original setup in a dispersed information model and focus on the best-expectation coordination game entailed by the maps $\{a_i(a)\}_{i\in(0,1)}$. In contrast to the original Guesnerie setting, here agents agree on the unconditional expectation of their idiosyncratic fundamental, which is exogenous to the average behavior, but are uncertain about the precision of the information they are looking at. Nothing in the model guarantees that agents will use the same conditional distribution to forecast their idiosyncratic fundamental. Below, we check whether the assumption of common knowledge of rationality is sufficient to restrict the agents' strategic space to the rational expectations equilibrium prescriptions.

Initially, we take a local point of view. Suppose it is common knowledge that the individual weights on the signal lie in a neighborhood $\mathcal{F}\left(\hat{a}\right)$ of $\hat{a}$. Is this a sufficient condition for convergence in higher-order beliefs to $\hat{a}$? The process of iterated deletion of never-best replies works as follows. Let τ index the iterative round of deletion. If $a_{i,0} \in \mathcal{F}\left(\hat{a}\right)$ for each i, then $a_0 \in \mathcal{F}\left(\hat{a}\right)$. Nevertheless, the latter implies that second-order beliefs are justified in which $a_{i,1} = a_i\left(a_0\right)$ for each i, so that $a_{i,1} \in a_i\left(\mathcal{F}\left(\hat{a}\right)\right)$. As a consequence, $a_1 \in a_i\left(\mathcal{F}\left(\hat{a}\right)\right)$. One can iterate the argument showing that $a_{i,\tau} \in a_i^\tau\left(\mathcal{F}\left(\hat{a}\right)\right)$. Hence, we have the following.

Definition 3. *A rational expectations equilibrium $\hat{a}$ is a locally unique rationalizable outcome if and only if there exists a neighborhood $\mathcal{F}\left(\hat{a}\right)$ of $\hat{a}$ such that $\lim_{\tau\to\infty} a_i^\tau\left(\mathcal{F}\left(\hat{a}\right)\right) = \hat{a}$.*

When a rational expectations equilibrium is a locally unique rationalizable outcome, we can conclude that the equilibrium is stable to a sufficiently small higher-order-beliefs perturbation (or is eductively stable, in Guesnerie's language). In other words, the equilibrium is robust to beliefs that others could locally deviate from it, as agents conclude that no rational conjecture can sustain such a deviation.

Higher-order beliefs may also be globally stable if the best response function entails a contraction for each point of the domain of a, that is, when $\lim_{\tau\to\infty} a_i^{\tau}(\Re) = \hat{a}$. When an equilibrium is the globally unique rationalizable outcome, then it represents the only profile of strategies that rational agents will play. In this sense, the theory provides a complete account of how out-of-equilibrium beliefs converge to the unique equilibrium.

Belief convergence requires that $a_i(a)$ is a contraction mapping. For a locally rationalizable rational expectations equilibrium, a necessary and sufficient condition is $|a_i'(\hat{a})| < 1$. Proposition 7 states the result.

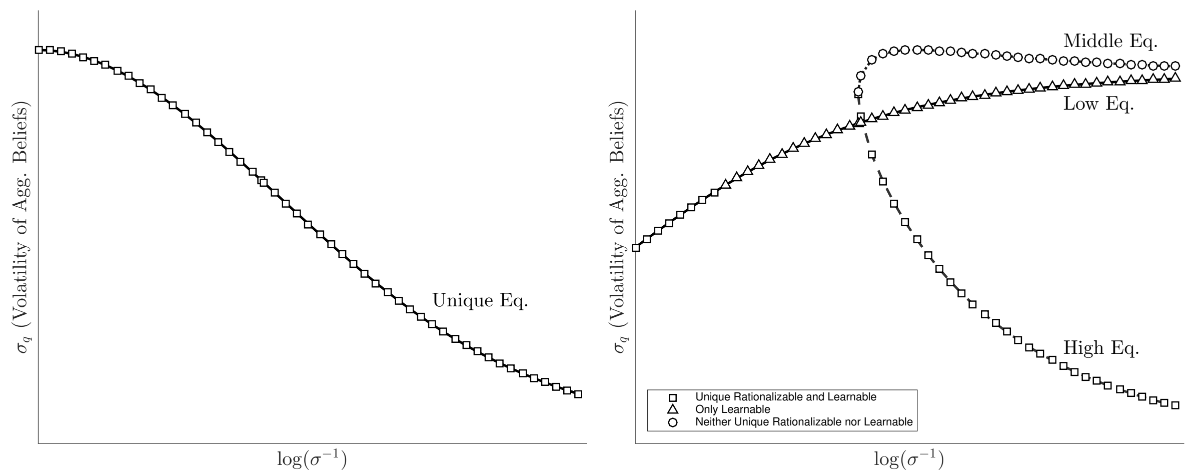
Proposition 7. *The low and unique equilibrium are locally unique rationalizable equilibrium provided σ is large enough. Whenever the middle and the high equilibria exist, the latter is always a locally unique rationalizable equilibrium, whereas the former is never. In the limit of $\sigma \to 0$, the middle and the low equilibria are never stable under higher-order beliefs dynamics, whereas the unique equilibrium is.*

Proof. Given in Appendix A.1 ∎

One can easily show that $a_\circ$ is never a locally unique rationalizable outcome from the qualitative properties associated with the equilibria. First, $a_i(1-\gamma)^{-1} = 0$ lies below the 45-degree line. Second, for $a > (1-\gamma)^{-1}$, the best-weight function is always monotonically increasing. These two observations taken together require that $a_i'(a_\circ) > 1$, thus proving that whenever the middle equilibrium $a_\circ$ exists, it is not locally unique rationalizable.

A second result is that whenever a_+ exists and is distinct from $a_\circ$, it is always a locally unique rationalizable outcome, since the first derivative at this equilibrium has to be bounded below one to meet the 45-degree line. In the knife-edge case in which $a_\circ = a_+$, the fixed-point mapping is tangent to the bisector, meaning that $a_i'(a_+) = 1$, which does not satisfy the condition for rationalizability.

To establish the convergence properties of a_-, one needs to check that there is a threshold $\underline{\sigma}$ such that for any $\sigma \in (0, \underline{\sigma})$, this equilibrium is not locally rationalizable; otherwise it is. To give an intuition, notice that in the case of the two limit equilibrium outcomes we have $\lim_{\sigma\to 0} a_- = \lim_{\sigma\to 0} a_\circ = (1-\gamma)^{-1}$, for which the derivative of the best-response function is $\lim_{\sigma\to 0} a_i' = \pm\infty$. On the other hand, $a_i'(a_-)$ increases in σ with 0 as an upper bound so that there exists a $\underline{\sigma}$ such that for any $\sigma > \underline{\sigma}$, the low a_- is always rationalizable. Therefore there could be a multiplicity (two) of rationalizable rational expectations equilibrium for



(a) $\gamma = 0.75$ (b) $\gamma = 0.25$

Figure 8: Volatilities and equilibrium stability approaching the limit.

intermediate values of $\sigma \in (\underline{\sigma}, \bar{\sigma})$. To examine when this is the case, we perform the following numerical analysis.

The first panel of Figure 8 illustrates the size of expectations volatility generated by the unique-equilibrium economy as a function of the inverse of σ, and shows that equilibrium is always rationalizable. The second panel plots beliefs volatility and stability properties for the three equilibria (whenever they exist) of the model with $\gamma = 0.25$. For high enough σ, only the low equilibrium exists. The volatility generated at that equilibrium is monotonically decreasing in σ. The low equilibrium is a locally unique rationalizable outcome, provided σ is sufficiently large. With sufficiently low σ, the middle and high equilibria exist as well. The latter is always a locally unique rationalizable outcome, whereas the former never is. The same picture shows that the unique equilibrium is always a unique locally rationalizable outcome.

Notice that in the example illustrated in Figure 8, there is no region in which multiple locally unique rationalizable outcomes exist. Moreover, there exists a region in which the low equilibrium is the only equilibrium, but it is not a locally unique rationalizable outcome. Finally, only for sufficiently small σ can a globally unique rationalizable outcome arise, originating in the low equilibrium.

In Figure 9, we show through numerical investigation that for sufficiently low values of γ, there is a region in which two equilibria, the high and the low, emerge as locally unique

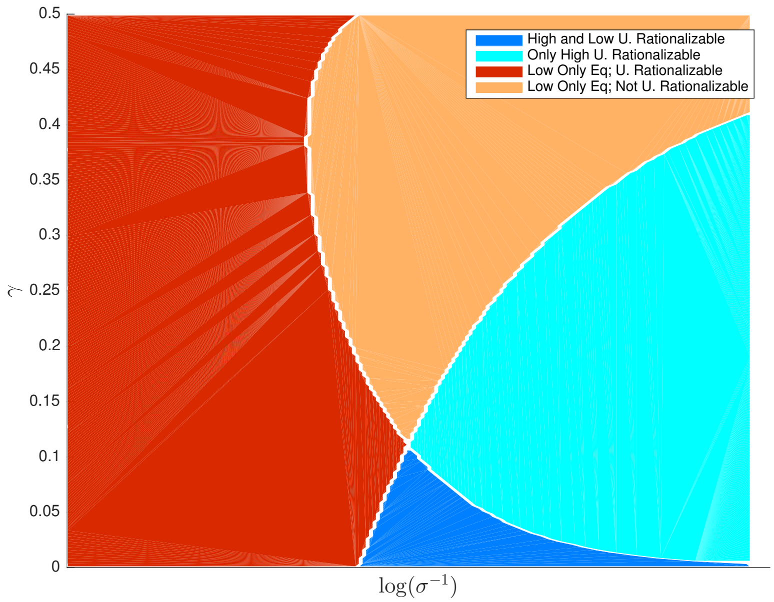


Figure 9: Stability properties for different signal weights.

rationalizable outcomes. Nevertheless, this would not arise in the limit of infinite precision, in which case only the high equilibrium remains a locally unique rationalizable equilibrium.

5.2 Adaptive learning

We now suppose that agents behave like econometricians rather than game theorists. That is, agents individually set their weights to be consistent with data generated by possibly out-of-equilibrium replications of the signal extraction problem, without internalizing the effect of the ongoing process of learning in the economy. At time t they set a weight $a_{i,t}$, which is estimated from the sample distribution of signals collected from past repetition of the signal extraction problem.

The asymptotic behavior of statistical learning algorithms can be analyzed by stochastic approximation techniques (for details, refer to Marcet and Sargent, 1989a,b and Evans and Honkapohja, 2001). To see how this works in our context, consider the case in which agents learn about the optimal weight according to an optimal adaptive learning scheme:

$$a_{i,t} = a_{i,t-1} + \gamma_t \, S_{i,t-1}^{-1} \, p_{i,t} \left(\mu_{i,t} - a_{i,t-1} p_{i,t} \right) \tag{58}$$

$$S_{i,t} = S_{i,t-1} + \gamma_{t+1} \left(p_{i,t}^2 - S_{i,t-1} \right), \tag{59}$$

where γ_t is a decreasing gain with $\sum \gamma_t = \infty$ and $\sum \gamma_t^2 = 0$, and matrix $S_{i,t}$ is the estimated variance of the signal rewritten with a convenient time index. The following formally defines adaptive stability.

Definition 4. *A rational expectations equilibrium $\hat{a}$ is a locally learnable equilibrium if and only if there exists a neighborhood $F(\hat{a})$ of $\hat{a}$ such that, given an initial estimate $a_{i,0} \in F(\hat{a})$, it is $\lim_{t\to\infty} a_{i,t} \overset{a.s}{=} \hat{a}$.*

Adaptive learning therefore represents an alternative description of out-of-equilibrium dynamics, which can explain how agents can converge (or fail to converge) to a rational expectations equilibrium.

An equilibrium is globally learnable whenever almost-sure convergence obtains irrespective of the initial condition—that is, when $\lim_{t\to\infty} a_{i,t} \overset{a.s}{=} \hat{a}$ for any $a_{i,0} \in \Re$. Notice that, in contrast to the rationalizability criterion, there could exist a unique globally learnable equilibrium despite the existence of multiple rational expectations equilibria. This is because the stochasticity of the learning process will always displace estimates temporarily away from equilibrium values. Nevertheless, if there exists only one rational expectations equilibrium, then if it is learnable it must be globally learnable.

To check local learnability of the rational expectations equilibrium, suppose we are already close to the resting point of the system. That is, consider the case $\int \lim_{t\to\infty} a_{i,t}\, di = \hat{a}$, where $\hat{a}$ is one of the equilibrium points $\{a_-, a_\circ, a_+\}$, and so

$$\lim_{t\to\infty} S_{i,t} = \sigma_s^2(\hat{a}) = \gamma^2\sigma_\mu^2 + \frac{(1-\gamma)^2}{(1-\hat{a}(1-\gamma))^2}\sigma_\zeta^2. \tag{60}$$

According to stochastic approximation theory, we can write the associated ODE governing the stability around the equilibria as

$$\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}t} &= \int \lim_{t\to\infty} \mathrm{E}\left[S_{i,t-1}^{-1} p_{i,t}\left(\mu_{i,t} - a_{i,t-1}p_{i,t}\right)\right] di \\
&= \sigma_s^2(\hat{a})^{-1} \int \mathrm{E}\left[p_{i,t}\left(\mu_{i,t} - a_{i,t-1}p_{i,t}\right)\right] di \\
&= \sigma_s^2(\hat{a})^{-1}\left(\gamma\sigma_\mu^2 - a_{i,t-1}\left(\gamma^2\sigma_\mu^2 + \frac{(1-\gamma)^2}{(1-a_{t-1}(1-\gamma))^2}\sigma_\zeta^2\right)\right) \\
&= a_i(a) - a.
\end{aligned} \tag{61}$$

For asymptotic local stability to hold, the Jacobian of the differential equation in (61) must be less than zero at the conjectured equilibrium. The relevant condition for stability is therefore $a_i'(a) < 1$. The result is stated by the following proposition.

Proposition 8. *The unique equilibrium is always learnable. Whenever they exist, the high equilibrium is locally learnable, whereas the middle equilibrium is not. The low equilibrium is always locally learnable, except at the limit* $\sigma \to 0$, *and it is globally learnable provided* σ *is large enough.*

Proof. Given in Appendix A.1 ∎

Referring to Figure 2, the slopes of the curves at the intersection of the bisector reflect the stable or unstable nature of the equilibrium. In particular, notice that the middle equilibrium defines two distinct basins of attraction for the learnable equilibria. As σ decreases, the basin of attraction of the high equilibrium grows from below. This means that estimates are more and more likely to converge to the high equilibrium—the equilibrium without sentiments—as σ gets smaller. At the limit $\sigma \to 0$, the low equilibrium is no longer learnable from above, meaning that any initial estimate a larger than a_-, no matter how close it is to a_-, is fated to trigger convergence to the high equilibrium. This would suggest that although sufficiently negative shocks to the estimates can lead to a persistent deviation in the lower basin of attraction of the low sentiment equilibrium, long run-convergence can only obtain at the high sentiment-free equilibrium.

Our learnability results contrast with the original stability analysis of Benhabib et al. (2015). In their approach, agents treat the signal as exogenous, conjecturing a common precision and then updating dynamically. In our characterization, however, agents' learning incorporates the endogenous relationship between signal precision and the average action, and this endogeneity generates a coordination issue not contemplated by Benhabib et al. (2015). The contrast between our results suggests that small differences in the microfoundations underpinning sentiment fluctuations can lead to opposing conclusions about their stability, and thus deserve additional attention in this literature.

6 Conclusion

Endogenous structures of asymmetric information can deliver strong multipliers on common disturbances, and thus offer a potential foundation for expectations-driven economic fluctuations. Here we have demonstrated that a single analysis can address such fluctuations, whether they originate in common fundamentals or a mixture of fundamentals and common noise. Because of the amplification power of this mechanism, sentiment equilibria may, paradoxically, originate from economic fundamentals themselves and need not originate with shocks disconnected from the physical environment. Instead, expectations-driven fluctuations can be initiated by small changes in fundamentals that, under full information, would trigger far smaller reactions.

The mechanism behind this result is a strong feedback loop that arises when agents observe, and draw inference from, endogenous variables. We microfounded such endogenous signals as competitive prices. The essential features for our mechanism are (i) shocks to local demand conditions and (ii) agents that learn from prices that reflect a combination of aggregate and local conditions. Our approach provides foundations for the correlated signal structures that are essential for sentiment-driven fluctuations to arise.

In this economy, shoppers observing higher prices partially attribute those prices to favorable local-demand conditions. When the effect of prices on inference is strong enough, observation of a higher price leads shoppers to demand more, rather than fewer, inputs. Through this process, the informational effects of prices can lead to upward-sloping aggregate demand, reversing the typical comovements associated with supply shocks.

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A Appendix

A.1 Proofs of Propositions

Proof of Proposition 1. To prove uniqueness for $\gamma \geq 1/2$, observe that the function $a_i(a)$ is continuous, bounded above by γ^{-1}, and monotonically decreasing in the range $(-\infty, (1-\gamma)^{-1})$. From $\gamma \geq 1/2$, we have $(1-\gamma)^{-1} > \gamma^{-1}$. Thus $a_i(a)$ intersects the 45-degree line a single time.

To prove the existence of a_-, notice that $\lim_{a \to -\infty} a_i = \gamma^{-1}$ and $a_i((1-\gamma)^{-1}) = 0$. By continuity, an equilibrium $a_- \in (0, (1-\gamma)^{-1})$ must always exist. Moreover a_- must be monotonically decreasing in σ^2 as a_i is monotonically decreasing in σ^2.

We now assess the conditions under which additional equilibria may also exist. Because $\lim_{a \to \infty} a_i = \gamma^{-1}$, the existence of a second equilibria (crossing the 45-degree line in Figure

1) implies the existence of a third. Thus, we must determine whether the difference $a_i(a) - a$ is positive anywhere in the range $a > (1-\gamma)^{-1}$. Such a difference is positive if and only if

$$\Phi(\sigma) \equiv \gamma (1 - a(1-\gamma))^2 (1 - \gamma a) - a (1-\gamma)^2 \sigma^2 > 0, \tag{62}$$

which requires $a < \gamma^{-1}$ as a necessary condition. Therefore, if two other equilibria exist they must lie in $((1-\gamma)^{-1}, \gamma^{-1})$. Fixing $a \in ((1-\gamma)^{-1}, \gamma^{-1})$, $\lim_{\sigma \to 0} \Phi(\sigma)$ is clearly positive, implying that there always exists a threshold $\bar{\sigma}$ such that two equilibria $a_+, a_\circ \in ((1-\gamma)^{-1}, \gamma^{-1})$ exist with $a_+ \geq a_\circ$ for $\sigma^2 \in (0, \bar{\sigma}^2)$. ∎

Proof of Proposition 2. Notice that $\partial\Gamma/\partial a > 0$ if and only if $\gamma < \min\{(1-\gamma)^{-1}, \gamma^{-1}\}$. The left-hand side of the fixed-point expression in (25) is downward-sloping in a and falling in σ, implying that the fixed-point intersection a_u and a_- must increase as σ falls. Similarly, $a_\circ$ falls and a_+ grows as σ falls, which implies amplifying feedback for the former and dampening feedback for the latter. ∎

Proof of Proposition 3. To prove the limiting statement for $\gamma \geq 1/2$, consider any point $a_\delta = \frac{1-\delta}{1-\gamma}$ such that $\delta > 0$. We then have

$$a_i(a_\delta) = \frac{\gamma \delta^2}{\gamma^2 \delta^2 + \sigma^2 (1-\gamma)^2.} \tag{63}$$

Since $\lim_{\sigma^2 \to 0} a_i(a_\delta) = \frac{1}{\gamma}$ for any δ, the unique equilibrium must converge to the same point. That the variance of this equilibrium approaches zero follows from equation (21).

To prove the limiting statement for $\gamma < 1/2$, recall the monotonicity of $a_i(a)$ on the range $(0, (1-\gamma)^{-1}$. Following the logic of proposition 1, for any point a_δ in that range, $\lim_{\sigma^2 \to 0} a_i(a_\delta) = \gamma^{-1}$, while $a_i((1-\gamma)^{-1}) = 0$. Thus, the intersection defining a_- must approach $(1-\gamma)^{-1}$. An analogous argument for the point just to the right of $(1-\gamma)^{-1}$ establishes that a_- converges to the same value. Finally, the bounded monotonic behavior of $a_i(a)$ establishes that for the high equilibrium $\lim_{\sigma^2 \to 0} a_+ = \gamma^{-1}$.

That the output variance of the high equilibrium in the limit $\sigma \to 0$ is zero follows from equation (22). The limiting variance of the two other limit equilibria can be established by noticing that (25) implies

$$\frac{\sigma^2}{(1 - a(1-\gamma))^2} = \frac{\gamma(1 - a\gamma)}{(1-\gamma)} \tag{64}$$

which, substituted into (22), gives (27) for $a \to (1-\gamma)^{-1}$. ∎

Proof of Proposition 6. We can prove that a sentiment-free equilibrium with no aggregate variance exists for $a = \gamma^{-1}$ by simple substitution in (56). The limiting variance of the other limit equilibrium at the singularity $a \to (1-\gamma)^{-1}$ can be established by noticing that (56) implies that

$$\frac{\sigma^2}{(1-a(1-\gamma))^2} = \frac{1-a\gamma}{a\gamma} + \frac{1-a(1-\gamma)}{a\gamma}\frac{\sigma^2}{(1-a(1-\gamma))^2},$$

which gives

$$\frac{\sigma^2}{(1-a(1-\gamma))^2} = -\frac{1-a\gamma}{1-a}.$$

Substituted into (55), this gives (27) for $a \to (1-\gamma)^{-1}$. ∎

Proposition 7. The derivative of $a_i(a)$ with respect to a is given by:

$$a_i'(a) = -\frac{2\gamma\,(1-\gamma)^3\,(1-(1-\gamma)\,a)\,\sigma^2}{\left((1-\gamma)^2\,\sigma^2 + (1-(1-\gamma)a)^2\,\gamma^2\right)^2}, \tag{65}$$

which is positive whenever $a > 1/(1-\gamma)^{-1}$. Then, necessarily, $a_i'(a_\circ) > 1$ and $a_i'(a_+) \in (0,1)$. Concerning the stability of a_-, notice that $\lim_{\sigma\to\infty} a_i'(a_-) = 0$ and

$$\lim_{\sigma^2\to 0, a\to(1-\gamma)^{-1}_{\pm}} a_i'(a) = \pm\infty,$$

given that, at the limit $\sigma \to 0$, σ^2 and $(1-a\,(1-\gamma))^2$ go to zero at the same speed. On the other hand, concerning the stability of a_u—which exists when $1-\gamma < \gamma$—notice that $\lim_{\sigma\to\infty} a_i'(a_u) = 0$ and

$$\lim_{\sigma^2\to 0, a\to\gamma^{-1}} a_i'(a) = \lim_{\sigma^2\to 0, a\to\gamma^{-1}} -\frac{2\gamma\,(1-\gamma)^3\,(1-(1-\gamma)\,a)\,\sigma^2}{\left((1-\gamma)^2\,\sigma^2 + (1-(1-\gamma)a)^2\,\gamma^2\right)^2} = 0_-,$$

which proves that the unique equilibrium a_u is locally unique rationalizable at the limit. ∎

Proposition 8. The derivative $a_i'(a)$ has been already studied. We know that $a_i'(a_u) < 0$, $a_i'(a_+) \in (0,1)$, $a_i'(a_-) < 0$ and $a_i'(a_\circ) > 1$. Nevertheless at the limit $\sigma \to 0$, where $a_+ = a_\circ$ coincide, there is no neighborhood to qualify a_+ a locally learnable rational expectations equilibrium. ∎

A.2 Derivations of the model

Proof. In this section we derive equilibrium in a version of the model extended to include convexity in the disutility of labor and money in the utility function. The baseline model in the text is obtained in the cashless limit with linear disutility of labor. The extended model demonstrates our claim in the text that using the Lagrange multiplier as a numeraire is equivalent to a more standard monetary numeraire.

The utility function of the representative family is

$$\int e^{\mu_i}\left(\log C_i - \phi N_i^{1+\alpha}\right) di + \varphi \log\left(M/P\right) \tag{66}$$

and the budget constraint is

$$M + \int P_i C_i di = \int W_i N_i di + QZ + \int \Pi_i di + M_-, \tag{67}$$

where $\alpha > 0$ is the inverse of the Frisch labor elasticity, φ is a constant parameter, M is the money holding of the family, and M_- is an exogenous initial endowment of the numeraire good "money." Money choice is delegated to a family member, the "money holder".

The information assumptions in the main text can be support as equilibrium outcomes when *all* agents observe the prices, whether posted or shadow cost, of the resources they use. In this case, producers see the local wage and aggregate endowment price Q, allowing them to fully infer both local and aggregate outcomes. Workers observe the utility cost of their labor input, which is enough for them to post an optimal labor supply schedule conditioned on the local wage. Shoppers observe the local price P_i, as well as the shadow cost of budget resources. The money-holder also observes the shadow cost of budget resources.

Under this more general information structure, the maximization problems of the representative family members become

$$\begin{aligned}
\text{money holder} &: \max_{M}\left\{\varphi\log(M) - \Lambda M\right\} \\
\text{producer} &: \max_{N_i, Z_{(i)}}\left\{P_i N_i^{\gamma}\left(e^{-\zeta} Z_{(i)}\right)^{1-\gamma} - W_i N_i - Q Z_{(i)}\right\} \\
\text{worker} &: \max_{N_i}\left\{E[\Lambda|W_i] W_i N_i - e^{\mu_i}\phi N_i^{1+\alpha}\right\} \\
\text{shopper} &: \max_{C_i}\left\{E[e^{\mu_i}|P_i]\log C_i - \Lambda P_i C_i\right\}
\end{aligned}$$

where Λ denotes the Lagrange multiplier associated with the budget constraint (67).

The money holder's first-order condition is

$$\frac{\varphi}{M} = \Lambda, \tag{68}$$

while market-clearing implies that $M = M_-$. The static structure of the economy means no further assumptions are needed to prevent rational bubbles from forming in the market for money. When the exogenous money supply is fixed *ex ante*, Λ is constant so that normalizing it is equivalent to assuming a monetary numeraire.

The other log-linear first-order conditions of the economy are given by:

$$
\begin{aligned}
w_i &= p_i + (\gamma - 1)n_i + (1-\gamma)z_{(i)} - (1-\gamma)\zeta \\
q &= p_i + \gamma n_i - \gamma z_{(i)} - (1-\gamma)\zeta \\
c_i &= \gamma n_i + (1-\gamma)z_{(i)} - (1-\gamma)\zeta \\
w_i &= \mu_i + \alpha n_i \\
c_i &= E[\mu_i|p_i] - p_i
\end{aligned}
$$

plus the market-clearing condition for the endowment $\int z_{(i)} di = 0$. Fixing $\alpha = 0$, these first order conditions are identical to the baseline version of the model. Moreover, notice that in the cashless limit of this economy, i.e. in the limit $\varphi \to 0$ with the utility weight φ and money supply M_{-} in constant proportion, (66) and (67) exactly match their analogues in the baseline model.

Aggregate variables. Averaging the two sides of the labor supply condition, we have $w = \alpha n$. Thus, we have

$$
\begin{aligned}
\alpha n &= p + (\gamma - 1)n - (1-\gamma)\zeta \\
q &= p + \gamma n - (1-\gamma)\zeta \\
c &= \gamma n - (1-\gamma)\zeta \\
c &= \int E[\mu_i|p_i] - p.
\end{aligned}
$$

This is a linear system in four unknown p, q, n, c, which can be expressed as functions of two states $\int E[\mu_i|p_i]di, \zeta$. Writing in matrix notation, we have

$$
\begin{bmatrix} p \\ q \\ n \\ c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1-\gamma+\alpha & 0 \\ 1 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \gamma^{-1} \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ q \\ n \\ c \end{bmatrix} + \begin{bmatrix} 0 & 1-\gamma \\ 0 & \gamma - 1 \\ 0 & (1-\gamma)\gamma^{-1} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \int E[\mu_i|p_i]di \\ \zeta \end{bmatrix},
$$

whose solution is

$$
\begin{aligned}
p &= \frac{1+\alpha-\gamma}{1+\alpha} \int E[\mu_i|p_i]di + (1-\gamma)\zeta \\
q &= \int E[\mu_i|p_i]di \\
n &= \frac{1}{1+\alpha} \int E[\mu_i|p_i]di \\
c &= \frac{\gamma}{1+\alpha} \int E[\mu_i|p_i]di - (1-\gamma)\zeta.
\end{aligned}
$$

Island-specific variables. The relevant system of equations is

$$\begin{aligned} E[\mu_i|p_i] - c_i &= p_i \\ c_i &= \gamma n_i + (1-\gamma)z_{(i)} - (1-\gamma)\zeta \\ w_i &= p_i + (\gamma - 1)n_i + (1-\gamma)z_{(i)} - (1-\gamma)\zeta \\ q &= p_i + \gamma n_i - \gamma z_{(i)} - (1-\gamma)\zeta, \end{aligned}$$

which constitutes a linear system in four unknown $p_i, c_i, n_i, z_{(i)}$ that can be expressed as functions of four states $\mu_i, \zeta, q, E[\mu_i|p_i]$. This system can be written as

$$\begin{bmatrix} p_i \\ c_i \\ n_i \\ z_{(i)} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & \gamma & 1-\gamma \\ \frac{1}{1+\alpha-\gamma} & 0 & 0 & \frac{1-\gamma}{1+\alpha-\gamma} \\ \gamma^{-1} & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} p_i \\ c_i \\ n_i \\ z_{(i)} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & \gamma - 1 & 0 & 0 \\ -\frac{1}{1+\alpha-\gamma} & -\frac{1-\gamma}{1+\alpha-\gamma} & 0 & 0 \\ 0 & (\gamma-1)\gamma^{-1} & -\gamma^{-1} & 0 \end{bmatrix} \begin{bmatrix} \mu_i \\ \zeta \\ q \\ E[\mu_i|p_i] \end{bmatrix}$$

where we already used $w_i = \mu_i + \alpha\mu_i$. The solution of the system is

$$\begin{aligned} c_i &= -\frac{\gamma}{1+\alpha}\mu_i + \frac{1+\alpha(1-\gamma)}{1+\alpha}E[\mu_i|p_i] - (1-\gamma)(q+\zeta) \\ p_i &= \frac{\gamma}{1+\alpha}\mu_i + \frac{\alpha\gamma}{1+\alpha}E[\mu_i|p_i] + (1-\gamma)(q+\zeta) \\ n_i &= \frac{1}{1+\alpha}\left(E[\mu_i|p_i] - \mu_i\right) \\ z_{(i)} &= -q + E[\mu_i|p_i], \end{aligned}$$

which is consistent with the expression for their relative aggregate variables.

In the case $\alpha \neq 0$, notice that the price signal can equivalently be written as

$$\tilde{p}_i = \frac{1+\alpha}{1+\alpha(1-\gamma)}\left(p_i - \frac{\alpha\gamma}{1+\alpha}E[\mu_i|p_i]\right) = \tilde{\gamma}\mu_i + (1-\tilde{\gamma})(q+\zeta),$$

where $\tilde{\gamma} = \gamma/(1+\alpha(1-\gamma))$. Notice that limit sentiment equilibria now exist with $\tilde{\gamma} < 1/2$, which could well obtain even with $\gamma > 1/2$ for a sufficiently high α. ∎